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# Applying Flexible Parameter Restrictions in Markov-Switching Vector Autoregression Models☆

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**Abstract**

We present a new method for imposing parameter restrictions in Markov-Switching Vector Autoregression (MS-VAR) models. Our method is more flexible than competing methodologies and easily handles a range of parameter restrictions over different equations, regimes and parameter types. We also expand the range of priors used in the MS-VAR literature. We demonstrate the versatility of our approach using three appropriate examples.

*Keywords:* Parameter Restrictions, MS-VAR estimation, Block Exogeneity, Zero Restrictions, Bayesian estimation

## 1. Introduction

Econometricians often possess prior knowledge or beliefs about model parameters before they have confronted the data. These assumptions or beliefs can come from theory, experience or instinct. Imposing these beliefs will often improve the estimation results, help parameter identification and allow the econometrician to disentangle the different channels operating in an economy. Sometimes these beliefs can be incorporated into the prior parameter distributions, other times practitioners must formulate these beliefs in terms of parameter restrictions and impose them directly on the model parameters. The primary contribution of this paper is to develop flexible methods for incorporating a range of parameter restrictions in Markov-Switching Vector Autogression (MS-VAR) and Bayesian Vector Autoregression (BVAR) models. Its secondary contribution is the development of more flexible and intuitive methods for estimating MS-VAR models. We demonstrate these methods using three relevant examples.

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Many types of parameter restrictions have been applied to MS-VAR and BVAR models, but the most common is likely to be the zero restriction. Block exogeneity restrictions are a particularly popular application of zero restrictions typically imposed when estimating small open economy VAR models. Several approaches have been suggested to impose this assumption, namely: the block recursive Gibbs sampling procedure of Zha (1999), applying linear restrictions directly (see Waggoner & Zha, 2003 and Sims et al., 2008), and imposing the Independent Normal-Wishart prior (see Robinson, 2013, for example).[1] The restricted least squares method of de Wind & Gambetti (2014) could also be applied to this type of problem. While these methods are effective for the specific problems they have been developed for, they are unable to handle more general restrictions such as cross-parameter, cross-equation, cross-regime and transition probability restrictions that the practitioner may wish to include in addition to the block exogeneity restrictions.

The estimation of MS-VAR models adds additional complexity by expanding the size of the parameter space and the type of parameters in the model. Imposing parameter restrictions can ease the burden of estimation, and improve the results and their interpretation in constant parameter models. This holds even more true in the case of MS-VAR models. In particular efforts have focused on applying zero restrictions to the impact matrix, and constraining the parameters in the transition matrix. Sims et al. (2008) (hereinafter SWZ) propose separate algorithms for each problem. They give examples of some common and not so common restrictions to impose on transition matrices, and provide a reasonably general procedure for applying them. SWZ modify the algorithms of Waggoner & Zha (2003) to apply zero restrictions in MS-VAR models. Their approach is however somewhat limited. In particular they focus their attention on zero restrictions applied to the regression coefficients, within the same equation and the same regime, and they require a separate algorithm to impose restrictions on the transition matrix, which eliminates the possibility of restrictions across the regression and transition probability parameters.

We propose a new method of imposing parameter restrictions, that is flexible enough to handle linear equality constraints (zero and non-zero) over many different parameter types, regimes, equations and transition matrices. Notably our procedure is independent of the type of posterior sampler used and can easily accommodate a range of prior parameter distributions. We also provide a method of estimating MS-VAR models that is more flexible in the prior parameter distributions and more intuitive in its interpretation. We demonstrate our methodology using three relevant examples. The codes are made available in Matlab as part of the RISE toolbox.

This paper is structured as follows: in Section 2 we present our MS-VAR estimation procedure, and compare it with the methodology developed by SWZ. We note that constant parameter models are just a special case of the MS-VAR model. In addition we propose a very flexible procedure for imposing a range of restrictions on MS-VAR parameters. We compare our approach to parameter restrictions with those outlined in SWZ. In Section 3

[1] Frequentist constant parameter models have been estimated using Maximum Likelihood (see Cushman & Zha, 1997) and SUR (see Buckle et al., 2002).

we demonstrate the versatility of our estimation and parameter restriction methodologies by applying them to three diverse examples. Section 4 concludes.

## 2. Methodology: Markov Switching VAR Models and Parameter Restrictions

We are concerned with estimating Structural Markov Switching VAR models of the form:

$$A_{0,s_t} Y_t = C_{s_t} D_t + A_{1,s_t} Y_{t-1} + \cdots + A_{p,s_t} Y_{t-p} + \Sigma_{s_t} \varepsilon_t, \tag{1}$$

$$\mathcal{Q} = \left\{ \mathcal{Q}_{s_t, s_{t+1}} \right\}, \tag{2}$$

where:

