A Quantitative Discursive Dilemma*

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Abstract

The typical judgment aggregation problem in economics and other fields is the following: A group of people has to judge/estimate the value of an uncertain variable y which is a function of k other variables, i.e. $y = D(x_1, ...x_k)$. We analyze when it is possible for the group to arrive at collective judgements on the variables that respect D. We consider aggregators that fulfill Arrow's IIA-condition and neutrality. We show how possibility and impossibility depend on the functional form of D, and generalize Pettit's (2001) binary discursive dilemma to quantitative judgements.

Keywords: Judgment aggregation, Dependent variables, Impossibility, Possibility

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1 Introduction

The typical decision problem – or judgment aggregation situation – in economics and other fields is the following: A group of people has to judge or estimate the size of a variable which is a function of some other variables. For example, a monetary policy committee's interest rate decisions depend on judgments about inflationary pressures and financial fragility; a cabinet's judgment of the future budget balance depends on its judgments of future revenues and costs; a corporate board's investment decisions depend on judgments of future cash flows and cost of capital. In this paper we analyze when a group of people with different judgments on the variables can make a collective judgment on the variables that respects the dependence between the variables.

To illustrate the issue and its importance, consider a corporate board assessing the profitability of an investment project. The profitability is measured as the project's expected net present value (NPV), which per definition is the discounted cash flow (DCF) less the investment cost (IC). Thus, the dependence between the variables is given by $NPV = DCF - IC$. Suppose that the corporate board has three members with estimates as in Table 1. Let the members of the board vote on the size of each variable, and assume that the outcome of the vote is the median of the individual estimates.

Table 1: Example of aggregate inconsistency for a corporate board assessing the net present value of an investment project.

	Discounted cash flow (DCF)	Investment cost (IC)	Net present value (NPV)
Member A	10	8	2
Member B	10	11	-1
Member C	13	12	1
Board	10	11	1

Then a vote on the conclusion-variable gives $NPV = 1$. But, this is not consistent with the majority's judgments on the two 'premise-variables', since $10 - 11 = -1$. Thus, the aggregate judgments do not respect the dependence between the variables. As a consequence the board faces a discursive dilemma (Pettit 2001). A premise-based procedure, where they vote on the two premise variables and let the conclusion follow, gives $NPV = -1$. A conclusion-based procedure, where they vote directly on NPV, gives $NPV = 1$.

In this paper we investigate if the example illustrates a general problem for groups aggregating judgments on dependent variables. We therefore construct a general social choice theoretic model and ask the following question: Under which conditions are there combinations of individual judgments that give aggregate judgments that do not respect the dependence between the variables (impossibility), and under which conditions are there no such combinations (possibility)? By using a general social choice theoretic framework we can treat all aggregation methods fulfilling some general conditions simultaneously. Examples of aggregators fulfilling our conditions are pairwise majority voting over the alternative values for each variable, and an agenda-setting method whereby the aggregate judgment of any two alternatives for one variable is the judgment of the agenda setter (the same each time) unless a supermajority has another judgment.

In our model a group of people has to conclude on the value of a dependent variable x_{k+1} when the value of this variable depends on the value of k independent variables

x_1,..., x_k by some general 'dependence function' D:

$$x_{k+1} = D(x_1, ..., x_k)$$

The dependence function can be a reaction function derived from maximizing an objective function, it can be a rule-of-thumb, a causal relationship between economic variables, a definition, or any mapping from values of the independent variables to the dependent variable. The arguments in the dependence function are the variables and parameters on which the members of the group may have different judgments. Variables and parameters that are relevant for the dependent variable, but which the members of the group always agree on, may be represented by the functional form of D. Suppose, for instance, that $y = \alpha x$ is a policy rule where y is a policy instrument (e.g. the central bank's key interest rate), x is an economic variable (e.g. the rate of underlying inflation), and α is a parameter that says how much a change in x should affect y. Then the dependence function is the policy rule (with x as the argument) if all individuals always agree on the value of α. Otherwise the dependence function has two arguments: x and α.

When modelling the judgments on each variable we follow the social choice tradition. We assume that each member holds a strict order over the alternative values for each of the $k + 1$ variables: an order over the alternative values of variable 1, an order over the alternative values for variable 2, ..., and an order over the alternative values for variable $k + 1$. The group uses an aggregator that takes profiles of individual orders (one order for each member) as inputs. The aggregator produces an aggregate relation (not necessarily an order), and fulfills a set of standard general conditions. The conditions are 'unanimity/Pareto', 'independence' and a strong and a weaker form of 'neutrality'. We derive characterization results for when there exist non-dictatorial aggregators such that the peaks of the aggregate relations respect the dependence function. It is seen that possibility arises only in the special case when $k = 1$ and the dependence function is strictly monotonic.

The paper has four sections. In Section 2 we present the model. In Section 3 we give the main characterization. We conclude with a discussion of our framework and key assumptions in Section 4.

Relation to the literature

Considering the aggregation of different interconnected variables is not new. A variety of aggregation problems has been proposed and solved in production theory, see Blackorby & Schworm (1984) for an overview. In opinion pooling the probability assignments of different individuals are to be merged into collective probability assignments. Genest & Zidek (1986) give an overview of classical results in opinion pooling. See Mongin (1995) and Dietrich & List (2007b) for more recent results. Rubinstein & Fishburn (1986) consider the problem of aggregating the entries in n rows in an $n \times m$ matrix into a summary row, where every entry is an element in an algebraic field. They find that if the entries always form a hyperplane, then every consistent aggregator is an aggregator whereby the aggregate estimate of a variable is the (normalized) linear sum of the individual estimates. If the entries do not form a hyperplane there is no consistent non-dictatorial aggregator.

In an earlier paper on quantitative discursive dilemmas we (Claussen and Røisland 2005) study a situation somewhat similar to the situation studied by Rubinstein & Fishburn (1986), but where we assign one variable the role as a dependent variable and the other variables the role as independent variables. Furthermore we have less strict domain restrictions. In that paper we find that if the group aggregates by taking the mean of the individual estimates, then the boundary between possibility and impossibility lies in whether or not the dependence function is linear. If the group aggregates by taking the median of the individual estimates, the boundary lies in whether or not the dependence function is strictly monotonic. In the current paper we step out of the model of our previous paper and the literature on the aggregation of different interconnected variables by considering the aggregation of $k+1$ orders rather than aggregation of $k+1$ estimates. By this move we are able to study the situation when a group aggregates by some voting method, rather than by just combining estimates.

