The generic properties of equilibrium correction mechanisms*

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6 August 1999

Abstract

Linear dynamic equilibrium correction mechanisms are shown to follow from the discretisation of continuous economic processes with steady-state solutions. In addition, the proposed procedure provides testable restrictions on the coefficients of the dynamic econometric model.

Keywords

Equilibrium correction, dynamic modelling, difference schemes, power series representations

JEL Classification *C50*

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*We would like to thank Øyvind Eitrheim and Fredrik Wulfsberg for helpful comments.

Introduction

Since the publication of Davidson, Hendry, Srba and Yeo (1978), equilibrium correction mechanisms (ECMs) such as

$$\triangle y_t = a(y - bx - c)_{t-1} + p\triangle y_{t-1} + q\triangle x_{t-1} + u_t \qquad (1)$$

have been used successfully to model dynamic economic relationships. The rationale for such econometric specifications has been derived either from economic theory, as in Nickell (1985), or from the properties of time series models, as in the seminal work on cointegration by Engle and Granger (1987). Hendry (1995, pp. 286-294) provides an introduction to and overview of these motivations, while Alogoskoufis and Smith (1991) give an extensive list.

In contrast, we point out that a dynamic ECM follows naturally from the discretisation of a differential equation with a steady-state solution. In addition, the linear case provides testable restrictions on the coefficients of the ECM formulation.

Backward-difference schemes

Let $t_1,\ t_2,\ \cdots,\ t_k,\ \cdots$ be a sequence of times spaced h apart and let $y_1,\ y_2,\ \cdots,\ y_k,\ \cdots$ be the values of a continuous real variable $y(t)$ at these times. The backward-difference operator $\triangle$ is defined by the rule

$$\triangle y_k = y_k - y_{k-1}\ , \qquad k \geq 1\ . \qquad (2)$$

By observing that $y_k = (1-\triangle)^0\, y_k$ and $y_{k-1} = (1-\triangle)^1\, y_k$, the value of y at the intermediate point $t = t_k - sh$ $(0 < s < 1)$ may be estimated by the interpolation formula

$$y(t_k - sh) = y_{k-s} = (1-\triangle)^s\, y_k\ , \qquad s \in [0,1]\ . \qquad (3)$$

When s is not an integer, $(1-\triangle)^s$ should be interpreted as the power series in the backward-difference operator obtained from the binomial expansion of $(1-x)^s$. This is an infinite series of differences. Specifically

$$(1-\triangle)^s = 1 - s\triangle - \frac{s(1-s)}{2!}\triangle^2 - \frac{s(1-s)(2-s)}{3!}\triangle^3 - \cdots . \qquad (4)$$

With this preliminary background, the differential equation

$$\frac{dy}{dt} = f(y,x)\ , \qquad x = x(t)\ , \qquad (5)$$

may be integrated over the time interval $[t_k, t_{k+1}]$ to obtain

$$y(t_{k+1}) - y(t_k) = \triangle y_{k+1} = \int_{t_k}^{t_{k+1}} f(y(t), x(t))\, dt \qquad (6)$$

in which the integral on the right hand side of this equation is to be estimated by using the backward-difference interpolation formula given in equation (4). The substitution $t = t_k + sh$

is now used to change the variable of this integral from $t \in [t_k, t_{k+1}]$ to $s \in [0, 1]$. The details of this change of variable are

$$\int_{t_k}^{t_{k+1}} f(y(t), x(t))\, dt = \int_0^1 f(y(t_k + sh), x(t_k + sh))\,(h\, ds) = h \int_0^1 f_{k+s}\, ds$$

where $f_{k+s} = f(y(t_k + sh), x(t_k + sh))$. The value of this latter integral is now computed using the interpolation formula based on (4). Thus

$$\begin{aligned} \int_0^1 f_{k+s}\, ds &= \int_0^1 (1 - \triangle)^{-s} f_k\, ds \\ &= \int_0^1 \left(f_k + s\triangle f_k + \frac{s(1+s)}{2!} \triangle^2 f_k + \frac{s(1+s)(2+s)}{3!} \triangle^3 f_k + \cdots \right) ds \\ &= f_k + \frac{1}{2} \triangle f_k + \frac{5}{12} \triangle^2 f_k + \frac{3}{8} \triangle^3 f_k + \cdots . \end{aligned}$$