- $p$ is lag length;
- $Y_t$ is a $k \times 1$ vector of date $t$ endogenous variables;
- $D_t$ is an $m \times 1$ vector of date $t$ exogenous variables;
- $\varepsilon_t$ is a $k \times 1$ vector of date $t$ disturbances;
- $s_t = 1, 2, \ldots, h$;
- $s_{t+1} = 1, 2, \ldots, h$;
- $A_{0,s_t}$ is an invertible $k \times k$ coefficient matrix and $A_{i,s_t}$ is a $k \times k$ coefficient matrix for $s_t = 1, 2, \ldots, h$ and $i = 1, 2, \ldots, p$;
- $C_{s_t}$ is a $k \times m$ coefficient matrix for $s_t = 1, 2, \ldots, h$;
- $\Sigma_{s_t}$ is a $k \times k$ matrix;
- $\mathcal{Q}_{s_t, s_{t+1}} = (q_{i,j})_{(i,j) \in h \times h}$ is an $h \times h$ transition matrix where the elements satisfy:
- $q_{i,j} \geq 0 \quad$ and $\quad \sum_{j \in h} q_{i,j} = 1$, so that the rows sum to one.

Reduced form and constant parameter models are just special cases of the Structural Markov Switching VAR model.

### *2.1. Our Approach to MS-VAR Models*

The MS-VAR estimation methodology developed in this paper improves on the methodology used by SWZ in two key areas. First, we assume that $A_{0,s_t}$ has ones on its diagonal:

$$\operatorname{diag}(A_{0,s_t}) = \begin{bmatrix} 1, & 1, & \cdots, & 1 \end{bmatrix}.$$

This assumption allows the diagonal elements of $\Sigma_{s_t}$ to be interpreted as the standard deviations of the shocks. As a consequence each shock standard deviation is represented by a single parameter and a single Markov chain when the shock standard deviations are allowed to switch.[2]

[2]The SWZ specification could result in the shock standard deviations being determined by two Markov chains, if they are allowed to switch.

The second difference lies in the types of priors permitted and how they are specified. Our procedure allows for Minnesota, Jeffrey's, Normal-Wishart and Independent Normal-Wishart priors, which are applied directly to the coefficient matrices; $A_{0,s_t}, \ldots, A_{p,s_t}$, in equation (1).[3] The main advantage of specifying the priors in this way is that the Likelihood function does not need to be modified. In contrast SWZ only permit the Sims-Zha prior (Sims & Zha, 1998), and their specification necessitates the Likelihood function be rewritten in terms of auxiliary parameters.

The Structural MS-VAR model can be estimated using the Metropolis-Hastings algorithm, or in the case of this paper, the Dynamic Striated Metropolis Hastings Algorithm (DSMH) (see Waggoner et al., 2014, for a similar algorithm).

The MS-VAR methodology outlined in this paper has sufficient flexibility to estimate structural and reduced form Markov switching and constant parameter models. In the case of a structural model with Markov Switching, $A_{0,s_t}$ and $\Sigma_{s_t}$ have $k(k-1)$ and $k$ free parameters to be estimated in each regime. In the reduced form model, $A_{0,s_t} = \underset{k\times k}{I}$ across regimes, and $\Sigma_{s_t}$ has $k(k-1)/2$ free parameters to be estimated in each regime. If a structural MS-VAR is estimated with all parameters allowed to switch on a single chain with $h$ states, then there are $h.k(d + k(p + 1)) + h^2$ parameters to estimate. If a reduced form MS-VAR model is estimated where again all parameters are allowed to switch on a single chain with $h$ states, then there are $h.k(d+k.p+(k-1)/2)+h^2$ parameters to estimate. The individual coefficient matrix contributions for both scenarios mentioned can be found in Table 1. below.

Table 1: Number of Estimated Parameters

| Coefficient Matrices | Structural MS-VAR | Reduced Form MS-VAR |
|---|---|---|
| $A_{0,s_t}$ | $h.k(k-1)$ | 0 |
| $C_{s_t}$ | $h.k.d$ | $h.k.d$ |
| $A_{1,s_t}, \ldots, A_{p,s_t}$ | $h.k^2.p$ | $h.k^2.p$ |
| $\Sigma_{s_t}$ | $h.k$ | $h.k(k-1)/2$ |
| $\mathcal{Q}_{s_t,s_{t+1}}$ | $h^2$ | $h^2$ |

### *2.2. A Comparison with SWZ*

SWZ make more restrictive assumptions when specifying and estimating Markov-Switching VAR models. As discussed earlier, their setup differs from our approach in two key areas. First they assume all parameters in $A_{0,s_t}$ can be freely estimated and hence allowed to switch. Combining this with the assumption that $\Sigma_{s_t}$ is diagonal means the shock standard deviations are determined by $A^{-1}_{0,s_t}\Sigma_{s_t}$. This notation is more cumbersome, because the shock standard deviations are functions of parameters in both $A_{0,s_t}$ and $\Sigma_{s_t}$, as opposed to being to being determined by a single parameter. It also means the shock standard deviations could be determined by two Markov chains, depending on the model's setup.

[3]For a detailed description of the details of these priors, see Koop & Korobilis (2010).