The setting of this paper is also somewhat parallel to a setting where a group of people aggregates judgments on interconnected propositions. In such situations an aggregation inconsistency akin to the inconsistency in the example of Table 1 may arise. Pettit (2001) coined that inconsistency the 'discursive dilemma'. Recently, researchers have built general social choice theoretic models to study the aggregation of judgments on propositions. The first example is List & Pettit (2002). They also provided the first impossibility result which was quickly followed by several stronger impossibility and possibility results. Roughly speaking, the impossibility results say that if the propositions under consideration are interlinked, then there is no aggregator that fulfils requirements similar to, but not exactly equal to, the Arrovian requirements that aggregate consistent individual judgments on propositions into consistent collective judgments on these propositions. See Dietrich (2007) for a generalized model of judgment aggregation, and List & Puppe (2009) for an overview of the literature. Compared to the judgment aggregation literature the important novelty our current paper is that variables need not be binary. Thus, we introduce a generalization of Pettit's (2001) discursive dilemma to non-binary and continuous variables.

2 The Model[1]

We consider a group, where $N = \{1, ..., n\}$ denotes the set of members, and $n > 2$.[2] Each member $i \in N$ will be referred to as a 'member' or an 'individual' depending on the context. The group has to evaluate real-valued variables $j = 1, ..., k+1$ where $k \geq 1$ and each variable j takes values in a non-empty set $X_j \subseteq \mathbb{R}$. This set has at least two elements and might be finite or infinite. Examples are $X_j = \mathbb{R}$, $X_j = [0, 1]$, and the binary case where $X_j = \{0, 1\}$ as in standard judgment aggregation. The variables $1, ..., k$ will be denoted 'independent variables', and variable $k+1$ the 'dependent variable'.

Let

$$D : X_1 \times ... \times X_k \to X_{k+1}$$

be a surjective function, the *dependence function*, representing how the dependent variable $k+1$ depends on the independent variables $1, ..., k$.[3]

A *preference relation* on a set X_j is an arbitrary binary relation $\succeq$ on X_j.[4] Its asymmetric part (representing strict preference) is as usual denoted by $\succ$ and defined as the binary relations on X_j given by $x \succ y \Leftrightarrow [x \succeq y$ and not $y \succeq x]$ for all $x, y \in X_j$. Its symmetric part (representing indifference) is as usual denoted by $\sim$ and defined as the

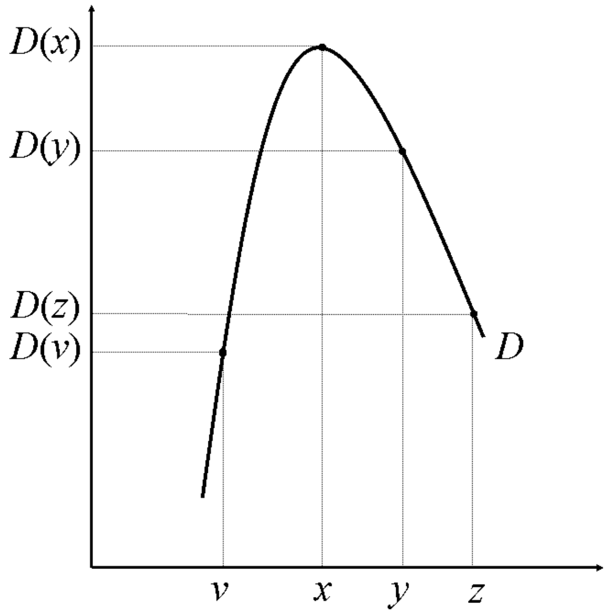
[1] This and the following section have benefitted greatly by detailed comments and suggestions from one of the referees.

[2] We assume $n > 2$ to make the propositions clear-cut (not contingent on n). If $n = 2$, systematicity (see section 3) implies that the dependence must be dictatorial, regardless of the functional form of D.

[3] A function D is said to be *surjective* or *onto*, if its values span its whole codomain; that is, for every $x_{k+1} \in X_{k+1}$, there is at least one vector $(x_1, .., x_k) \in X_1 \times ... \times X_k$ such that $D(x_1, .., x_k) = x_{k+1}$.

[4] The term 'preference' should not be taken literally and is not meant as a restriction. The model applies in many contexts; see Section 4 for examples and a discussion.

Figure 1: Illustration with $k = 1$ and concave dependence function.



binary relations on X_j given by $x \sim y \Leftrightarrow [x \succeq y$ and $y \succeq x]$ for all $x, y \in X_j$. Let $\mathcal{G}^*_{X_j}$ be the set of complete and anti-symmetric preference relations on X_j (i.e. all preference relations on X_j that satisfy $[x \succ y$ or $y \succ x]$ for all distinct $x, y \in X_j$ and $x \sim x$ for all $x \in X_j$). Let $\mathcal{G}_{X_j}$ be the set of complete, anti-symmetric and transitive preference relations on X_j, (i.e. complete and anti-symmetric preference relations that also satisfy $[x \succcurlyeq y$ and $y \succcurlyeq z] \Rightarrow x \succcurlyeq z$ for all $x, y, z \in X_j$). An element in $\mathcal{G}_{X_j}$ is called a *(strict) order*. Note that $\mathcal{G}_{X_j} \subsetneq \mathcal{G}^*_{X_j}$ as the relations in $\mathcal{G}^*_{X_j}$ need not be transitive.

A value $x \in X_j$ is the *peak* of variable j under $\succcurlyeq_{X_j} \in \mathcal{G}^*_{X_j}$ if $x \succ_{X_j} y$ for all $y \in X_j \backslash x$. Peaks will sometimes, depending on the context, be called estimates. A sequence of relations $(\succcurlyeq_{X_1}, ..., \succcurlyeq_{X_{k+1}}) \in \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ is said to *respect the dependence function* D if $x_{k+1} = D(x_1, ..., x_k)$ whenever $x_1, ..., x_{k+1}$ are peaks of $\succcurlyeq_{X_1}, ..., \succcurlyeq_{X_{k+1}}$, respectively.

