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Underidentified SVAR models: A framework for combining short and long-run restrictions with sign-restrictions

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Abstract

I describe a new method for imposing zero restrictions (both short and long-run) in combination with conventional sign-restrictions. In particular I extend the Rubio-Ramírez et al. (2010) algorithm for applying short and long-run restrictions for exactly identified models to models that are underidentified. In turn this can be thought of as a unifying framework for short-run, long-run and sign restrictions. I demonstrate my algorithm with two examples. In the first example I estimate a VAR model using the Smets & Wouters (2007) data set and impose sign and zero restrictions based on the impulse responses from their DSGE model. In the second example I estimate a BVAR model using the Mountford & Uhlig (2009) data set and impose the same sign and zero restrictions they use to identify an anticipated government revenue shock.

Keywords: SVAR, Identification, Impulse responses, Short-run restrictions, Long-run restrictions, Sign restrictions

1. Introduction

Vector Autoregression (VAR) models have become an integral part of most macroeconometricians' toolkits. Not only have they proven their worth in forecasting but they have also proven useful in uncovering the transmission mechanisms of key macroeconomic shocks. Guided by economic theory, the econometrician imposes restrictions on how the structural shocks impact variables within the model system transforming the VAR model into a Structural Vector Autoregression (SVAR) model. This paper deals with how econometricians can impose zero restrictions on their VAR models when there are not enough restrictions to identify a unique SVAR model (i.e. the model is not globally identified). In particular I develop

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[1]Any opinions expressed here do not necessarily reflect the views of the management of the Norges Bank.

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an algorithm for combining short-run and long-run restrictions with sign restrictions.

Sims (1980) was the first to consider using a recursive identification scheme (through the Cholesky decomposition) to impose zero constraints on the short-run impact matrix of a VAR model. Under such a scheme, the ordering of the variables (chosen by the econometrician) determines which variables are allowed to respond to a given shock upon impact, and which variables have a one period delayed response. Blanchard & Quah (1989) extended this recursive identification scheme to identifying the model through long-run restrictions (see Gali (1999) and Christiano et al. (2006) for applications). As with the short-run restrictions, the ordering of the variables determines which variables are allowed to have a non-zero cumulative response to a given shock and which variables are forced to have a cumulative response equal to zero. In both cases the Cholesky decomposition results in an exactly identified model. Gali (1992) continued this line of research by using both short-run and long-run restrictions to identify an SVAR model using zero restrictions derived from an IS-LM model. However combining zero restrictions on both short and long-run impulse response functions results in a highly non-linear problem that must be solved using numerical optimisation routines, as demonstrated in Gali (1992). This limits the usefulness of such methods if the short-run and long-run restrictions are imposed on many matrices which is the case when the model is simulated under many parameterisations. For example Bayesian VAR models include parameter uncertainty which means model based analysis requires taking a large number of parameter draws from the posterior distribution and then simulating to produce the impulse responses or moments of interest. In a recent paper Rubio-Ramírez et al. (2010) (RWZ from now on) propose a more efficient algorithm for imposing short and long-run restrictions in exactly identified models. They recast the problem in terms of finding an appropriate rotation matrix that satisfies the zero restrictions. This eliminates the covariance constraint, which is nonlinear in the coefficients of the problem, making the problem a linear one which allows the use of more efficient linear algebra. In the same paper RWZ establish conditions under which a model is globally identified and exactly identified.

Short-run and/or long-run restrictions have been primarily applied to exactly identified SVAR models. Exact identification imposes strict assumptions on the number of zero restrictions and their location in the impact matrix. Such "incredible" identifying assumptions may be inconsistent with the identification of many shocks. Sign restrictions have been proposed as an alternative method for identifying SVAR models (see Faust (1998), Uhlig (2005) and Canova & De Nicolo (2002)). Sign restrictions can be used to identify underidentified SVAR models by sampling from all possible SVAR models that are consistent with the reduced form VAR model. As a consequence a "band" of impulse responses is generated which can be pruned using an acceptance/rejection criteria based on the sign of selected impulse responses. The sign restrictions are chosen by the econometrician and are usually derived from economic theory. Canova & De Nicolo (2002) propose the use of Givens rotation matrices to draw from the set of models consistent with the reduced form VAR model. RWZ (in the same paper they describe solving exactly identified SVAR models) describe a more efficient algorithm for imposing sign-restrictions using the QR decomposition.

Combining zero restrictions with sign restrictions when the model is underidentified has until recently been quite difficult and only used in a limited number of applications. A common application occurs with small open economy SVAR models that use the block exogeneity assumption (see Liu et al., 2011). While popular this approach has limitations in terms of where the zero restrictions can occur. Mountford & Uhlig (2009) present another application that combines zero restrictions with sign restrictions to identify anticipated or announced fiscal policy shocks. Under their identification scheme, an anticipated fiscal policy shock results in no change to the fiscal variable (government revenue or expenditure) for the first four quarters, followed by an increase in the fiscal variable for the next four quarters. This identifying assumption can be represented as a combination of sign and zero restrictions. Mountford & Uhlig (2009) impose these restrictions by setting up a penalty function based on the desired sign restrictions, subject to the covariance constraint, an orthogonality constraint on the identified shocks and zero constraints on the fiscal impulses for the first four quarters. The shocks are solved in a recursive fashion, so that the ordering determines the importance of the shocks. Those ordered first are likely to explain more of the variation than those ordered later.[3] This approach has the advantage that draws are more likely to match the sign restrictions and that the ordering of shocks can be used to weight shocks importance. However orthogonality is only imposed on a small subset of identified shocks and because of the recursive solution the results are likely to be sensitive to the shock ordering.[4] This method also relies on numerical optimisation methods which can be slow to implement.

Another recent approach to combining simple zero restrictions with sign restrictions uses special rotation matrices known as the Householder transformation matrix and the Givens rotation matrix. Baumeister & Benati (2012) show how to impose a single zero restriction on impact in combination with sign restrictions using Givens rotation matrices. The Givens rotation matrix rotates a matrix along two axes until a particular entry is equal to zero. Entire blocks can be zeroed out with just a single rotation using block Givens rotations. In a series of papers Benati & Lubik (2012a), Benati & Lubik (2012b), Benati (2013a) and Benati (2013b) show how to impose zero restrictions on the long-run impact matrix in combination with sign restrictions using the Householder transformation. The Householder transformation zeros out an entire row or column of a matrix below a given entry. With some additional matrix manipulation the Householder transformation can be used to zero out any adjacent entries in a row or column of the impact matrix. Applying various combinations of these rotation matrices would make it possible to simultaneously apply short and long-run

[3] Mountford & Uhlig (2009) impose orthogonality on the business cycle shock, monetary policy shock and the fiscal shocks. They also assume that business cycle shocks explain the largest amount of the variation in GDP, Consumption and Investment so they find this shock first. Then monetary policy shocks are expected to explain the second largest amount of variation, so they are found second. Fiscal shocks are expected to have the smallest contribution to the business cycle out of the three identified shocks, so they are found last. Such an identifying assumption ensures the natural variation of government spending and taxation due to the business cycle is not attributed to fiscal shocks.

[4] As Mountford & Uhlig (2009) mention, the shock ordering is key to identifying fiscal policy shocks. However there may be similar examples where the econometrician wants the identification scheme to be invariant to the shock ordering, in which case this algorithm would need to be modified. The resulting constrained optimisation problem is likely to be more difficult to solve.

zero restrictions with sign restrictions.[5] In this paper I present an alternative method for combining zero restrictions with sign restrictions.