Secondly, they only allow for the Sims-Zha prior in their treatment of MS-VAR models. Moreover their specification of the prior is cumbersome and requires the Likelihood be rewritten in terms of auxiliary parameters. We illustrate this below. SWZ rewrite equation (1) as

$$A_{0,s_t} Y_t = F_{s_t} X_t + \Sigma_{s_t} \varepsilon_t, \tag{3}$$

where

$$X_t = \begin{bmatrix} Y_{t-1} \\ \vdots \\ Y_{t-p} \\ D_t \end{bmatrix} \quad \text{and} \quad F_{s_t} = \begin{bmatrix} A_{1,s_t} & \cdots & A_{p,s_t} & C_{s_t} \end{bmatrix}.$$

The parameters are redefined in terms of the auxiliary parameters $G_{s_t}$:

$$\underset{k\times(p.k+m)}{F_{s_t}} = \underset{k\times(p.k+m)}{G_{s_t}} + \underset{k\times k}{A_{0,s_t}} \underset{k\times(p.k+m)}{\bar{S}},$$

where

$$\bar{S} = \begin{bmatrix} \underset{k\times k}{I} & \underset{k\times(p.(k-1)+m)}{0} \end{bmatrix}.$$

If the auxiliary parameters $G_{s_t}$ are mean zero, this redefinition is consistent with the reduced form random walk Minnesota prior. We circumvent the problem of modifying the Likelihood by applying the priors directly to the $A_{0,s_t}, \ldots, A_{p,s_t}$ matrices in equation (1).

### *2.3. Our Approach to Imposing Parameter Restrictions*

We develop versatile tools for imposing both linear and non-linear parameter restrictions in MS-VAR & constant parameter BVAR models. In particular parameter restrictions across equations, regimes and parameter types are allowed.[4] Competing methods by SWZ are unable to handle all the scenarios covered by our methodology.

Our approach permits non-linear parameter restrictions involving inequalities. This is achieved by evaluating the restrictions for each parameter draw, and in the event they are violated, the draw is assigned a low Likelihood value.

The procedure also handles linear parameter restrictions of the form

$$R\theta = r,$$

where $R$ is a $q \times z$ matrix with full row rank, $\theta$ is a $z \times 1$ vector (the vectorised coefficient matrix) and $r$ is a $q \times 1$ vector of restrictions, $q$ being the number of linear constraints imposed and $z$ the number of coefficients in the model. Note that $r$ can contain zero and non-zero equality restrictions. $z$ will vary with the type of model estimated, the number of Markov chains and the number of coefficients that are regime dependent.

[4] For example, it is possible to impose parameter restrictions that involve combinations of regression coefficients, shock standard deviations and the transition probabilities.

For example, in the case of a Markov switching SVAR model with a single Markov chain where all coefficients switch,

$$A_{s_t} \equiv \left[A_{0,s_t}, A_{1,s_t}, \ldots, A_{p,s_t}, \Sigma_{s_t}\right],$$

$$A \equiv \left[A_1, A_2, \ldots, A_h\right],$$

$$\alpha = \text{vec}\left(A\right), \qquad q = \text{vec}\left(\mathcal{Q}\right), \quad \theta = \left[\alpha', q'\right]',$$

$$z = h(k(k-1) + k.d + k^2.p + k) + h^2,$$

and in the case of a reduced form Markov switching VAR model with a single Markov chain where all coefficients switch,

$$A_{s_t} \equiv \left[A_{1,s_t}, \ldots, A_{p,s_t}, \Sigma_{s_t}\right],$$

$$A \equiv \left[A_1, A_2, \ldots, A_h\right],$$

$$\alpha = \text{vec}\left(A\right), \qquad q = \text{vec}\left(\mathcal{Q}\right), \quad \theta = \left[\alpha', q'\right]',$$

$$z = h(k.d + k^2.p + k(k-1)/2) + h^2.$$

Typically $R$ will not be invertible, but the columns of $R$ can be permuted so that it can be partitioned into a square matrix that is invertible, and a rectangular matrix that is singular. This follows from $R$ being a matrix of full row rank. The QR decomposition with column permutation can be used to achieve such a partition

$$QR^* = RP.$$

$R^*$ can be further partitioned as follows

$$QR^* = Q\left[\begin{array}{cc} R_1, & R_2 \end{array}\right] = \left[\begin{array}{cc} \tilde{R}_1, & \tilde{R}_2 \end{array}\right] = RP,$$

where $R_1$ is $q \times q$ and invertible and so is the orthogonal matrix $Q$, so that

$$\tilde{R}_1 = QR_1, \quad \tilde{R}_2 = QR_2.$$

Using the change of parameters $P\tilde{\theta} = \theta$, the constraint on the parameters becomes

$$RP\tilde{\theta} = \left[\begin{array}{cc} \tilde{R}_1 & \tilde{R}_2 \end{array}\right]\left[\begin{array}{c} \tilde{\theta}_1 \\ \tilde{\theta}_2 \end{array}\right] = r,$$

$$\tilde{R}_1\tilde{\theta}_1 + \tilde{R}_2\tilde{\theta}_2 = r.$$

Making use of the fact that $\tilde{R}_1$ is invertible allows $\tilde{\theta}_1$ to be written as a function of $\tilde{\theta}_2$

$$\tilde{\theta}_1 = -\tilde{R}_1^{-1}\tilde{R}_2\tilde{\theta}_2 + \tilde{R}_1^{-1}r.$$