We will also use a (rationality) requirement for alternatives ranked lower than the peaks. The requirement will rule out sequences that respect D but where the preference relations are otherwise somewhat arbitrary. To give an illustration, put $k = 1$, $X_1 = \{v, x, y, z\}$, and suppose D is concave as illustrated in Figure 1. Let the preference relation on X_1 be $v \succ x \succ y \succ z$. Our requirement will then allow for individual sequences with orders over the corresponding alternatives in X_2 like

$$D(v) \succ D(x) \succ D(y) \succ D(z)$$
or
$$D(v) \succ D(z) \succ D(y) \succ D(x),$$

but it will rule out arbitrary individual sequences like sequences where the order on X_2 is

$$D(v) \succ D(y) \succ D(x) \succ D(z)$$
or
$$D(v) \succ D(x) \succ D(z) \succ D(y).$$

Formally, a sequence of relations $\left(\succcurlyeq_{X_1}, ..., \succcurlyeq_{X_{k+1}}\right) \in \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ is called *arbitrary* if there exists $\boldsymbol{x}', \boldsymbol{x}'', \boldsymbol{x}''' \in X_1 \times ... \times X_k$ and $m \in \{1, ..., k\}$ with x'_m, x''_m, x'''_m three pairwise distinct elements in X_m and $x'_j = x''_j = x'''_j$ for all $j \neq m$, such that (i) D is strictly monotonic for $\boldsymbol{x}', \boldsymbol{x}'', \boldsymbol{x}'''$, (ii) $x'_m \succ_{X_m} x''_m \succ_{X_m} x'''_m$ and (iii) $\succcurlyeq_{X_{k+1}}$ranks $D(\boldsymbol{x}'')$ strictly above or below both $D(\boldsymbol{x}''')$ and $D(\boldsymbol{x}')$. A sequence is *non-arbitrary* if it is not arbitrary.

Let $\mathcal{G}$ be the set of all non-arbitrary sequences $(\succcurlyeq_{X_1}, ..., \succcurlyeq_{X_{k+1}}) \in \mathcal{G}_{X_1} \times ... \times \mathcal{G}_{X_{k+1}}$ that respect D. A *profile (of sequences)*, denoted g, is an $(k+1)n$-tuple in $\mathcal{G}^n$ with one sequence for each member.

An *aggregator* f is a mapping

$$f : \mathcal{G}^n \to \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}.$$

Notice that the aggregator takes (profiles of) *orders* as inputs but produce *relations*. Thus, we do not require the outcome of the aggregation to be transitive. Furthermore, we do not require the outcome of the aggregation to be non-arbitrary. The *aggregator respects the dependence function* D if, for every profile $g \in \mathcal{G}^n$, $f(g)$ respects D.

Denote individual orders by $\succcurlyeq_{i,X_j}$ and aggregate relations by $\succcurlyeq_{N,X_j}$. We say that the aggregator is *non-dictatorial* if there is no $i \in N$ such that for all profiles $g \in \mathcal{G}^n$, $f(g) = (\succcurlyeq_{i,X_j})_{j=1,...,k+1}$.

3 The Characterization

We will now see when there is a $g \in \mathcal{G}^n$ such that $f(g)$ does not respect the dependence function (impossibility), and when $f(g)$ respects the dependence function for all $g \in \mathcal{G}^n$ (possibility).

We will consider aggregators that fulfill a set of standard conditions. The first condition is the unanimity principle which says that if every member of the group finds that $x \in X_j$ is better than $y \in X_j$, then the collective view should also be that x is better than y. The second condition is Arrow's independence condition (IIA). This condition says that the aggregator obtains aggregate relations by comparing two alternatives at a time taken in isolation from the other alternatives. Thus, the aggregate preference of any pair of alternatives for a variable will depend exclusively on the individual preferences over that pair. Consequently, the aggregation is independent between variables and independent for each variable seen in isolation. Formally, the conditions are as follows.

Unanimity principle/Pareto: For all $(\succcurlyeq_{i,X_j})_{i\in N, j=1,...,k+1} \in \mathcal{G}^n$, all $j \in \{1, ..., k+1\}$ and all $x, y \in X_j$, if $x \succ_{i,X_j} y$ for all $i \in N$, then $x \succ_{N,X_j} y$.

Independence (of Irrelevant Alternatives): For any two profiles in $\mathcal{G}^n$, $g^a = (\succcurlyeq^a_{i,X_j})_{i\in N, j=1,...,k+1}$, $g^b = (\succcurlyeq^b_{i,X_j})_{i\in N, j=1,...,k+1}$, any variable j and any alternatives $x', x'' \in X_j$, if for all individuals i $[x' \succcurlyeq^a_{i,X_j} x'' \Leftrightarrow x' \succcurlyeq^b_{i,X_j} x'']$, then $[x' \succcurlyeq^a_{N,X_j} x'' \Leftrightarrow x' \succcurlyeq^b_{N,X_j} x'']$.

In addition to fulfilling independence, we require the aggregator to be neutral in two respects. First, if the aggregate preference over two alternatives for one variable is determined by some method, for example a pair-wise majority vote, then the aggregate preference on any other two alternatives for the same variable shall be determined by the same method. Second, if the aggregate preference relation on one variable is determined by some method, then the aggregate preference relation on any other variable shall be determined by the same method. Neutrality and independence give the following condition:[5]

Systematicity: For any two profiles in $\mathcal{G}^n$, $g^a = (\succcurlyeq^a_{i,X_j})_{i \in N, j=1,...,k+1}$, $g^b = (\succcurlyeq^b_{i,X_j})_{i \in N, j=1,...,k+1}$, any two variables j and m, and any alternatives $x'_j, x''_j \in X_j$ and $x'_m, x''_m \in X_m$, if for all individuals i $[x'_j \succcurlyeq^a_{i,X_j} x''_j \Leftrightarrow x'_m \succcurlyeq^b_{i,X_m} x''_m]$ then $[x'_j \succcurlyeq^a_{N,X_j} x''_j \Leftrightarrow x'_m \succcurlyeq^b_{N,X_m} x''_m]$.

The following proposition holds.

Proposition 1 *A non-dictatorial aggregator* $f : \mathcal{G}^n \to \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ *that satisfies the Unanimity principle and Systematicity respects the dependence function* D *if and only if* $k = 1$ *and* D *is strictly monotonic.*

Proof. The proof is in the appendix. ■

It might come as a surprise that the boundary between possibility and impossibility is somewhat simpler in our framework where the group aggregates orders than in a model of aggregating just estimates, c.f. Rubinstein & Fishburn (1986). In the latter case, a crucial question is whether the dependence function is linear. The reason why we get the simpler boundary is that in our case only the relative ranking of the estimates for a variable matter in the aggregation. In the literature on the aggregation of estimates the relative size of the individuals' estimates matters for the aggregate estimate. This is because the authors assume that the aggregate estimate is some linear or non-linear combination of the individual estimates. An exception is Claussen and Røisland (2005) where we assume that the aggregate estimate is the median of the individual estimates. With this aggregator, the crucial question is whether the dependence function is strictly monotonic or not, as it is in the case with aggregating orders.