My contribution in this paper is the extension of the RWZ algorithm for imposing short and long-run restrictions on exactly identified models to the case where the SVAR models are underidentified. In particular I combine the RWZ algorithm for short and long-run restrictions (for exactly identified models) with the RWZ algorithm for sign restrictions. This results in a very general algorithm that can handle short-run restrictions, long-run restrictions, sign restrictions and any combination of the three. In the case where the model is exactly identified, the algorithm reverts to the standard RWZ algorithm for exactly identified models. When there are no zero restrictions, the algorithm reverts to the RWZ sign restrictions algorithm, and when there are some zero restrictions, but not enough to exactly identify the model, a band of impulse responses is generated, and this band can be further pruned with the addition of some sign restrictions. This algorithm can be thought of as a theory of underidentified SVAR models, or as a way of including zero restrictions with sign restrictions.

I demonstrate my algorithm with two examples. In the first example I use the Smets & Wouters (2007) data set to estimate an SVAR model. I use identifying assumptions based on the SW DSGE model and I plot the impulse responses from my identified SVAR model against those from the SW DSGE model to compare the validity of the DSGE model and my identification assumptions. In the second example I demonstrate how my algorithm can be extended to include zero restrictions on the impact matrices for multiple periods. These are similar to the identifying assumptions used to find anticipated fiscal policy shocks in Mountford & Uhlig (2009). However I do not use their recursive identification scheme. I demonstrate this extension to the algorithm by estimating a BVAR using the Mountford & Uhlig (2009) data set and by applying the same sign and zero restrictions. As such this example can be used to see how sensitive the results from Mountford & Uhlig (2009) are to their recursive identification assumption. I provide Matlab code for my algorithm.

I proceed as follows, in section 2, I describe the general VAR setup along with some notation and in section 3, I discuss identification. In section 4, I describe the RWZ algorithm for exactly identified models and in section 5, I describe the RWZ algorithm for sign restrictions. Section 6 explains a new algorithm for underidentified models, while in section 7, I look at the implementation of the new algorithm, and in section 8, I conclude.

2. Preliminaries

In this section I give a brief description of a VAR model and I introduce some notation. For the $m \times 1$ vector of data Y_t, I define the VAR(q) model as follows

$$\begin{aligned} Y_{t+1} &= B(L)Y_t + u_{t+1}, \quad Eu_t u_t' = \Sigma, \\ B(L) &\equiv B_1 + B_2 L + \ldots + B_q L^{q-1}, \end{aligned} \tag{1}$$

[5] I thank Luca Benati for pointing out the additional utlity in these methods not presented in his papers.

where u_{t+1} is the date $t+1$ forecast error, q is the number of lags, Σ is the $m \times m$ covariance matrix of the forecast errors and L is the lag operator. Econometricians usually make the following additional assumptions regarding the relationship between the structural and the reduced form shocks:

$$u_t = Z\varepsilon_t,\ E\varepsilon_t\varepsilon_t' = \underset{m\times m}{I},\ ZZ' = \Sigma,$$

where Z is the short-run impact matrix and ε_t are the structural shocks. Typically there are many matrices, Z, that satisfy $ZZ' = \Sigma$, so additional information, usually economic theory, must be used to pin down Z, or in the case of sign restrictions, the Zs. This information can be imposed via short-run restrictions (see section 4), long-run restrictions (see section 4), sign restrictions (see section 5) or some combination of all three (to be explained in section 6).

3. Identification

I discuss what is meant by exact identification and underidentification in this section using results from RWZ and I demonstrate these using some examples. Having a clear understanding of the identification problem will allow us to choose an appropriate strategy when it comes to identifying an SVAR model. First I introduce some additional notation that will prove useful in the discussion in this section and the subsequent sections. Following RWZ the restrictions on the short-run and long-run impact matrices can be written as

$$f(Z,\mathbf{B}) = \begin{bmatrix} L_0 \\ L_\infty \end{bmatrix} = \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_m \\ v_1 \\ v_2 \\ \vdots \\ v_m \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & \cdots & s_m \end{array} \\ \left[\begin{array}{cccc} 0 & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \vdots & & & \vdots \\ \times & \times & \cdots & \times \\ \hdashline \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \vdots & & & \vdots \\ \times & \times & \cdots & \times \end{array}\right] \end{array}, \tag{2}$$

where L_0 is the $m \times m$ short-run impact matrix (such that $L_0 = Z$ and $L_0 L_0' = \Sigma$) and L_∞ is the $m \times m$ long-run impact matrix (such that $L_\infty = (I - \mathbf{B})^{-1} L_0$ where $\mathbf{B} = \sum_{j=1}^{q} B_j$). The labels on the rows represent the variables where the ith variable is denoted by v_i, and the labels on the columns represent shocks so that s_i is the ith structural shock. $\times$ represents a number that is typically not known at this stage and must be found. The econometrician imposes zeros in the $f(Z, \mathbf{B})$ matrix where she wants a shock to have no contemporaneous effect on a variable and/or no long-run effect on a variable. However the zeros cannot be placed anywhere, there are conditions on how many zero restrictions can be applied and where they can occur. These conditions will be discussed in this section in terms of exact identification and underidentification.

Following RWZ, the zero restrictions imposed on each shock can be written in terms of an $m \times 2m$ matrix Q_j such that[6]

$$Q_j f(Z, \mathbf{B}) e_j = 0, \tag{3}$$

where e_j is the jth column of the $m \times m$ identity matrix. Given there are m shocks, there will also be m Q_j matrices, one for the set of restrictions placed on each shock. Letting $q_j = \text{rank}(Q_j)$, I follow RWZ by ordering the columns in $f(Z, \mathbf{B})$ in descending order of the ranks for the corresponding Q_j matrices. This is consistent with the theory they develop and the algorithm they use to find the solution. I now define what is meant by exact identification by reproducing Theorem 7 from RWZ:

Theorem 1. *Consider an SVAR with restrictions represented by R. The SVAR is exactly identified if and only if $q_j = m - j$ for $1 \leq j \leq m$.*

Proof See RWZ. ■

Theorem 1 defines a rank condition for checking whether a model is exactly identified. It also implies that the total number of restrictions imposed is equal to $m(m-1)/2$. The solution will also be unique so that by RWZ's definitions it will be globally identified. When a model is exactly identified, RWZ's algorithm for exactly identified models can be used to find the unique solution (see section 4).

It follows from RWZ's definition of exact identification that if $q_j \leq m - j$ for $1 \leq j \leq m$ and for some $j : q_j < m - j$ for $1 \leq j \leq m$ then the model is underidentified. When the model is underidentified, there will typically be multiple solutions that are consistent with the reduced form VAR model. Sign restrictions could be used with an underidentified model to remove some of the spurious solutions. This will be discussed further in sections 5 and 6 along with a method of imposing zero short and long-run restrictions.

To illustrate the concepts of exact identification and underidentification I follow RWZ and specify restrictions on a VAR model with four variables; GDP growth ($\Delta \log \text{Y}$), inflation ($\log \text{P}$), interest rates (R), and the change in the exchange rate ($\Delta \log \text{Ex}$), to identify four shocks; the exchange rate shock (Ex), the monetary policy shock (P), the demand shock (D) and the supply shock (S).

Example 1: Exact Identification

[6] This is the case where both short and long-run restrictions are imposed on the model. I only deal with this case because it is the most general case and it allows for only short-run restrictions, only long-run restrictions and the combination of both. In one of the examples in section 7, I extend my algorithm to impose zero restrictions on the impact matrices for the first four quarters. The same conditions for the number of zero restrictions that can be imposed on a model will also apply to this example.