This allows the vector of all coefficients to be written as a function of the $(z - q)$ subset of parameters $\tilde{\theta}_2$

$$\tilde{\theta} = \begin{bmatrix} -\tilde{R}_1^{-1}\tilde{R}_2 \\ \underset{(z-q)}{I} \end{bmatrix} \tilde{\theta}_2 + \begin{bmatrix} \tilde{R}_1^{-1} r \\ \underset{(z-q)\times 1}{0} \end{bmatrix},$$

or in more compact notation

$$\theta = P\left(c \cdot \tilde{\theta}_2 + d\right),$$

where

$$c = \begin{bmatrix} -\tilde{R}_1^{-1}\tilde{R}_2 \\ \underset{(z-q)}{I} \end{bmatrix}, \qquad d = \begin{bmatrix} \tilde{R}_1^{-1} r \\ \underset{(z-q)\times 1}{0} \end{bmatrix}.$$

*2.4. A Comparison with SWZ*

SWZ also provide tools for imposing linear restrictions on the coefficient and transition matrices of MS-VAR models. However their approach is generally more restrictive and less flexible than the approach outlined here. In general they consider linear restrictions on the coefficient matrices of the form

$$\mathfrak{R}_{\ell,s_t} \left[\begin{array}{cc} a_{\ell,s_t} & f_{\ell,s_t} \end{array}\right]' = 0 \tag{4}$$

where $a_{\ell,s_t}$ is the $\ell$th row of $A_{0,s_t}$ and likewise $f_{\ell,s_t}$ is the $\ell$th row in $F_{s_t}$ for $s_t \in 1, \ldots, h$, where $F_{s_t}$ is the same matrix in equation (3). $\mathfrak{R}_{\ell,s_t}$ is a $(k + k.p + m) \times (k + k.p + m)$ matrix that specifies the relevant linear restrictions to be imposed. Several points are worth making about equation (4). First, the restrictions only apply within a single equation.[5] Second, the restrictions only apply within a single regime. Third, the restrictions can only be applied to the regression coefficients and not across coefficient types. And fourth, only zero restrictions can be applied. The type of restrictions we consider in section 2.3 are more general and less restrictive.

To impose restrictions on the transition probabilities, SWZ require a separate procedure, we outline this below. Let $q_i$ be the $i$th row of $\mathcal{Q}$ where $1 \leq i \leq h$ and $q$ be an $h^2 \times 1$ column vector stacking $q_i'$s.

For $1 \leq i \leq v$, $w_i$ is a $d_i$ dimensional vector representing the number of coefficients in the $i$th row of $\mathcal{Q}$ where $v$ may be greater or less than $h$, $w_i \geq 0$ and $\sum_{i=1}^{v} w_i = 1$. $w$ is a $d \times 1$ column vector stacking $w_i$s where $d = \sum_{i=1}^{v} d_i$.

$$q = Mw,$$

[5] SWZ's method could be extended to cover cross equation and cross regime restrictions, but they do not do this.

where $M$ is an $h^2 \times d$ matrix such that

$$M = \begin{bmatrix} M_{1,1} & \cdots & M_{1,v} \\ \vdots & \ddots & \vdots \\ M_{h,1} & \cdots & M_{h,v} \end{bmatrix},$$

and $M_{j,i}$ is an $h \times d_i$ matrix. For each $(j,i)$, all the elements of $M_{j,i}$ are non-negative and each row of $M$ has at most one non-zero element.

As SWZ show, this methodology can handle many different types of restrictions on the transition matrix. However it appears a little more cumbersome to implement than the methodology we have outlined in this paper, and it cannot be used to impose restrictions across transition and regression coefficients.

## 3. Applications

We use three examples to demonstrate our methodology. The first two examples illustrate how parameter restrictions can be applied in the estimation of MS-VAR models. In particular we use the methodology to estimate a small closed economy VAR model with many parameter restrictions à la Rudebusch & Svensson (1999) where the monetary policy shock variance is allowed to switch. We also apply the methodology to a large closed economy model where all shock standard deviations can switch, similar to the models outlined in Sims & Zha (2006). Our third example, inspired by Cushman & Zha (1997), illustrates the versatility of our parameter restrictions tools by applying block exogeneity restrictions in the estimation of a small open economy BVAR model with constant parameters.

### *3.1. A Small Closed Economy Model*

We estimate a small closed economy model using the same variables and restrictions applied in Rudebusch & Svensson (1999). We add an extra dimension to the problem by allowing the standard deviation of the monetary policy shock to switch between a high volatility regime and a low-volatility regime. The model is a three equation backward-looking New Keynesian model consisting of a simple backward-looking Phillips curve relationship, a backward-looking IS curve and a Taylor type rule. It explains the relationship between the output gap ($y_t$), inflation ($\pi_t$) and the fed funds rate ($i_t$).