4 Discussion

We will conclude by a discussion of our framework and some key assumptions.

Preference relations

We use the term 'preference relation' and not e.g. 'judgment relation', as 'preference relation' is the well established term in the literature. The term 'preference' should not be taken literally and is not meant as a restriction. The model applies in many contexts. To see this, remember that the definition of a preference relation only says that each member can, for any two distinct alternatives $x, y \in X_j$, say that she 'prefers' x to y (or y to x). The definition does not say anything about *why* she 'prefers' x to y. Member i could, for instance, prefer x to y because she finds that x gives her higher utility than y, she could prefer x to y because she believes that x is closer to the true value of the variable than y (it is a "better estimate"), or – if variable j is a policy variable – she could prefer x to y because she finds that x gives higher social welfare than y.

Another question is if the members actually hold preference relations. Empirically it is clear that members of many groups do. In monetary policy, for instance, the minutes of the meetings of the monetary policy committees reveal that the members disagree and

[5] The condition is inspired by a similar concept from the literature on the aggregation of judgments on propositions where it was first introduced by List & Pettit (2002). It will be relaxed somewhat in Section 4.

have preference relations over the relevant alternatives for the key interest rate and the premise-variables. Similarly, minutes and reports of other expert panels reveal that the members have preference relations over relevant alternatives for the relevant variables. In formal models any cardinal utility function embodies a preference relation.

Strict preference relations

The assumption of *strict* preference relations may at first glance seem strong. But, it is usually a reasonable assumption, in particular if variables are continuous. If a variable is continuous, it is in practice impossible for the group to consider all possible alternatives in the aggregation. What groups normally do is to perform an aggregation – the 'ote' – over a limited set of alternatives. These alternatives will typically be each member's preferred value, i.e. the peaks of the individual preference relations. The combination of a continuous variable and aggregation over peaks only imply that it is reasonable to assume strict preferences. To see this, suppose first that $Z_j (\neq \emptyset)$ is a convex subset of $\mathbb{R}$ representing all possible values variable j can take. Let $\{x_1, ..., x_n\} \in Z_j^n$ be the set of peaks with one peak for each member of N (orders may be weak). As the group only consider the set of peaks when aggregating we put $X_j = \{x_1, ..., x_n\}$, i.e. the set of alternatives that is up for a vote is X_j (and not Z_j). Let member i's preferences over the alternatives in Z_j be described by the 'preference' function $u = -(x - x_i)^2$, where $x_i \in Z_j$ is the most preferred alternative of member i. Notice that the function imply a single peaked (weak) preference relation. Suppose that x_i is drawn from a distribution described by a continuous density function $h_i(x)$ over Z_j (allowing for different continuous density functions for each member). Now, suppose for contradiction that member i is indifferent between two distinct alternatives $x_s, x_m \in X_j \backslash x_i$. It then follows from the preference function that $x_s = 2x_i - x_m$. However, as all elements in X_j are drawn from continuous distributions, we have that per definition $\int_{x_m}^{x_m} h_s(x)\, dx = \int_{x_s}^{x_s} h_m(x)\, dx = 0$. Thus, there is zero probability that $x_s = 2x_i - x_m$, i.e. there is zero probability that one of the members of N is indifferent between two alternatives in X_j.

Discrete variables are often discrete for practical reasons (e.g. rounding), not because the variable is discrete in nature. An argument similar to the argument above therefore applies. The interest rate decisions of monetary policy committees are illustrative. The key policy interest rate of a central bank is a continuous variable that in theory can take any value in $\mathbb{R}$. However, in practice monetary committees only consider alternatives in the set $Q = \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, ...\}$. Suppose that the members' preferences are described by $u = -(x - x_i)^2$ as above. Suppose that a member of a monetary policy committee find 2 to be the best level of the key rate among the elements in Q. Would she then be indifferent between $1\frac{3}{4}$ and $2\frac{1}{2}$? If $x_i = 2$ she would. However the preference relation over $\mathbb{R}$ would typically have its peak at another value than 2. When considering the true set of values that the key rate may take, the member's best estimate would be, say, 1.90, but the preferred rate of the alternatives in Q is 2 (the closest feasible one). But then, with a symmetric preference relation, the member would not be indifferent between $1\frac{3}{4}$ and $2\frac{1}{2}$. She would prefer $1\frac{3}{4}$ to $2\frac{1}{2}$.

The aggregator

The aggregator used in the model takes the individual orders as inputs, and produces an aggregate relation for each variable. We think it is relevant to study the properties of this aggregator for at least three reasons: First, if the members of the group cannot agree, but have to reach a decision, they have to use some aggregation method. Many groups resort to majority voting or some other method that implicitly take ordinal preferences as the input and output of the aggregation. They do this even though the primary interest is the highest ranked alternative. The method is often 'implicit' as the group does not pursue an explicit aggregation and spell out the aggregate preference over pairs where the aggregate preference is obvious. Furthermore, the aggregate preferences of some pairs may not be of interest and are therefore not explicitly spelled out. Monetary policy committees, for instance, will never make an explicit aggregation over all possible values for the key interest rate, but only pursue an explicit vote over the pairs where there is disagreement.[6] Second, in theoretical models of economics and political economy, methods where the alternatives are cast against each other in a pairwise vote is typically assumed to be the aggregation method (see e.g. Persson & Tabellini (2000)). It is therefore useful to have a characterization for such aggregation methods. Third, we want to relate to the existing literature on binary judgment aggregation and introduce a generalization of Pettit's (2001) binary discursive dilemma to non-binary quantitative judgments.