In the first example I impose the following restrictions

$$f(Z,\mathbf{B}) = \begin{bmatrix} L_0 \\ L_\infty \end{bmatrix} = \begin{array}{rc} & \begin{matrix} \mathrm{Ex} & \mathrm{P} & \mathrm{D} & \mathrm{S} \end{matrix} \\ \begin{matrix} \Delta\log \mathrm{Y} \\ \log \mathrm{P} \\ \mathrm{R} \\ \Delta\log \mathrm{Ex} \\ \Delta\log \mathrm{Y} \\ \log \mathrm{P} \\ \mathrm{R} \\ \Delta\log \mathrm{Ex} \end{matrix} & \begin{bmatrix} 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & 0 & 0 & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \end{array}.$$

These restrictions translate into the following Q_j matrices

$$Q_1 = \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \; Q_2 = \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$Q_3 = \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \; Q_4 = \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Applying Theorem 1 to Example 1, it is easy to verify that this model is exactly identified. The rank for the restrictions on the exchange rate shock is $q_1 = 3$, and the ranks for the monetary policy, aggregate demand and the aggregate supply shocks are $q_2 = 2$, $q_3 = 1$, and $q_4 = 0$ respectively. Thus the rank condition for exact identification is satisfied.

Example 2: Underidentification

I impose the following short-run restrictions on my model

$$\begin{array}{rc} & \begin{matrix} \mathrm{Ex} & \mathrm{P} & \mathrm{D} & \mathrm{S} \end{matrix} \\ \begin{matrix} \Delta\log \mathrm{Y} \\ \log \mathrm{P} \\ \mathrm{R} \\ \Delta\log \mathrm{Ex} \\ \Delta\log \mathrm{Y} \\ \log \mathrm{P} \\ \mathrm{R} \\ \Delta\log \mathrm{Ex} \end{matrix} & \begin{bmatrix} 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \end{array}$$

In this particular case the rank for the restrictions on the exchange rate shock is $q_1 = 2$, for the monetary policy shock $q_2 = 1$, for the aggregate demand shock $q_3 = 0$ and for the aggregate supply shock $q_4 = 0$. This model is underidentified because the rank condition from Theorem 1 says the rank for the exchange rate shock should be 3, for the monetary policy shock it should be 2, for the demand shock it should be 1 and for the supply shock it should be zero. There are not enough restrictions imposed on each of the shocks to uniquely identify them. There are in fact many models that are consistent both with this identification scheme and the reduced form VAR model.

Example 3: Underidentification

I impose the following zero restrictions on the model

$$\begin{array}{r c}
 & \begin{array}{cccc} \text{Ex} & \text{P} & \text{D} & \text{S} \end{array} \\
\begin{array}{r} \Delta\log \text{Y} \\ \log \text{P} \\ \text{R} \\ \Delta\log \text{Ex} \\ \Delta\log \text{Y} \\ \log \text{P} \\ \text{R} \\ \Delta\log \text{Ex} \end{array} &
\begin{bmatrix} 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}
\end{array}$$

In this example the rank condition for exact identification is met by the exchange rate shock ($q_1 = 3$). However the rank conditions for the monetary policy shock ($q_2 = 1$), the aggregate demand shock ($q_3 = 0$), and the aggregate supply shock ($q_4 = 0$) are not met resulting in a model that is underidentified. Note that because the rank condition is met for the exchange rate shock, the exchange rate shock will be unique, while their will be many solutions for the other shocks that will be compatible with the reduced form model.

Example 4: Over/Underidentified

In this example, I impose the following short and long-run restrictions

$$\begin{array}{r c}
 & \begin{array}{cccc} \text{Ex} & \text{P} & \text{D} & \text{S} \end{array} \\
\begin{array}{r} \Delta\log \text{Y} \\ \log \text{P} \\ \text{R} \\ \Delta\log \text{Ex} \\ \Delta\log \text{Y} \\ \log \text{P} \\ \text{R} \\ \Delta\log \text{Ex} \end{array} &
\begin{bmatrix} \times & 0 & \times & 0 \\ \times & \times & \times & \times \\ 0 & \times & \times & 0 \\ \times & \times & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}
\end{array}$$

The ranks for each shock are given by $q_1 = 2$, $q_2 = 2$, $q_3 = 0$ and $q_4 = 2$. This violates the condition for ordering the columns (from highest rank to lowest rank), so the third and the fourth columns are switched to obtain

$$\begin{array}{r} \\ \Delta\log \mathrm{Y} \\ \log \mathrm{P} \\ \mathrm{R} \\ \Delta\log \mathrm{Ex} \\ \Delta\log \mathrm{Y} \\ \log \mathrm{P} \\ \mathrm{R} \\ \Delta\log \mathrm{Ex} \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{Ex} & \mathrm{P} & \mathrm{S} & \mathrm{D} \end{array} \\ \begin{bmatrix} \times & 0 & 0 & \times \\ \times & \times & \times & \times \\ 0 & \times & 0 & \times \\ \times & \times & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \end{array}$$

The first and second shocks satisfy the condition for the model to be underidentified. However because the rank of the third shock is $q_3 = 2$, which is greater than the rank condition for the model to be exactly identified, this shock is overidentified. The RWZ algorithm cannot be used to solve this model.

4. The RWZ algorithm for exactly identified models

In this section I outline the pseudocode for the RWZ algorithm for imposing short and long-run restrictions on exactly identified models. The novelty of the RWZ algorithm is its ability to solve highly non-linear problems by recasting them as linear. Instead of trying to find the individual elements of Z that are consistent with the short and long-run restrictions subject to the covariance matrix (as is done in Gali (1992)), an orthogonal rotation matrix is found that is consistent with rotating an initial short-run impact matrix until the zero restrictions are met. Because the zero restrictions on each shock can be written as linear restrictions in a matrix (these are the Q_j matrices) and the covariance constraint (which is non-linear in the unknown coefficients in Z) is eliminated, the problem of finding an orthogonal rotation matrix is linear allowing linear algebra to be used to solve the problem.[7] This also improves the speed of the algorithm because the linear algebra is implemented much faster than the optimisation routines required to solve the nonlinear problem.

My implementation of the RWZ algorithm proceeds as follows. For a given short-run and long-run impact matrix $f(Z, \mathbf{B})$, the RWZ algorithm finds an orthogonal matrix that will rotate an initial impact matrix until it satisfies the zero restrictions placed on L_0 and L_∞. I let L_0^* denote the initial short-run impact matrix and L_∞^* the initial long-run impact matrix.