Backward-looking Phillips curve relationship:

$$\pi_t = c_\pi + \alpha_{\pi 1}\pi_{t-1} + \alpha_{\pi 2}\pi_{t-2} + \alpha_{\pi 3}\pi_{t-3} + \alpha_{\pi 4}\pi_{t-4} + \alpha_y y_{t-1} + \sigma_\pi \epsilon_t \tag{5}$$

Backward-looking IS curve relationship:

$$y_t = c_y + \beta_{y1} y_{t-1} + \beta_{y2} y_{t-2} - \beta_r \left(\bar{i}_{t-1} - \bar{\pi}_{t-1}\right) + \sigma_y \eta_t \tag{6}$$

where

$$\bar{\pi}_{t-1} = \sum_{j=1}^{4} \pi_{t-j}, \quad \bar{i}_{t-1} = \sum_{j=1}^{4} i_{t-j}$$

Taylor type rule:

$$i_t = c_i + h i_{t-1} + g_\pi \bar{\pi}_{t-1} + g_y y_{t-1} + \sigma_{i,s_t} u_t \tag{7}$$

where $\sigma_{i,s_t}$ is allowed to switch. This model can be written more compactly as a structural VAR model of the form:

$$A_0 Y_t = C + A_+(L) Y_{t-1} + \Sigma_{s_t} \varepsilon_t$$

where $Y_t = \begin{pmatrix} \pi_t, & y_t, & i_t \end{pmatrix}'$ and $\varepsilon = \begin{pmatrix} \epsilon_t, & \eta_t, & u_t \end{pmatrix}'$, and the following coefficient restrictions are applied:

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} c_\pi \\ c_y \\ c_i \end{bmatrix}, \; A_1 = \begin{bmatrix} \alpha_{\pi 1} & \alpha_y & 0 \\ \beta_r/4 & \beta_{y1} & -\beta_r/4 \\ g_\pi & g_y & h \end{bmatrix}, \; A_2 = \begin{bmatrix} \alpha_{\pi 2} & 0 & 0 \\ \beta_r/4 & \beta_{y2} & -\beta_r/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} \alpha_{\pi 3} & 0 & 0 \\ \beta_r/4 & 0 & -\beta_r/4 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} \alpha_{\pi 4} & 0 & 0 \\ \beta_r/4 & 0 & -\beta_r/4 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Sigma_{s_t} = \begin{bmatrix} \sigma_\pi & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_{i,s_t} \end{bmatrix}.$$

The model is estimated on quarterly US data from 1954:4-2015:2, the sources are listed in Appendix A. It is estimated with four lags and a constant using Bayesian methods. In particular we use a standard Minnesota prior, and the transition probabilities follow a Dirichlet distribution. The posterior distribution is estimated using a version of the Dynamic Striated Metropolis Hastings sampling algorithm. Figure 1. displays the median probability of the high volatility monetary policy regime over history. Figure 2. presents the generalized impulse response functions for a monetary policy shock using the model.