Systematicity

Systematicity implies that the characterization only regards situations where the aggregation on each variable is independent of the aggregation on the other variables. However, our result is relevant also if this condition is violated, especially from a normative perspective. Suppose, for instance, that $k = 1$, D is concave as in Figure 1, and that the group aggregates by pairwise majority voting. Then a premise-based procedure – a procedure where the aggregate estimate for the conclusion variable follows from the aggregate estimates of the independent variable and the dependence function – will tend to give a higher estimate of the dependent variable than a conclusion-based procedure where the group aggregates directly over the judgments for the dependent variable. Thus, even though the group may in fact use a premise-based procedure (which violates variable wise independence), there exists an alternative procedure, the conclusion-based procedure, which will tend to give a higher estimate than a premise-based procedure. Similar effects arise when $k > 1$. Thus, our results highlight when groups face a choice between different procedures that may give different expected outcomes.

Systematicity also implies that the characterization only regards situations where the group uses the same aggregation method for each variable. Relaxing this neutrality condition will give impossibility. Consider the following weakening of Systematicity allowing for different aggregation methods on different variables.

Weak systematicity: For any two profiles in $\mathcal{G}^n$, $g^a = (\succsim^a_{i,X_j})_{i\in N, j=1,\ldots,k+1}, g^b = (\succsim^b_{i,X_j})_{i\in N, j=1,\ldots,k+1}$, any variable j, and any alternatives $x, y \in X_j$ and $x', y' \in X_j$, if for all individuals i $[x \succsim^a_{i,X_j} y \Leftrightarrow x' \succsim^b_{i,X_j} y']$ then $[x \succsim^a_{N,X_j} y \Leftrightarrow x' \succsim^b_{N,X_j} y']$.

Notice that the set of aggregators fulfilling Systematicity is entailed in the set of aggregators fulfilling Weak systematicity. A straight forward corollary from our proposition is therefore that there is impossibility for $k > 1$ and for non-monotonic dependence functions also under Weak systematicity. Furthermore, a corollary (less straight forward) is that there is impossibility also for cases when there is possibility under Systematicity.

[6] Notice also that our framework does not require the members to have judgments on irrelevant values of a variable, and it does not require the group to aggregate judgments over irrelevant alternatives. To see this, consider again the example of Table 1. For our model to apply, it is sufficient that each member have an order over the estimates in the table, and that these are the alternatives considered in the aggregation. Thus, the set X_j may be the set of alternatives for the variable that has been put on the table, or the set of all values that the variable may take in any hypothetical world. Our results apply in both cases.

Corollary 1 *If* $|X_j| > 2$ *for some* $j \in \{1, ..., k+1\}$*, no non-dictatorial aggregator* $f : \mathcal{G}^n \rightarrow \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ *that satisfies the Unanimity principle and Weak systematicity respects the dependence function.*

Proof. The proof is in the appendix. ■

It is easy to come up with examples where aggregate peaks do not respect the dependence function if $|X_1| = |X_2| = 2$. However, there is no general corollary for this case (c.f. the proof in the appendix).

Notice also that there are sometimes normative or epistemic reasons for using the same aggregation method for all variables. If all members are more or less equally skilled in judging all variables then the aggregator that is optimal for one variable is presumably also optimal for the other variables.

Non-arbitrariness

The need for non-arbitrariness as a minimal rationality requirement is clear. We will now argue that a stronger requirement whereby the orders on the independent variables 'pin down' the order on X_{k+1} is too strong.[7]

Consider again the illustration in Figure 1 where $k = 1$, $X_1 = \{v, x, y, z\}$ and $X_2 = \{D(v), D(x), D(y), D(z)\}$. Say that the order on X_1 pins down the order on X_2 if $x' \succ_{X_1} x'' \Leftrightarrow D(x') \succ_{X_2} D(x'')$ where $x', x'' \in X_1$. Suppose someone has the order $v \succ_{X_1} x \succ_{X_1} y \succ_{X_1} z$. If this order pins down the order on X_2, we have that $D(v) \succ_{X_2} D(x) \succ_{X_2} D(y) \succ_{X_2} D(z)$. These two orders forms a sequence that often will not make sense. In particular, it will not make sense if the dependent variable is the value of a policy instrument (a key interest rate, a tax rate, etc.). To see this, let φ be a target variable (inflation, pollution, etc.). Let the relationship between the policy instrument p and the target variable φ be given by $\varphi = \alpha p + \varepsilon$, where ε is a factor that affects φ which is exogenous to the policy and α is the effect of policy. Let the committee's aim be to maximize a standard objective function $W = -[(\varphi - \varphi^*)^2 + \lambda p^2]$, where φ^* is the desired (target) level of variable φ, and λ is the cost changing the policy instrument. Then optimal policy is given by (first order condition)[8]

$$p = \frac{\alpha}{\alpha^2 + \lambda} (\varphi^* - \varepsilon). \qquad (1)$$

If the members agree on ε and λ, but disagree on α, we have that $p = D(\alpha)$, where $D(\alpha)$ is given by (1). The function $D(\alpha)$ is non-monotonic and concave. Turning back to Figure 1, let D illustrate $D(\alpha)$. Suppose a member has alternative v as the peak of his order over the alternatives $\{v, x, y, z\}$. For this person we have that $W = [(vp + \varepsilon - \varphi^*)^2 + \lambda p^2]$ which has its minimum at $p = D(v)$. Furthermore,

$$\frac{dW}{dp} = 2[(v - \lambda) p + v (\varepsilon - \varphi^*)] > 0.$$

Then $D(v) < D(x) < D(y) < D(z)$, and the order on X_2 must be $D(v) \succ_{X_2} D(z) \succ_{X_2} D(y) \succ_{X_2} D(x)$, not the order that is 'pinned down'.

[7] We might formalize such a condition as follows. Call $(x_1, ..., x_k) \in X_1 \times ... \times X_k$ an *explanation* for $x \in X_{k+1}$ if $x = D(x_1, ..., x_k)$. Given the orders $\succcurlyeq_{X_1} \in \mathcal{G}_{X_1}, ..., \succcurlyeq_{X_{k+1}} \in \mathcal{G}_{X_{k+1}}$, say that $(x_1, ..., x_k) \in X_1 \times ... \times X_k$ dominates $(x'_1, ..., x'_k) \in X_1 \times ... \times X_k$ if $x_1 \succ_{X_1} x'_1 \& ... \& x_k \succ_{X_k} x'_k$. Say that a sequence $(\succcurlyeq_{X_1}, ..., \succcurlyeq_{X_{k+1}}) \in \mathcal{G}_{X_1} \times ... \times \mathcal{G}_{X_{k+1}}$ *strongly respects* D if, for all $x, x' \in X_{k+1}$ we have that $x \succ_{X_{k+1}} x'$ whenever x has an explanation that dominates every explanation of x'. 'Strongly respects D' entails 'respects D' and the stronger non-arbitrariness condition discussed in this paragraph.