One particular candidate for the initial impact matrix is the lower Cholesky decomposition of the covariance matrix, Σ, so that

$$C = \mathrm{chol}(\Sigma)', \quad \text{where} \quad CC' = \Sigma,$$

[7]Reframing the problem in terms of rotation matrices removes the non-linear covariance constraint and replaces it with an orthogonality constraint (the rotation matrix to be found must be orthogonal). This orthogonality constraint on the rotation matrix can be included in the problem in a linear fashion.

$$L_0^* = C. \tag{4}$$

The corresponding long-run impact matrix consistent with this short-run impact matrix is given by

$$L_\infty^* = \Big[\underset{m\times m}{I} - \mathbf{B}\Big]^{-1} C. \tag{5}$$

The matrix of short and long-run impacts consistent with the Cholesky decomposition is then given by

$$F = \begin{bmatrix} L_0^* \\ L_\infty^* \end{bmatrix}. \tag{6}$$

This will form the initial impact matrix. The RWZ algorithm finds an orthogonal rotation matrix P so that F is consistent with the restrictions imposed in equation (2).

$$FP = f(Z, \mathbf{B}), \quad \text{where} \quad PP' = \underset{m\times m}{I}.$$

This implies

$$L_0 = L_0^* P, \quad L_\infty = L_\infty^* P, \quad L_0^* P P' (L_0^*)' = \Sigma.$$

The first step of my implementation of the RWZ algorithm is to translate equation (2) into a matrix of zeros and ones, with the zeros representing the location of the zero restrictions and ones located everywhere else. One such example could be

$$f = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{7}$$

Note that the zero restrictions need to satisfy the rank condition in Theorem 1 so that the model is exactly identified. Now the bulk of the algorithm is written as the following block of pseudocode:

Algorithm 1 RWZ Short and Long Run Restrictions

1: **for** i = 1, m **do**
2: $H = fe(:, i) == 0$ ▷ create an index for the zero entries in each column of f
3: $Q_i = \text{diag}(H)$ ▷ turn this index into a Q_i matrix
4: $R_i = \text{rank}(Q_i)$ ▷ store the rank of the Q_i matrix
5: **end for**
6: Sort R so that the ranks are in descending order, use this ordering to change the order of the Q_i matrices (the reordered matrices will be denoted by Q_j). Create an index to map the new ordering back into the old ordering.
7: Check the rank condition is satisfied.
8: $F = \begin{bmatrix} L_0^* \\ L_\infty^* \end{bmatrix}$ where $L_0^* = C$, $L_\infty^* = [I - \mathbf{B}]^{-1} C$ and $C = \text{chol}(\Sigma)'$.
9: Initialise P matrix: $P = \underset{m \times m}{0}$
10: **for** j = 1, m **do**
11: **if** j = 1 **then**
12: $\tilde{Q}_j = Q_j F$
13: **else**
14: $\tilde{Q}_j = \begin{bmatrix} Q_j F \\ P' \end{bmatrix}$
15: **end if**
16: $\left[\hat{Q}, \hat{R}\right] = \text{qr}(\tilde{Q}_j')$
17: $P(:, j) = \hat{Q}(:, m)$
18: **end for**
19: Reorder the columns of P using the index so that they are consistent with the original column ordering.
20: $Z = CP$, $ZZ' = \Sigma$
21: Produce the impulse responses

The For loop on lines 1 to 5 produces the m Q_i matrices, one for the restrictions on each shock in the model. The ranks of these matrices are also stored so that the Q_i matrices and the columns in the covariance matrix can be reordered from highest rank to lowest rank (line 6) as required. A mapping from the old ordering to the new ordering is also created, so that the columns of the P matrix can be made consistent with the original ordering. The ranks of the linear restrictions are checked to see if the model is exactly identified (line 7). An initial impact matrix is created on line 8. The next step is to find the rotation matrix, P, column by column. This is done in the For loop on lines 10 to 18. In each step of the loop I solve for $P(:, j)$, or the jth column of P. Because it is an underdetermined system (there are more unknowns than equations at each iteration of the loop), the QR decomposition is used to find the minimum norm solution (see Appendix A for a more indepth description). The rank of $\tilde{Q}_j$ is only $m - 1$, which means the last column of Q will be the solution to the system of equations.[8]

By appending the $j-1$ columns of P that have been solved to the $\tilde{Q}_j$ matrix on each iteration of the loop, the orthogonality of P is ensured. The columns of P are then reordered using the mapping created, so that they are consistent with the original ordering of the shocks. Finally the initial short-run impact matrix (from the Cholesky decomposition) is post multiplied by P and impulse responses are produced. Even though the algorithm finds the minimum norm solution for each column of P the resulting P matrix will be unique because the model is exactly identified (assuming the rank condition is satisfied) and because P has to be an orthogonal matrix.[9]

5. The RWZ algorithm for imposing sign restrictions

In this section I outline the RWZ algorithm for imposing sign restrictions. As before

$$C = \mathrm{chol}(\Sigma)', \quad CC' = \Sigma$$

The RWZ sign restrictions algorithm proceeds as follows

Algorithm 2 RWZ Sign Restrictions

while $j <$ required number of draws **do**
    $\underset{m\times m}{N} = \mathrm{randn}\left(0, \underset{m\times m}{I}\right)$
    $[Q^*, R^*] = \mathrm{qr}(N)$
    for i = 1, m **do**
        if $R^*(i,i) < 0$ **then**
            $Q^*(:,i) = -Q^*(:,i)$
        end if
    end for
    $Z = CQ^*$
    Shock the model, produce impulse responses
    if sign restrictions satisfied **then**
        keep the draw
        $j = j + 1$
    end if
end while

[8] The solution requires $\tilde{Q}_j P(:,j) = 0$. The QR decomposition gives $\tilde{Q}_j = \hat{R}'\hat{Q}'$. The proposed solution is: $\tilde{Q}_j P(:,j) = \tilde{Q}_j \hat{Q}(:,m) = \hat{R}'\hat{Q}'\hat{Q}(:,m) = 0$, because $\hat{Q}'\hat{Q}(:,m) = \left[\underset{1\times m-1}{0}, 1\right]'$, and $\hat{R} = \left[\underset{m-1\times m-1}{\hat{R}_1'}, \underset{m-1\times 1}{0}\right]'$.

[9] This is because the orthogonality condition implicity adds an additional constraint to each shock/column. Any scalar multiple of the vector $\hat{Q}(:,m)$ will be a solution to $\tilde{Q}_j P(:,j) = 0$, but only $\hat{Q}(:,m)$ will result in $P(:,j)$ being orthogonal to the other columns in P. This means there are m constraints for each $\tilde{Q}_j$ resulting in a unique solution even though this matrix only has rank equal to $m-1$.

The algorithm begins by randomly drawing an $m \times m$ matrix from a normal distribution. The QR decomposition of this matrix is taken to produce a randomly drawn orthogonal matrix Q^*. By ensuring that the diagonal elements of R^* are positive (and the corresponding columns of Q^* are consistent with the diagonal elements of R^*) the normality of Q^* is maintained. Draws of Z are then generated by post multiplying the initial impact matrix by the random orthogonal draw. Impulse responses are produced, if they satisfy the sign restrictions they are kept. This continues until the required number of draws is obtained.

6. An algorithm for underidentifed models

This section describes an algorithm for combining zero restrictions with sign restrictions when the model is underidentified. RWZ show that their algorithm for imposing restrictions on the short and long-run impact matrices is valid in exactly identified models. The RWZ algorithm also finds a valid solution when the model is underidentified, however the solution is not unique, there will in general be many P matrices that are consistent with the zero restrictions. For the given initial short-run impact matrix C, the RWZ algorithm will find the minimum norm orthogonal matrix P that rotates C to match the zero restrictions. To generate a random draw from all possible Z matrices consistent with the restrictions, random draws could be generated for the initial impact matrix in the RWZ algorithm for exactly identified models (that is use CQ^* from algorithm 2 in place of C in algorithm 1). Repeating this process many times will generate a band of impulse responses. Further restrictions could be applied through sign restrictions. The pseudocode for such an algorithm is presented below.