Figure 1: Probability of High Volatility Monetary Policy Regime

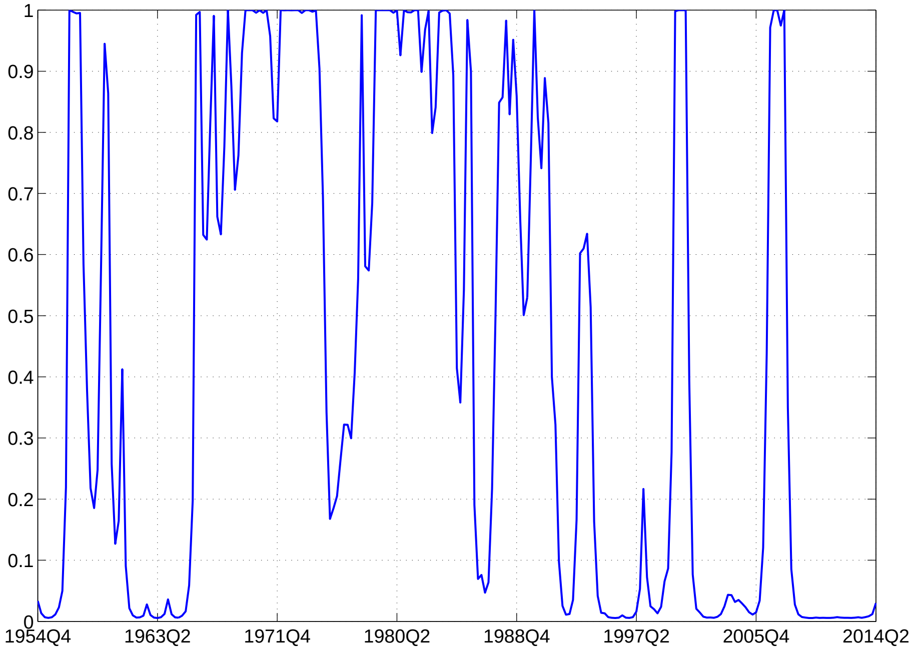
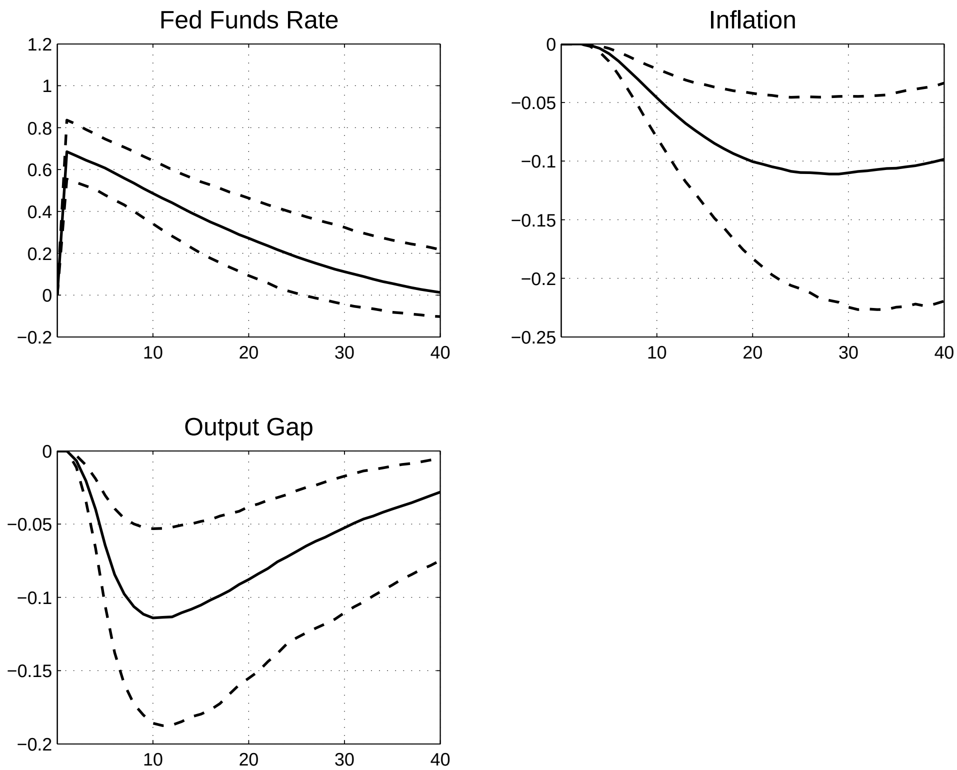



Figure 1. shows that the model is able to pick up the great moderation because more time has been spent in the low volatility regime from the late 1980s through to the present. However we do see that the recession in the early 2000s and the global financial crisis occurred at the same time as a return to a more volatile regime.

Figure 2: Monetary Policy Shock (GIRF)




*Notes:* The solid black line represents the median impulse and the dashed line represents the 16% and 84% posterior probability bands.

The generalized impulse responses for the monetary policy shock show the conventional pattern of a decline in inflation and the output gap following the shock.

*3.2. A Large Closed Economy Model*

In this application we demonstrate how the methodology can be applied to a larger MS-VAR model. We estimate a model similar to those in Sims & Zha (2006) which have been used to investigate the sources of the great moderation. More specifically we use monthly U.S. data from 1959:1-2015:5 to estimate an MS-VAR model with 13 lags and a constant. We use a Minnesota prior, with a Dirichlet prior on the transition probabilities. The posterior distribution is sampled using a version of the Dynamic Striated Metropolis Hastings algorithm.

The model consists of six endogenous variables, namely: the change in log commodity prices ($\Delta \log (Pcom_t)$), the change in log of the M2 money supply ($\Delta \log (M2_t)$), the federal funds rate ($R_t$), the change in log interpolated monthly real GDP ($\Delta \log (GDP_t)$), the change in the log PCE price deflator ($\Delta \log (P_t)$) and the unemployment rate ($U_t$). The data sources are listed in Appendix A. The vector of date $t$ endogenous variables is

$$Y_t = \left( \Delta\log\left(Pcom_t\right), \quad \Delta\log\left(M2_t\right), \quad R_t, \quad \Delta\log\left(GDP_t\right), \quad \Delta\log\left(P_t\right), \quad U_t \right)'$$