[8] The same type of dependence function appears if only the first argument $(\varphi - \varphi^*)^2$ enters the objective function, but where the members take uncertainty into account, see Brainard (1967).

Similarly, requiring the orders on the independent variables pin down the order on X_{k+1} will tend to be too strong if $k > 2$. Suppose, for instance, that an advisory committee of three persons is assessing the profitability of an investment project. The members are asked to summarize their assessment in one estimate of the net present value of the project. Each member knows that ex-post, his competence will be measured by some measure that is decreasing in the absolute distance between the committee's estimate and the actual profitability of the project. Thus, each member has a single peaked order over the alternative estimates for the net present value with his estimate being the peak. Let the committee members have estimates similar to the estimates in Table 1, and let member A has the following order over the estimates for the independent variables: DCF: $10 \succ_{DCF} 13$; IC: $8 \succ_{IC} 11 \succ_v 12$. If the order on the independent variables should dictate the order on the dependent variable it then follows that the order on the alternatives for the NPV is $2 \succ_{NPV} -1 \succ_{NPV} 1$. This order is clearly at odds with his single peaked orders.

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Appendix.

Proof of Proposition

Assume $f : \mathcal{G}^n \to \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ is non-dictatorial and satisfies the Unanimity principle and Systematicity. Call a coalition $C \subseteq N$ winning if for all $j \in \{1,..,k+1\}$, $x'_j, x''_j \in X_j$, and $g \in \mathcal{G}^n$ with $C = \{i : x'_j \succcurlyeq_{i,X_j} x''_j\}$ there is $x'_j \succcurlyeq_{N,X_j} x''_j$. By systematicity, 'all' can be replaced by 'some', and, denoting the set of winning coalitions by $\mathcal{C}$, the aggregator f is given by

$$x'_j \succcurlyeq_{N,X_j} x''_j \Leftrightarrow \{i : x'_j \succcurlyeq_{i,X_j} x''_j\} \in \mathcal{C}, \text{ for all } j \in \{1,..,k+1\}, \; x'_j, x''_j \in X_j, \; g \in \mathcal{G}^n.$$

Claim 1: $N \in \mathcal{C}$, and for every coalition $C \in N$, $C \in \mathcal{C}$ if and only if $N \backslash C \notin \mathcal{C}$.
Proof: The first part follows from the unanimity principle. The second part follows from complete aggregate relations and the universal domain.

Possibility: $f : \mathcal{G}^n \to \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ respects the dependence function if $k = 1$ and D is strictly monotonic.
Proof: Put $k = 1$ and D strictly monotonic. Let x, y be two distinct alternatives in X_1. Let $C_{x \succ y} = \{i : x \succ_{i,X_1} y \in \succcurlyeq_{i,X_1}\}$ and $C_{D(x) \succ D(y)} = \{i : D(x) \succ_{i,X_2} D(y) \in \succcurlyeq_{i,X_2}\}$. As the orders in any sequence in $\mathcal{G}$ respect the dependence function and are non-arbitrary (and D is strictly monotonic), we have that for any $g \in \mathcal{G}^n$, $x \succ_{i,X_1} y \Leftrightarrow D(x) \succ_{i,X_2} D(y)$ for all i. Consequently, for any $g \in \mathcal{G}^n$,

$$C_{x \succ y} = C_{D(x) \succ D(y)}. \qquad (*)$$

Consider $g \in \mathcal{G}^n$ with $f(g) = (\succcurlyeq_{N,X_1}, \succcurlyeq_{N,X_2})$ where x is the peak of $\succcurlyeq_{N,1}$. Then there is, for each $y \in X_1 \backslash x$, a coalition $C_{x \succ y} = \{i : x \succ_{i,X_1} y \in \succcurlyeq_{i,X_1}\} \in \mathcal{C}$. By (*) we then have that $C_{D(x) \succ D(y)} \in \mathcal{C}$. Then $f(g)$ must respect the dependence function by the second part of claim 1.

Impossibility: $f : \mathcal{G}^n \to \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ does not respect the dependence function if $k = 1$ and D is non-monotonic, or if $k > 1$.

For the proof of impossibility, notice first that if $k > 1$ and $|X_1| = ... = |X_{k+1}| = 2$ we have a standard binary judgment aggregation problem, and it follows from Proposition 1 in Dietrich & List (2007a) that f does not respect the dependence function. (If $k = 1$ and $|X_1| = |X_2| = 2$, f is strictly monotonic.) We have left to prove impossibility in the cases when $|X_j| > 2$ for at least one $j \in \{1, ..., k+1\}$. The proof will proceed through claim 2-5 below.

Claim 2: For any coalitions $C, C^* \subseteq N$, if $C \in \mathcal{C}$ and $C \subseteq C^*$ then $C^* \in \mathcal{C}$.
Proof: Suppose X_j has three pairwise distinct $x, y, z \in X_j$. Suppose $C \in \mathcal{C}$ and $C \subseteq C' \subseteq N$. We have to show that $C' \in \mathcal{C}$. A partition of N is $\{C, C' \backslash C, N \backslash C'\}$. Consider a profile in $\mathcal{G}^n$ where members have the following order on X_j.

$$\begin{array}{ll} z \succ_{i,X_j} x \succ_{i,X_j} y \succ_{i,X_j} ..., & \text{if } i \in C \\ z \succ_{i,X_j} y \succ_{i,X_j} x \succ_{i,X_j} ..., & \text{if } i \in C' \backslash C \\ y \succ_{i,X_j} z \succ_{i,X_j} x \succ_{i,X_j} ..., & \text{if } i \in N \backslash C' \end{array}$$

where for each member, x, y, z are ranked above all other alternatives in X_j as indicated by "$\succ_{i,X_j}$ ...". Then all rank z over x, hence $z \succ_{N,X_j} x$ by $N \in \mathcal{C}$. Further, exactly those in C rank x over y, so $x \succ_{N,X_j} y$ by $C \in \mathcal{C}$. Thus, if $\succcurlyeq_{N,X_j}$ is to have a peak, $z \succ_{N,X_j} y$. Hence $C' \in \mathcal{C}$, since exactly the members of C' rank z over y.