Algorithm 3 An Algorithm for Underidentified Models

for i = 1, m **do**
    $H = fe(:, i) == 0$
    $Q_i = \text{diag}(H)$
    $R_i = \text{rank}(Q_i)$
end for
Sort R so that the ranks are in descending order, use this ordering to change the order of the Q_i matrices (the reordered matrices will be denoted by Q_j). Create an index to map the new ordering back into the old ordering.
Check the rank condition is satisfied.
$C = \text{chol}(\Sigma)'$
while $j <$ draws **do**
    $\underset{m \times m}{N} = \text{randn}$
    $[Q^*, R^*] = \text{qr}(N)$
    for i = 1, m **do**
        if $R^*(i, i) < 0$ **then**
            $Q^*(:, i) = -Q^*(:, i)$
        end if
    end for

Algorithm 3 An Algorithm for Underidentified Models cont.

```
17:     F = [ L*_0 ; L*_∞ ] where L*_0 = CQ* and L*_∞ = [I − B]^{-1} CQ*.
18:     Initialise P* matrix: P* = 0_{m×m}
19:     for i = 1, m do
20:         if i = 1 then
21:             Q̃_j = Q_j F
22:         else
23:             Q̃_j = [ Q_j F ; (P*)' ]
24:         end if
25:         [Q̂, R̂] = qr(Q̃'_j)
26:         P*(:, j) = Q̂(:, m)
27:     end for
28:     Reorder the columns of P* using the index so that they are consistent with the
        original column ordering.
29:     Z = CQ*P*
30:     Shock the model, produce impulse responses
31:     if sign restrictions satisfied then
32:         keep the draw
33:         j = j + 1
34:     end if
35: end while
```

Where line 17 is $F = \begin{bmatrix} L_0^* \\ L_\infty^* \end{bmatrix}$ where $L_0^* = CQ^*$ and $L_\infty^* = [I - \mathbf{B}]^{-1} CQ^*$; line 18 is $P^* = \underset{m\times m}{0}$; line 23 is $\tilde{Q}_j = \begin{bmatrix} Q_j F \\ (P^*)' \end{bmatrix}$; line 25 is $\left[\hat{Q}, \hat{R}\right] = \mathrm{qr}(\tilde{Q}_j')$; line 26 is $P^*(:, j) = \hat{Q}(:, m)$.

The key difference between algorithm 1 and algorithm 3 is the initial matrices used in F (line 17 of algorithm 3). In algorithm 3 the initial short-run impact matrix (the Cholesky decomposition of the covariance matrix) is randomised by post multiplying by a randomly drawn orthogonal matrix Q^*. This allows draws to be taken from all models consistent with the reduced form VAR model and the zero restrictions. Note that I differentiate between the rotation matrix P calculated in algorithm 1 and the second rotation matrix calculted in algorithm 3 by using the notation P^*, where P^* is consistent with the initial impact matrix CQ^*.

When the SVAR is exactly identified, the model is globally identified so that the impact matrix Z is unique. The RWZ algorithm for exactly identified models (algorithm 1) finds the unique orthogonal rotation matrix that satisfies the zero restrictions such that:

$$Z = CP$$

In the case of sign restrictions, the model is underidentified, so that there are many Z matrices that are consistent with the reduced form model. The RWZ algorithm (algorithm 2) draws random rotation matrices to sample from all possible Z matrices that are consistent with the reduced form model:

$$Z = CQ^*$$

When there are zero restrictions but not enough to exactly identify the model, the model is underidentified. In which case my algorithm (algorithm 3) produces two rotation matrices, the first randomises the initial impact matrix, the second rotates this matrix so that it matches the zero restrictions.

$$Z = CQ^*P^*$$

My algorithm combines the RWZ algorithm for exactly identified models with the RWZ algorithm for sign restrictions. If my algorithm is used when the model is exactly identified, the algorithm will always find the appropriate rotation matrix consistent with the unique Z matrix (that is $Q^*P^* = P$). If no zero restrictions are imposed, the algorithm will always set $P^* = \underset{m\times m}{I}$ so that it collapses to the RWZ sign restrictions algorithm.

7. Examples

In this section I demonstrate how my algorithm can be used to combine zero restrictions with sign restrictions using two simple examples.

7.1. Example 1: Smets and Wouters (2007) data

In this first example I estimate a VAR model for the US between 1966:1 and 2004:4 using the data set of Smets & Wouters (2007) (SW from now on). The set of observables used to estimate the model include interest rates (i_t), GDP growth ($\Delta\log(Y_t)$), CPI inflation (π_t), hours worked (H_t) and wage inflation ($\Delta\log(W_t)$), where all the data definitions are the same as those used in SW. The VAR is estimated using OLS with the lag length set to 2 based on the lowest BIC. The shocks I try to identify are the monetary policy shock (ε_t^{MP}), the aggregate demand shock (ε_t^{AD}), the aggregate supply shock (ε_t^{AS}) and the wage mark-up shock (ε_t^{WM}). A fifth shock (ε_t^{U}) is left unidentified. To help with specifying an identification scheme, I base some of my identifying assumptions on the impulse responses obtained from the DSGE model in SW. I will then compare the impulse responses from my SVAR model against those from the SW DSGE model as a cross check for both the validity of the identifying restrictions and the DSGE model. I use the following identification scheme

$$f(Z,\mathbf{B}) = \begin{array}{c} \\ i_0 \\ \Delta\log(Y_0) \\ \pi_0 \\ H_0 \\ \Delta\log(W_0) \\ i_\infty \\ \Delta\log(Y_\infty) \\ \pi_\infty \\ H_\infty \\ \Delta\log(W_\infty) \end{array} \begin{array}{c} \begin{array}{ccccc} \varepsilon^{MP} & \varepsilon^{AD} & \varepsilon^{AS} & \varepsilon^{WM} & \varepsilon^{U} \end{array} \\ \left[\begin{array}{ccccc} + & + & - & - & \times \\ - & + & + & + & \times \\ - & + & - & - & \times \\ \times & \times & \times & + & \times \\ \times & \times & \times & - & \times \\ \hline \times & \times & \times & \times & \times \\ 0 & 0 & \times & 0 & 0 \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{array}\right] \end{array}.$$

Upon impact it is assumed that a monetary policy shock will result in an increase in the interest rate and a fall in both GDP growth and inflation. The monetary policy shock will have no long-run effect on GDP growth. The aggregate demand shock is assumed to cause interest rates, GDP growth and inflation to increase upon impact. The aggregate demand shock will not have any long-run effect on GDP growth. The aggregate supply shock is assumed to cause interest rates and inflation to fall, and GDP growth to increase on impact. The fall in the interest rate on impact is based on impulse responses from the SW DSGE model. It is assumed that the aggregate supply shock is the only shock that has a long-run impact on GDP growth. This is a common assumption in papers that use long-run identifying restrictions (see Gali (1999), Blanchard & Quah (1989) and Christiano et al. (2006) for examples). The sign restrictions used to identify the wage mark-up shock come directly from the SW DSGE model. It is assumed that on impact the wage mark-up results in a decrease in interest rates, CPI inflation and wage inflation. Upon impact the wage mark-up shock is assumed to result in an increase in GDP growth and hours worked. The wage mark-up shock cannot affect GDP growth in the long-run. While the sign restrictions on the aggregate supply shock and the wage mark-up shock are very similar, the aggregate supply shock is unrestricted when it comes to its effects on hours and wage inflation on impact and the wage mark-up shock cannot have a permanent effect on GDP growth. The only assumption made on the unidentified shock is that it cannot affect GDP growth in the long-run.

The impulse responses for the monetary policy shock are presented in figure B.2 in Appendix B. The 95% bands for the SVAR model are plotted alongside the median impulse response function, the SVAR model nearest to the median of the impulse responses (as advocated by Fry & Pagan (2011)) and the impulse responses from the monetary policy shock in the SW DSGE model. The impulse responses from the SVAR and the SW model are quite similar. The responses of interest rates, GDP and hours worked are slightly stronger in the SW model, while the inflation and wage inflation responses are slightly weaker.