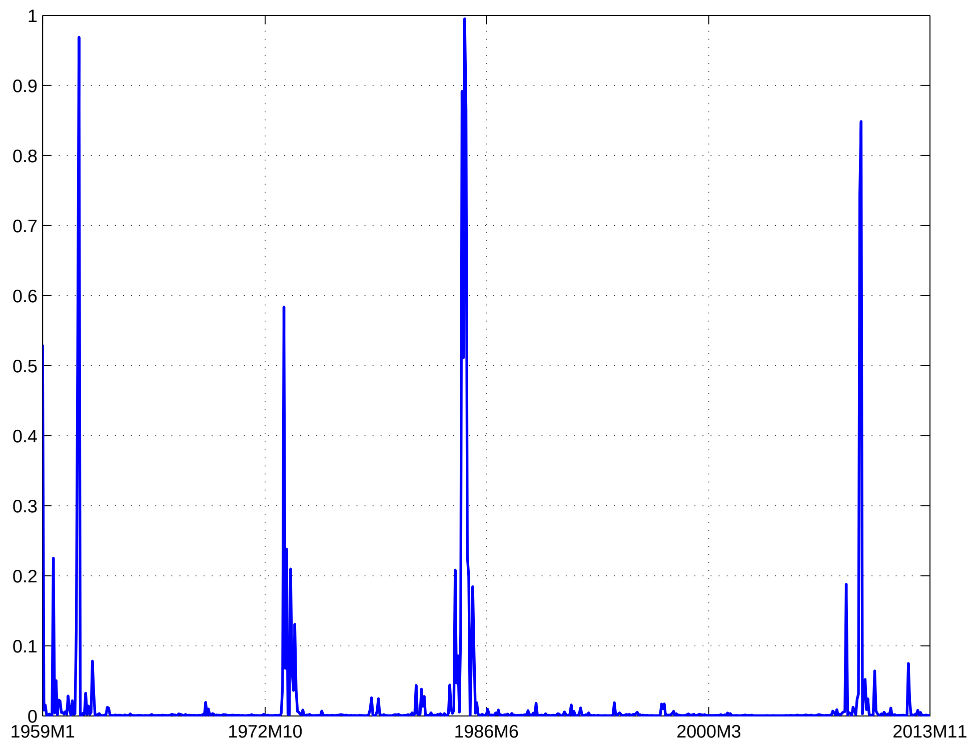
Another key difference between our model and the models in estimated in Sims & Zha (2006) is the endogenous variables with trends are differenced. Markov switching is introduced into the model by allowing all the diagonal elements of $\Sigma_{s_t}$ to switch on a two-state chain. Following Sims & Zha (2006) this model should allow us to determine whether the volatility of the shocks have declined in the great moderation period. To identify the shocks, we use our method for applying linear restrictions to the model to impose the following zero restrictions on $A_0$:

$$A_0 = \begin{array}{c} \\ \Delta\log\left(Pcom\right) \\ \Delta\log\left(M2\right) \\ R \\ \Delta\log\left(GDP\right) \\ \Delta\log\left(P\right) \\ U \end{array} \begin{array}{c} \begin{array}{cccccc} Inf & Fed & MD & Prod & Prod & Prod \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \times & 1 & \times & 0 & 0 & 0 \\ \times & \times & 1 & 0 & 0 & 0 \\ \times & 0 & \times & 1 & \times & \times \\ \times & 0 & \times & 0 & 1 & \times \\ \times & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

These are the same restrictions used in Sims & Zha (2006) to identify the shocks. We present the (median) historical probability of being in a high volatility regime in Figure 3. below.[6]

[6]The impulse responses are available from the authors upon request.

Figure 3: High Volatility Regime




We note that there are three short-lived periods of high volatility between 1959:1 and 1986:6, none between 1986:6 and 2007:12 and then one after 2007:12. This is consistent with the hypothesis that volatility fell during the great moderation.

### *3.3. A Small Open Economy VAR Model*

To illustrate how the linear restrictions can be used to impose block exogeneity restrictions, we estimate a small open economy BVAR model that is in many ways similar to the model estimated in Cushman & Zha (1997). The model is estimated on quarterly data for the US and Canada between 1972:2 and 2014:4. In the foreign block we include the US output gap ($\widehat{Y}_t^*$), the change in log US CPI index ($\Delta \log (P_t^*)$), the federal funds rate ($R_t^*$) and the change in a log commodity price index ($\Delta \log (Pcom_t)$). In the domestic block we use the Canadian output gap ($\widehat{Y}_t$), the log change in Canadian CPI index ($\Delta \log(P_t)$), Canadian interest rates ($R_t$), the log change in Canadian M1 money supply ($\Delta \log(M1_t)$) and the log change in the US/Canadian exchange rate ($\Delta \log(Exc_t)$). A full description of the data is available in Appendix A. We impose the Minnesota prior and estimate the model with a constant and 4 lags by maximizing the posterior. The shocks are identified by imposing the usual lower triangular Cholesky type restrictions. We report the impulse responses at the posterior mode for a Canadian monetary policy shock. The BVAR model takes the general form:

$$A(L)\,Y_t + C = \Sigma\varepsilon_t.$$

We impose block exogeneity on the model such that

$$Y_t = \begin{pmatrix} y_{1,t} \\ y_{1,2} \end{pmatrix}, \quad A(L) = \begin{pmatrix} A_{11}(L) & A_{12}(L) \\ 0 & A_{22}(L) \end{pmatrix}, \quad \varepsilon_t = \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix}.$$