Claim 3: $\mathcal{C}$ contains a ($\subseteq$-)minimal element C^* that has at least 2 members.
Proof: If $\mathcal{C}$ is non-empty, it has an element, hence has also a minimal element C^*. By claim 1, $C^* \neq \emptyset$. By claim 2, C^* cannot be singleton as then we obtain dictatorship.

By claim 3, any ($\subseteq$-)minimal element C^* in $\mathcal{C}$ can be partitioned into two non-empty disjoint sets C_1, C_2. Define $C_3 := N \backslash C^*$.

Claim 4: The coalitions C_1, C_2, C_3 are individually non-winning, but pairwise unions of them are winning.
Proof: As C^* is a ($\subseteq$-)minimal element, no partition of C^* can be winning, and hence C_1 and C_2 are not winning. As $C_3 = N \backslash C^*$, C_3 cannot be winning by claim 1. As $C_1 \cup C_2 = C^*$, $C_2 \cup C_1$ is winning. Furthermore, we have that since C_1 is not winning, $C_2 \cup C_3$ must be winning by claim 1. The union $C_1 \cup C_3$ must be winning for the same reasons.

Claim 5: There is a $g \in \mathcal{G}^n$ such that f does not respect D when $|X_j| > 2$ for at least one $j \in \{1, ..., k+1\}$.
Proof: For the proof we have to go through 5 cases. When going through the cases we consider profiles where the members rank all alternatives not explicitly mentioned below the alternatives explicitly mentioned, as indicated by "$\succ_{i,X_j}$ ...".

Let $\{C_1, C_2, C_3\}$ be a partition of N such that the coalitions C_1, C_2, C_3 are individually non-winning but pairwise unions of them are winning.

Case 1. $k = 1$ and D is non-monotonic and non-injective.

Let x, y, z be three pairwise distinct alternatives in X_1 where $D(x) = D(z)$ and $D(y)$ are the two corresponding distinct alternatives in X_2. Consider a profile $g \in \mathcal{G}^n$ where the following holds.

$$\begin{array}{lcll} x \succ_{i,X_1} y \succ_{i,X_1} z \succ_{i,X_1} ... & \text{and} & D(x) \succ_{i,X_2} D(y) \succ_{i,X_2} ... & \text{if } i \in C_1 \\ y \succ_{i,X_1} x \succ_{i,X_1} z \succ_{i,X_1} ... & \text{and} & D(y) \succ_{i,X_2} D(x) \succ_{i,X_2} ... & \text{if } i \in C_2 \\ z \succ_{i,X_1} y \succ_{i,X_1} x \succ_{i,X_1} ... & \text{and} & D(x) \succ_{i,X_2} D(y) \succ_{i,X_2} ... & \text{if } i \in C_3 \end{array}$$

Using claim 1 for the neglected parts we then have that

$$f(g) = \begin{pmatrix} y \succ_{N,X_1} x \succ_{N,X_1} z \succ_{N,X_1} ..., \\ D(x) \succ_{N,X_2} D(y) \succ_{N,X_2} ... \end{pmatrix},$$

a sequence that does not respect D.

Case 2. $k = 1$ and D is non-monotonic and injective.

Let x, y, z be three pairwise distinct alternatives in X_1 where $D(x), D(y)$ and $D(z)$ are the three corresponding pairwise distinct alternatives in X_2. Consider a profile $g \in \mathcal{G}^n$ where the following holds.

$$\begin{array}{lcll} x \succ_{i,X_1} y \succ_{i,X_1} z \succ_{i,X_1} .. & \text{and} & D(x) \succ_{i,X_2} D(z) \succ_{i,X_2} D(y) \succ_{i,X_2} ... & \text{if } i \in C_1 \\ y \succ_{i,X_1} x \succ_{i,X_1} z \succ_{i,X_1} ... & \text{and} & D(y) \succ_{i,X_2} D(z) \succ_{i,X_2} D(x) \succ_{i,X_2} ... & \text{if } i \in C_2 \\ z \succ_{i,X_1} y \succ_{i,X_1} x \succ_{i,X_1} ... & \text{and} & D(z) \succ_{i,X_2} D(x) \succ_{i,X_2} D(y) \succ_{i,X_2} ... & \text{if } i \in C_3 \end{array}$$

Using claim 1 for the neglected parts we then have that

$$f(g) = \begin{pmatrix} y \succ_{N,X_1} x \succ_{N,X_1} z \succ_{N,X_1} \ldots, \\ D(z) \succ_{N,X_2} D(x) \succ_{N,X_2} D(y) \succ_{N,X_2} \ldots \end{pmatrix},$$

a sequence that does not respect D.

Case 3. $k > 1$, and D is non-monotonic in one or more $j \in \{1, ..., k\}$.

Let D be non-monotonic in some variable $m \in \{1, ..., k\}$. Let $\succcurlyeq'_{X_j}$ be any order on variable j. Let $Y \subseteq \mathcal{G}^n$ be the set of profiles where $\succcurlyeq_{i,X_j} = \succcurlyeq'_{X_j}$ for all $i \in N$ and all $j \in \{1, ..., k\} \backslash m$. By Claim 1 we then have that for all profiles $g \in Y$, $\succcurlyeq_{N,X_j} = \succcurlyeq'_{X_j}$ for all $j \in \{1, ..., k\} \backslash m$. For all $g \in Y$ we may then plug the peaks (estimates) of $\succcurlyeq_{N,X_j}$ $j = \{1, ..., k\} \backslash m$ into D and consider D a function of x_m only. It then follows from case 1 and 2 that f does not respect the dependence function.

Case 4. $k > 1$, and D is strictly monotonic in all $j \in \{1, ..., k\}$ and non-injective.