The impulse responses for the demand shock in the SVAR are plotted against the government spending shock, the risk premia shock and the investment shock from the SW DSGE model in figure B.3 in Appendix B. These impulse responses are also quite similar. The DSGE model is slightly more persistent as can be seen in the responses of GDP and hours worked to the shocks. The only major difference between the SVAR and the DSGE model occurs with the response of wage inflation to the demand shocks. The median response of wage inflation falls following the shock, while wages increase in the DSGE model following the demand shocks.

The impulses for the aggregate supply shock in the SVAR are plotted against the impulses for a technology shock and a cost push shock from the SW DSGE model. These are presented in figure B.4 in Appendix B. In general the signs of the impulses from the SVAR are consistent with the signs of the impulses from the DSGE model (some of this is by construction). The fall in hours worked after the supply shock from the SVAR model is consistent with the response following a technology shock in the DSGE model. The interest rate, GDP, hours and inflation response are slightly stronger in the SVAR model.

The signs of the impulse responses used to identify a wage mark-up shock have been imposed using the same signs as the DSGE model (presented in figure B.5). However the impulses are quite different. The DSGE model is more persistent for nearly all the variables.

Using these sign and zero restrictions the SVAR model gives similar results to the demand and monetary policy shocks in the SW DSGE model. However while the signs are the same for the supply and wage markup shocks, there are some differences in the persistence and magnitudes of the responses.

7.2. Example 2: Mountford & Uhlig (2009) type restrictions

In this second example, I demonstrate how my algorithm can be extended to deal with Mountford & Uhlig (2009) (MU from now on) style sign and zero restrictions. In their paper MU discuss how anticipated government spending shocks can be identified by assuming that the response of government spending is zero for the first four quarters and then positive for the next four quarters, which is a combination of zero and sign restrictions. They adopt a recursive identification scheme by identifying business cycle and monetary policy shocks first using sign restrictions which they impose via a penalty function that penalises impulse responses with the wrong sign. By minimising the penalty function they allow these shocks to explain the maximum amount of variation in GDP and the GDP component variables. This is important because it separates the endogenous fluctuations of government spending and taxation due to the business cycle from the purely exogenous fiscal policy interventions. Then they identify the government spending shock using the same penalty function approach but with zero and orthogonality constraints in addition to the sign restrictions. They use a similar approach when identifying an anticipated government revenue shock.

In this example I demonstrate how my algorithm can be extended to include zero restrictions on multiple impact matrices in combination with sign restrictions following a shock. In particular I follow MU and identify an anticipated government revenue shock by estimating a BVAR model with an independent Normal-Wishart prior using the MU data set. The MU data set uses US data from 1955:1 to 2000:4 and includes: GDP, government expenditure, government revenue, the federal funds rate, adjusted reserves, the producer price index, private consumption, non-residential investment and real wages, where all the definitions are the same as those in MU. I use six lags and do not include a constant or a time trend. I apply the same zero and sign restrictions as MU do to identify an anticipated government revenue shock, a business cycle shock and a monetary policy shock. I also plot my results against the results from MU obtained used a recursive penalty function method, for comparison. Using my algorithm the shocks are identified simultaneously which differs from the recursive identification approach in MU. The recursive approach is key to the MU identification scheme and as such I do not advocate my method as a replacement for their method. However the MU type sign and zero restrictions without the recursive assumption are well suited to illustrate the types of problems my methodology can deal with. Comparing the results using my methodology against the results using the MU methodology also provides a nice test of the sensitivity of the MU identification scheme to the recursive assumption. There are likely to be similar problems where zero restrictions are required for multiple periods in combination with sign restrictions where shocks receive equal importance, in which case my algorithm would be a natural choice.

The MU sign restrictions and zero restrictions can be summarised as follows

$$L_i = \begin{array}{c} \\ \text{GDP} \\ \text{Govt. Expenditure} \\ \text{Govt. Revenue} \\ \text{Fed. FundsRate} \\ \text{Adjusted Reserves} \\ \text{PPIC} \\ \text{GDP Deflator} \\ \text{Private Consumption} \\ \text{Priv. Non-res. Inv.} \\ \text{Wages} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \varepsilon^{BC} & \varepsilon^{MP} & \varepsilon^{GR} & & & & & & & \end{array} \\ \begin{bmatrix} + & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ + & \times & 0 & \times & \times & \times & \times & \times & \times & \times \\ \times & + & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & - & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & - & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & - & \times & \times & \times & \times & \times & \times & \times & \times \\ + & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ + & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \end{array}.$$

for $i = 0, \cdots, 3$. Where ε^{BC} is the business cycle shock, ε^{MP} is the monetary policy shock and ε^{GR} is the government revenue shock. These are the sign and zero restrictions applied to the first four impact matrices following the shock. The sign restrictions for the next four quarters can be summarised as follows

$$L_i = \begin{array}{c} \\ \text{GDP} \\ \text{Govt. Expenditure} \\ \text{Govt. Revenue} \\ \text{Fed. FundsRate} \\ \text{Adjusted Reserves} \\ \text{PPIC} \\ \text{GDP Deflator} \\ \text{Private Consumption} \\ \text{Priv. Non-res. Inv.} \\ \text{Wages} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \varepsilon^{BC} & \varepsilon^{MP} & \varepsilon^{GR} & & & & & & & \end{array} \\ \begin{bmatrix} \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & + & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \end{array}.$$

for $i = 4, \cdots, 7$. Only the impact matrices for the first four quarters are required to solve the zero restrictions because it is only in these periods that the zero restrictions occur. Equation (2) can be modified as follows

$$f(Z, B(L)) = \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix}. \tag{8}$$

Equation (8) implies that both the SVAR model and the government revenue shock are underidentified. If the columns were reordered based on the ranks of the restriction matrices (the Q_js in equation (3)), the government revenue shock would be in the first column and have a rank of $q_1 = 4$, which is smaller than the maximum allowed of 9 by Theorem 1. The next step requires obtaining the initial impact matrices to be used in the algorithm. To derive these matrices, it is useful to rewrite the VAR model in equation (1) in moving average form

$$Y_t = \Phi(L)u_t, \tag{9}$$

where $\Phi(L) = \sum_{i=0}^{\infty} \Phi_i L^i$, $\Phi_0 = \underset{m\times m}{I}$, $\Phi_i = \sum_{j=1}^{i} \Phi_{i-j}B_j$. This allows equation (1) to be rewritten as

$$Y_t = \sum_{i=0}^{\infty} \Phi_i u_{t-i}. \tag{10}$$

This moving average form of the model can be used to write the impulse response function for the variables Y_t, h periods after a shock as

$$\begin{aligned} Y_h &= \Phi_h u_0, \\ &= \Phi_h Z \varepsilon_0. \end{aligned} \tag{11}$$