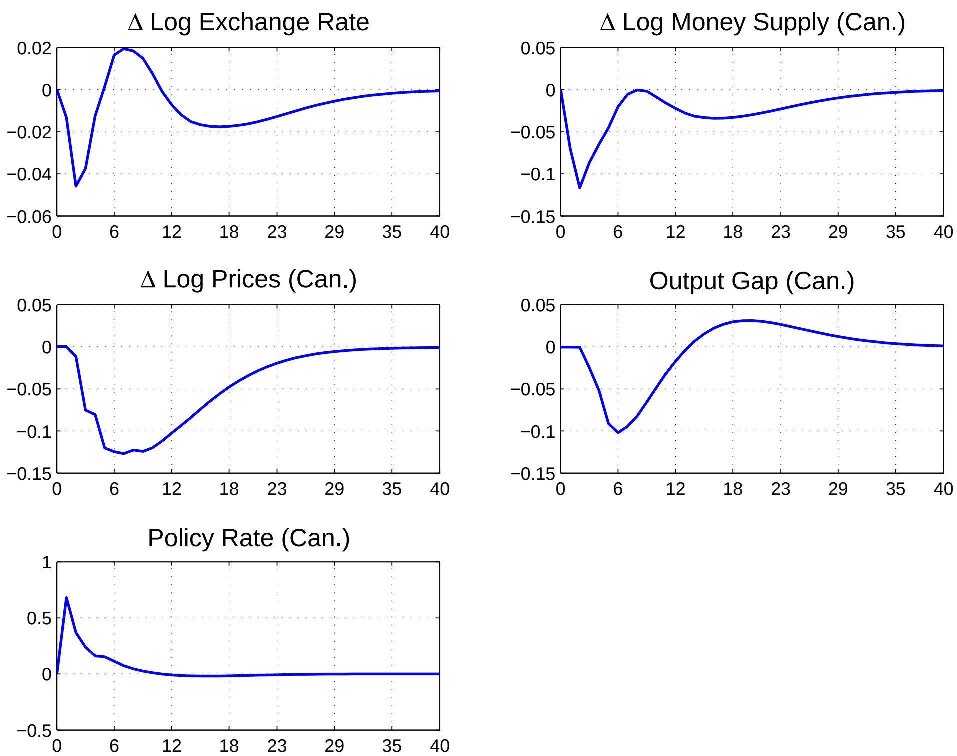
where

$$y_{1,t} = \left( \;\Delta \log(Exc_t)\,, \quad \Delta \log(M1_t) \quad R_t, \quad \Delta \log(P_t)\,, \quad \widehat{Y}_t \;\right)'$$

$$y_{2,t} = \left( \;\Delta \log(Pcom_t)\,, \quad R_t^*, \quad \Delta \log(P_t^*)\,, \quad \widehat{Y}_t^* \;\right)'$$

The impulse responses for a Canadian monetary policy shock in Figure 4. below.

Figure 4: A Canadian Monetary Policy Shock




The increase in Canadian interest rates results in a decrease in the Canadian money supply, output gap and inflation rates. The Canadian dollar initially appreciates against the US dollar before depreciating. We do not present the response of the US variables because the block exogeneity assumption means they do not respond to Canadian variables.

## 4. Conclusion

We present a new, more flexible method of imposing parameter restrictions in MS-VAR and BVAR models. In particular our method is capable of handling a range of linear equality constraints across many different parameter types, equations and regimes. A key advantage of our method is that it is independent of prior parameter distribution and the posterior sampling algorithm. Our secondary contribution is the development of a more flexible and intuitive approach to estimating MS-VAR models. More specifically our parameter decompositions allow for a more intuitive interpretation of the shock standard deviations and we are able to handle a range of prior parameter distributions that are applied directly to the model coefficients.

We demonstrate how to apply the parameter restrictions using three different examples. The first example illustrates how many parameter restrictions inspired by economic theory can be applied to a small MS-VAR model. The second demonstrates how parameter restrictions can be applied to a larger MS-VAR model. The third example shows how to impose block exogeneity restrictions to a small open economy VAR model. We make the codes available in Matlab as part of the RISE toolbox.

## Appendix A. Data

### *Appendix A.1. A Small Closed Economy Model*

All data is taken from the Federal Reserve Economic Database (FRED), where FRED pneumonics appear in parentheses. Inflation is calculated as the percent change in the GDP deflator (GDPDEF), the real GDP (GDPC1) gap is calculated using the Hodrick Prescott filter with a $\lambda$ of 1600 and the quarterly federal funds rate (FEDFUNDS) is calculated using the average of the monthly rate.

### *Appendix A.2. A Large Closed Economy Model*

We take the following variables from the FRED database: M2 money supply (M2SL), the federal funds rate (FEDFUNDS), Prices (PCEPI), the unemployment rate (UNRATE) and real GDP (GDPC1). Commodity prices (PSCCOM) are taken from the Commodity Research Bureau.

### *Appendix A.3. A Small Open Economy VAR Model*

We take the following variables from the FRED database: Canadian real GDP (NAEXKP01CAQ661S), Canadian CPI (CANCPIALLMINMEI), Canadian 90 day interest rates (IR3TIB01CAQ156N), Canadian M1 money supply (MANMM101CAQ189S), US Real GDP (GDPC1), US CPI (CPIAUCSL), federal funds rate (FEDFUNDS) and the Canada/US exchange rate (EXCAUS). Monthly commodity prices (PSCCOM) are taken from the Commodity Research Bureau, and are averaged to create the quarterly variable. The output gaps for Canada and the US are constructed using the Hodrick Prescott filter with a $\lambda$ of 1600 and the quarterly federal funds rate (FEDFUNDS) is calculated using the average of the monthly rate.

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