We consider the case when $k = 2$. The extention to cases when $k > 2$ is straight forward (c.f. case 3).
As D is strictly monotonic and non-injective, there exsists two distinct elements x'_1, x''_1 in X_1 and two distinct elements x'_2, x''_2 in X_2, such that $x'_3 = D(x'_1, x'_2)$, $x''_3 = D(x'_1, x''_2) = D(x''_1, x'_2)$ and $x'''_3 = D(x''_1, x''_2)$ are the three corresponding pairwise distinct alternatives in X_3. Consider the profile $g \in \mathcal{G}^n$ where

$$\begin{array}{lll} x'_1 \succ_{i,X_1} x''_1 \succ_{i,X_1} \ldots, x'_2 \succ_{i,X_2} x''_2 \succ_{i,X_2} \ldots & \text{and} \quad x'_3 \succ_{i,X_3} x''_3 \succ_{i,X_3} x'''_3 \succ_{i,X_3} \ldots & \text{if } i \in C_1, \\ x'_1 \succ_{i,X_1} x''_1 \succ_{i,X_1} \ldots, x''_2 \succ_{i,X_2} x'_2 \succ_{i,X_2} \ldots & \text{and} \quad x''_3 \succ_{i,X_3} x'_3 \succ_{i,X_3} x'''_3 \succ_{i,X_3} \ldots & \text{if } i \in C_2, \\ x''_1 \succ_{i,X_1} x'_1 \succ_{i,X_1} \ldots, x'_2 \succ_{i,X_2} x''_2 \succ_{i,X_2} \ldots & \text{and} \quad x''_3 \succ_{i,X_3} x'''_3 \succ_{i,X_3} x'_3 \succ_{i,X_3} \ldots & \text{if } i \in C_3. \end{array}$$

Using claim 1 for the neglected parts we then have that

$$f(g) = \begin{pmatrix} x'_1 \succ_{N,X_1} x''_1 \succ_{N,X_1} \ldots, \\ x'_2 \succ_{N,X_2} x''_2 \succ_{N,X_2} \ldots, \\ x''_3 \succ_{N,X_3} x'_3 \succ_{N,X_3} x'''_3 \succ_{N,X_3} \ldots \end{pmatrix},$$

a sequence that does not respect D.

Case 5. $k > 1$, and D is strictly monotonic in all $j \in \{1, ..., k\}$ and injective.

We consider the case when $k = 2$.The extention to cases when $k > 2$ is straight forward (c.f. case 3).
As D is strictly monotonic and injective, there exsists two distinct elements x'_1, x''_1 in X_1 and two distinct elements x'_2, x''_2 in X_2, such that $x'_3 = D(x'_1, x'_2)$, $x''_3 = D(x'_1, x''_2)$, $x'''_3 = D(x''_1, x'_2)$ and $x''''_3 = D(x''_1, x''_2)$ are the four corresponding pairwise distinct alternatives in X_3. Put $x'_3 < x''_3 < x'''_3 < x''''_3$ (The analysis of the other cases is similar when D is monotonic.). Consider a profile $g \in \mathcal{G}^n$ where

$$\begin{array}{lll} x'_1 \succ_{i,X_1} x''_1 \succ_{i,X_1} \ldots, x'_2 \succ_{i,X_2} x''_2 \succ_{i,X_2} \ldots & \text{and} \quad x'_3 \succ_{i,X_3} x''_3 \succ_{i,X_3} x'''_3 \succ_{i,X_3} x''''_3 \succ_{i,X_3} \ldots & \text{if } i \in C_1, \\ x'_1 \succ_{i,X_1} x''_1 \succ_{i,X_1} \ldots, x''_2 \succ_{i,X_2} x'_2 \succ_{i,X_2} \ldots & \text{and} \quad x''_3 \succ_{i,X_3} x'''_3 \succ_{i,X_3} x'_3 \succ_{i,X_3} x''''_3 \succ_{i,X_3} \ldots & \text{if } i \in C_2, \\ x''_1 \succ_{i,X_1} x'_1 \succ_{i,X_1} \ldots, x'_2 \succ_{i,X_2} x''_2 \succ_{i,X_2} \ldots & \text{and} \quad x'''_3 \succ_{i,X_3} x''''_3 \succ_{i,X_3} x''_3 \succ_{i,X_3} x'_3 \succ_{i,X_3} \ldots & \text{if } i \in C_3. \end{array}$$

Using claim 1 for the neglected parts we then have that

$$f(g) = \begin{pmatrix} x'_1 \succ_{N,X_1} x''_1 \succ_{N,X_1} \ldots, \\ x'_2 \succ_{N,X_2} x''_2 \succ_{N,X_2} \ldots, \\ x''_3 \succ_{N,X_3} x''''_3 \succ_{N,X_3} x'_3 \succ_{N,X_3} x''''_3 \succ_{N,X_3} \ldots \end{pmatrix},$$

a sequence that does not respect D.

Proof of Corollary

As the set of aggregators satisfying Systematicity is a subset of the set of aggregators satisfying Weak systematicity it follows from our proposition that no non-dictatorial aggregator $f : \mathcal{G}^n \to \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ that satisfies the Unanimity principle and Weak systematicity respects the dependence function when $k > 1$ or D is non-monotonic. Thus, it suffices for the proof to consider the case when $k = 1$ and D is strictly monotonic.

Put $k = 1$ and let D be strictly monotonic. Then there are two distinct elements $x, y \in X_1$ such $D(x) \neq D(y)$.

Let f_A and f_B be two different aggregators satisfying the Unanimity principle and Systematicity. Let $g \in \mathcal{G}^n$ be a profile where $f_A(g) \neq f_B(g)$. Denote the set of winning coalitions under f_A by $\mathcal{C}_{f_A}$, and the set of winning coalitions under f_B by $\mathcal{C}_{f_B}$ ('winning coalition' is defined as in the proof of the proposition). As $f_A(g) \neq f_B(g)$ we have that there is a coalition C_A such that that $C_A \in \mathcal{C}_{f_A}$ and $C_A \notin \mathcal{C}_{f_B}$. Put $|X_1| > 2$. As f_A and f_B satisfy the Unanimity principle and Systematicity claim 1 and 2 in the proof of our proposition also apply to f_A and f_B. By claim 1, $C_A \in \mathcal{C}_{f_A} \Rightarrow N \backslash C_A \notin \mathcal{C}_{f_A}$. By claim 1 and 2, $N \backslash C_A \in \mathcal{C}_{f_B}$. Thus, we have that there is a partition of N such that $N = \{C_A, N \backslash C_A\}$ and $C_A \in \mathcal{C}_{f_A}$ and $N \backslash C_A \in \mathcal{C}_{f_B}$. Let f_C be an aggregator where the aggregate relation on variable 1 is determined by the same aggregation method that is used in f_A and the aggregate relation on variable 2 is determined by the same aggregation method that is used in f_B. Let $g^* \in \mathcal{G}^n$ be a profile where all members of C_A rank x strictly above all other alternatives in X_1 and $D(x)$ strictly above all other alternatives in X_2, and the members of $N \backslash C_A$ rank y strictly above all other alternatives for X_1 and $D(y)$ strictly above all other alternatives in X_2. Then $f_C(g^*)$ have peaks $x, D(y)$.