The final form for the backward-difference approximation to the solution of this differential equation is therefore

$$\triangle y_{k+1} = h f_k + \frac{h}{2} \triangle f_k + \frac{5h}{12} \triangle^2 f_k + \frac{3h}{8} \triangle^3 f_k + \cdots . \tag{7}$$

Model specification

The backward-difference scheme (7) is valid for all suitably differentiable functions $f(y, x)$. For ease of exposition, consider the simplest case in which a constant input $x = X$ induces $y(t)$ to approach asymptotically a constant state Y as $t \to \infty$. Clearly X and Y satisfy $f(Y, X) = 0$. The usual procedure is to expand the differential equation about this steady-state solution (see Gandolfo, 1997, p260). Employing this procedure yields

$$f(y, x) = f(Y, X) + \frac{\partial f(Y, X)}{\partial y}(y - Y) + \frac{\partial f(Y, X)}{\partial x}(x - X) + R \tag{8}$$

where

$$R = \frac{1}{2!} \left(\frac{\partial^2 f(\xi, \eta)}{\partial x^2}(x - X)^2 + 2\frac{\partial^2 f(\xi, \eta)}{\partial x \partial y}(x - X)(y - Y) + \frac{\partial^2 f(\xi, \eta)}{\partial y^2}(y - Y)^2 \right)$$

and (ξ, η) is a point such that ξ lies between y and Y while η lies between x and X. Since Y and X are the steady-state values for y and x respectively, then the expression for $f(y, x)$ takes the simplified form

$$f(y, x) = a\,(y - Y) + \beta\,(x - X) + R \tag{9}$$

where $a = \partial f(Y, X)/\partial y$ and $\beta = \partial f(Y, X)/\partial x$ are constants.

If f is a linear function of y and x then $R = 0$ and so

$$f(x, y) = a \left(y - Y + \frac{\beta}{a}(x - X) \right) = a\,(y - bx - c)\ , \tag{10}$$

in which $b = -\beta/a$ and $c = Y + (\beta/a)X$. This expression for f, when applied to the backward-difference scheme (7), gives the ECM representation (with $k = t-1$ and $h = 1$)

$$\triangle y_t = a\,(y - bx - c)_{t-1} + \frac{1}{2}\,a(\triangle y_{t-1} - b\,\triangle x_{t-1}) + u_t\ , \qquad (11)$$

$$u_t = \frac{5}{12}\,a(\triangle^2 y_{t-1} - b\triangle^2 x_{t-1}) + \cdots\ . \qquad (12)$$

Equation (11), expressed in reduced form, is identical to the ECM equation (1). In particular, if f is a linear function then theory predicts that $p = a/2$ and $q = ab/2$. These restrictions should, in principle, be testable.

On the other hand, if f is a nonlinear function of y and x, the backward-difference analysis is still valid but now the appropriate scheme is

$$\triangle y_t = a\,(y - bx - c)_{t-1} + R_{t-1} + \frac{1}{2}\,a(\triangle y_{t-1} - b\,\triangle x_{t-1}) + \frac{1}{2}\,\triangle R_{t-1} + u_t\ ,$$

$$u_t = \frac{5}{12}\,a(\triangle^2 y_{t-1} - b\,\triangle^2 x_{t-1}) + \frac{5}{12}\,\triangle^2 R_{t-1} + \cdots\ .$$

It can now be expected that the presence of non-negligible contributions from R contaminates p and q so that the linear restrictions derived previously are no longer applicable.

Conclusion

This note shows that the ECM representation popular in the dynamic econometric literature follows from the discretisation of a continuous process with a steady-state solution. This recognition should prompt further research—perhaps the most immediate relates to the practical feasibility of testing the linear restrictions identified previously.

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