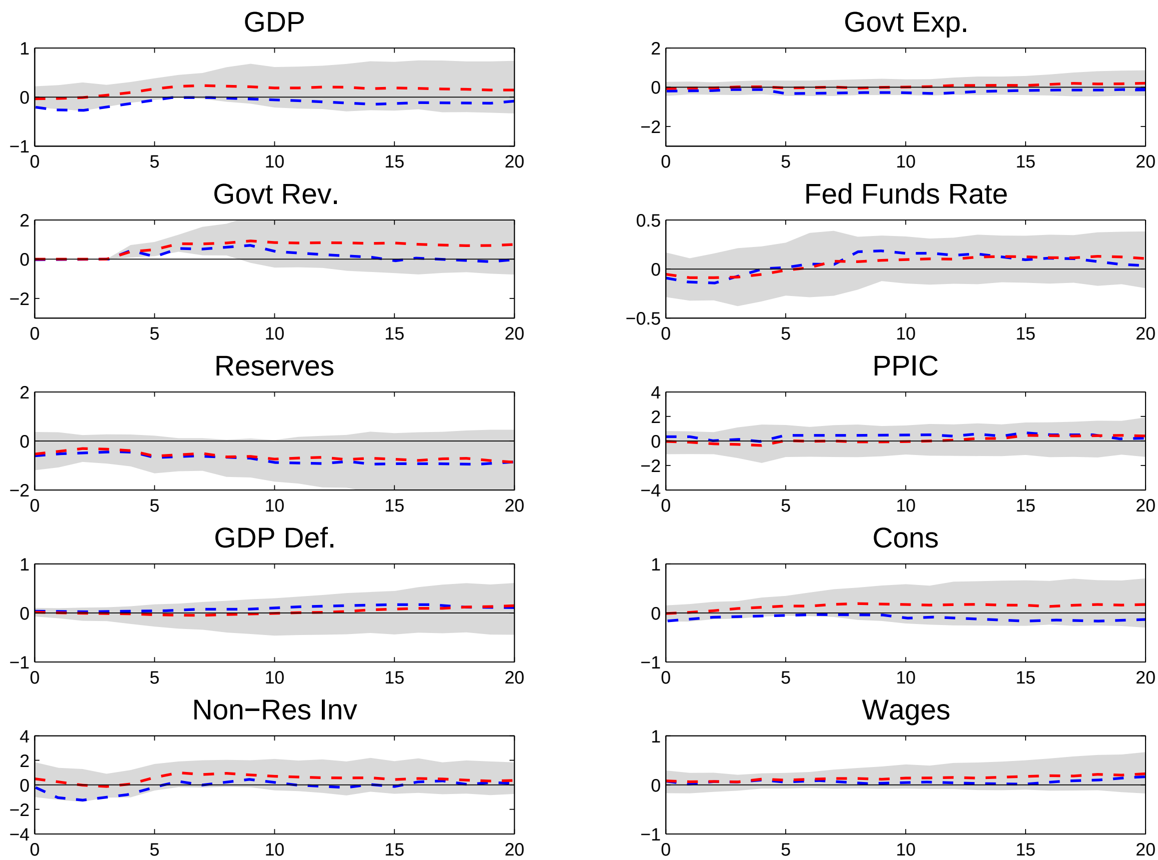
The initial impact matrices on line 17 in algorithm 3 can be replaced with

$$F = \begin{bmatrix} L_0^* \\ L_1^* \\ L_2^* \\ L_3^* \end{bmatrix}, \tag{12}$$

where $L_0^* = CQ^*$, $L_1^* = \Phi_1 CQ^*$, $L_2^* = \Phi_2 CQ^*$, and $L_3^* = \Phi_3 CQ^*$. As before, C is the lower Cholesky decomposition of the shock covariance matrix and Q^* is a randomly drawn orthogonal matrix. No further modifications to algorithm 3 are required.

Imposing such restrictions results in the following impulse responses to an anticipated government revenue shock.

Figure 1: Anticipated Government Revenue Shock



Note: The red dashed line is the median and the gray shaded area covers the 16th to 84th percentiles. The blue dashed lines are the median impulse responses from MU.

In general the results using both algorithms are quite similar. However there are some differences, in particular the MU approach results in a negative response to GDP over the entire period plotted, while the algorithm used in this paper leads to no GDP response initially followed by an increase in GDP. This is similar to the responses for consumption and non-residential investment. This illustrates the motivation for MU adopting a recursive identification scheme. In their paper GDP does not rise after an anticipated government revenue shock because they order the shocks to remove the endogenous response of government revenue to the business cycle which allows them to isolate what are likely to be exogenous policy interventions. Combining the MU sign and zero restrictions with my algorithm confirms the role the recursive identifying assumption plays in the identification of fiscal policy shocks. It also illustrates quite nicely how my algorithm can be generalised to included zero restrictions on multiple impact matrices.

8. Conclusion

In this paper I have presented a new method for combining zero restrictions with sign restrictions. More specifically, I have extended the RWZ algorithm for imposing zero restrictions on exactly identified models to models that are underidentified by combining it with the RWZ sign restrictions algorithm. This results in a fast and very general algorithm that can be applied to identify SVAR models under a range of identifying assumptions including standard zero restrictions and standard sign restrictions. I have demonstrated how my algorithm can be applied using two examples. In the first example I have shown how sign restrictions can be combined with long-run restrictions to identify an SVAR model estimated using the SW data set. The sign restrictions used are a combination of standard restrictions in addition to some that are derived from the SW DSGE model. In the second example I have demonstrated how the algorithm can be modified to impose zero restrictions on the impact matrices for the first four periods following a shock. This is very similar to the identification scheme used by Mountford & Uhlig (2009) to identify anticipated fiscal shocks except the shocks are found simultaneous and not recursively. In addition to the two examples presented in this paper there are many other problems where such an algorithm could be used. For example, those working with small open economy SVAR models will likely find this algorithm useful. In particular the algorithm could be used to impose short and/or long-run restrictions on the small open economy VAR model in combination with sign restrictions. The exogeneity of the foreign block on the domestic block could be maintained by imposing additional zero restrictions on the short-run impact matrix.

Appendix A. Solving underdetermined systems

This Appendix describes how to solve an underdetermined system using the QR decomposition and how this is applied in the RWZ algorithm. The first part of the example is taken from page 272 of Golub & Van Loan (1996). Given the linear system

$$Ax = b, \tag{A.1}$$

where A is $m \times n$, x is $n \times 1$ and b is $m \times 1$. This system is under determined because $\text{rank}(A) = m$, where $m < n$. Taking the QR decomposition of A' results in

$$A' = QR = \underset{n\times n}{Q} \begin{bmatrix} \underset{m\times m}{R_1} \\ \underset{n-m\times m}{0} \end{bmatrix}. \tag{A.2}$$

So that $Ax = b$ becomes

$$(QR)'x = [R_1', 0] \begin{bmatrix} \underset{m\times 1}{z_1} \\ \underset{n-m\times 1}{z_2} \end{bmatrix} = b, \tag{A.3}$$

where

$$Q'x = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \text{and} \quad x = Q \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

The minimum norm solution can then be obtained by setting $z_2 = 0$ and solving for z_1 and then using Q to find x.

$$R_1' z_1 = b, \tag{A.4}$$
$$x = Q_1 z_1, \tag{A.5}$$

where

$$Q = \left[\underset{n \times m}{Q_1}, \underset{n-m \times m}{Q_2} \right].$$

The application in this paper is slightly different because the right hand side is equal to 0. The underdetermined system in this paper is given by

$$\tilde{Q}_j P_j = 0, \tag{A.6}$$

where $\tilde{Q}_j$ is a known $(m-1) \times m$ matrix defined on line 12 or 14 of Algorithm 1, and P_j is an $m \times 1$ unknown vector. Taking the QR decomposition of $\tilde{Q}_j'$ gives

$$\tilde{Q}_j' = QR = Q \begin{bmatrix} \underset{m-1 \times m-1}{R_1} \\ \underset{1 \times m-1}{0} \end{bmatrix}. \tag{A.7}$$

Substituting equation (A.7) into (A.6) gives

$$[R_1', 0]\, Q' P_j = 0. \tag{A.8}$$

$Q' P_j$ can be rewritten as

$$Q' P_j = z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{A.9}$$

This implies

$$R_1' z_1 = 0. \tag{A.10}$$

Because R_1 has rank equal to $m-1$, a solution to (A.10) is

$$z_1 = 0. \tag{A.11}$$

Plugging this back into (A.9) gives

$$z = \left[\underset{1 \times m-1}{0}, 1 \right]'. \tag{A.12}$$

The last element in z has to be non-zero so that a solution exists. The solution will be the last column of the orthogonal matrix created at each iteration of the algorithm. Setting the value of the last element in z to one ensures that P_j, the solution to the problem, will be orthogonal to the other columns in the rotation matrix P. This implies the solution for P_j is given by

$$P_j = Qz = Q(:, m). \tag{A.13}$$

Appendix B. Examples: Impulses

Figure B.2: Monetary Policy Shock

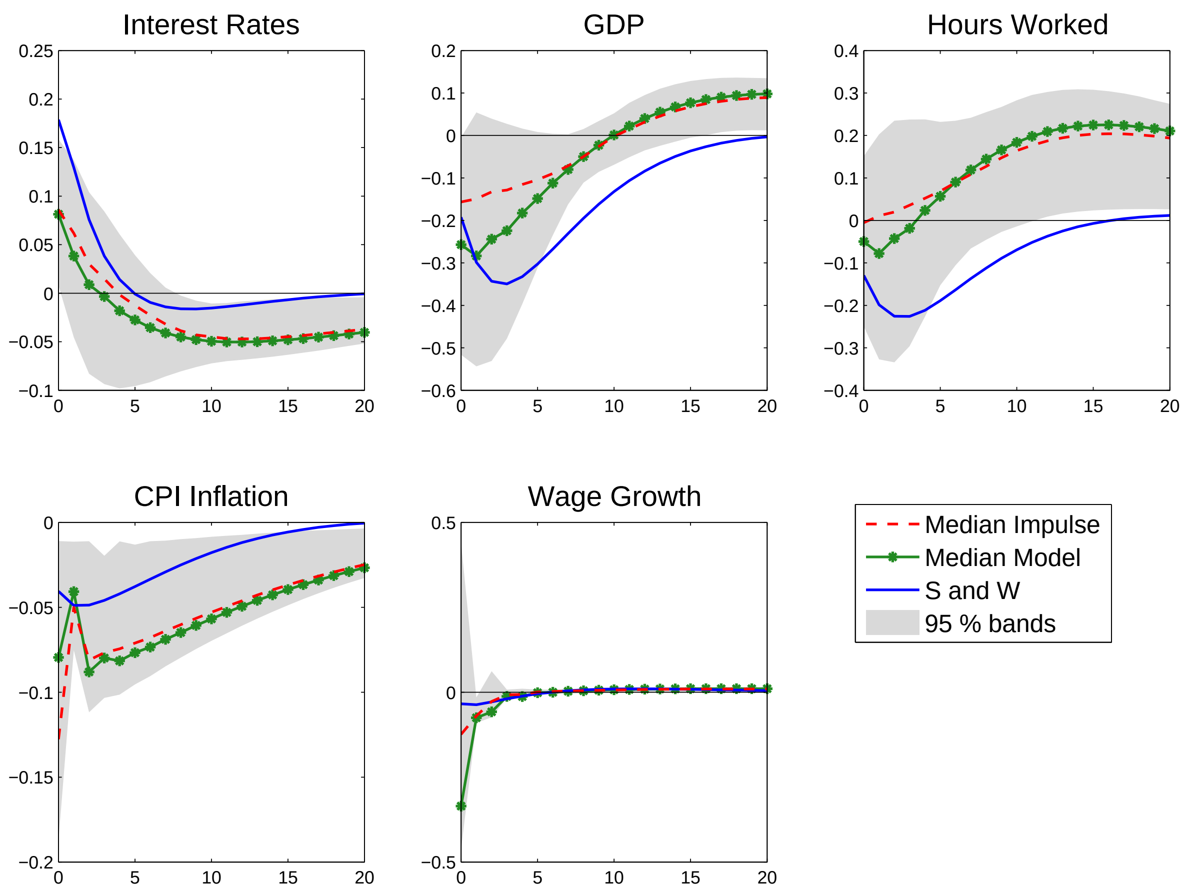


Figure B.3: Demand Shock

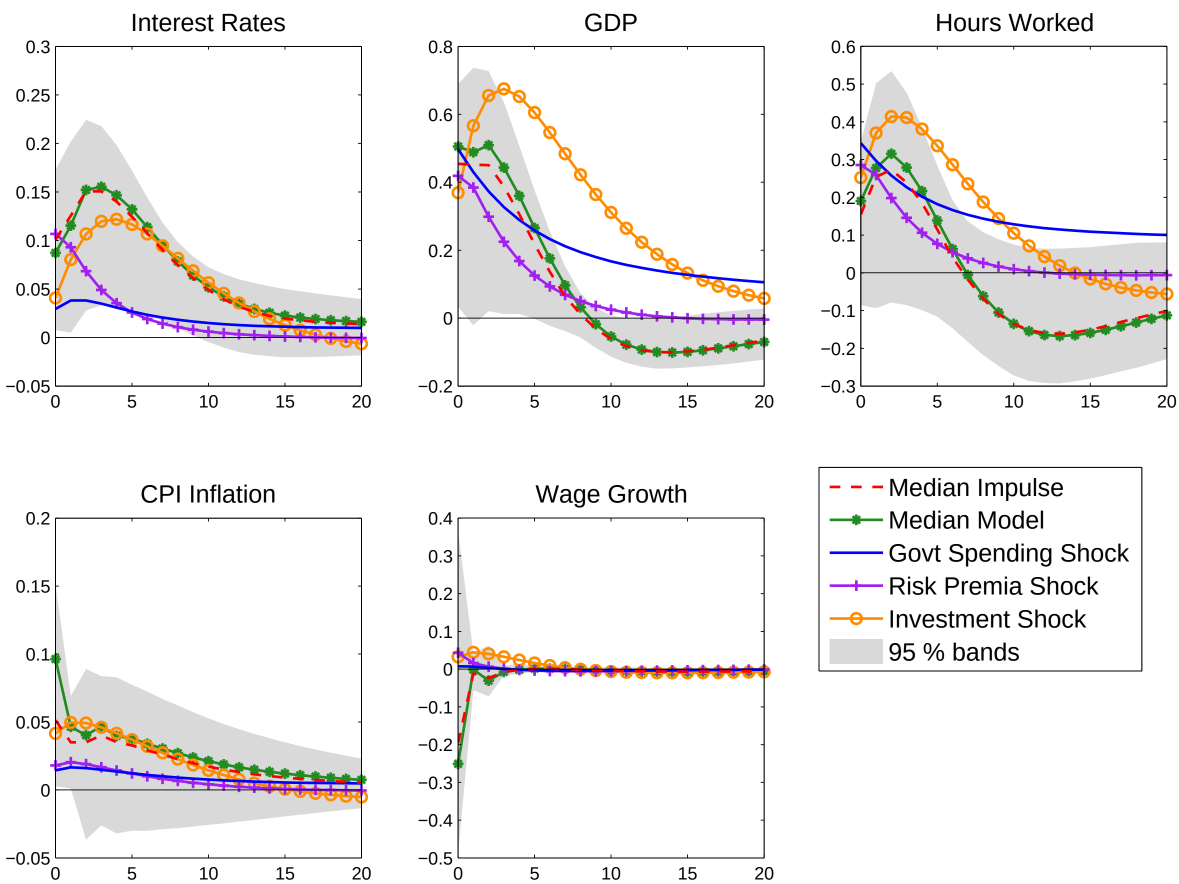


Figure B.4: Supply Shock

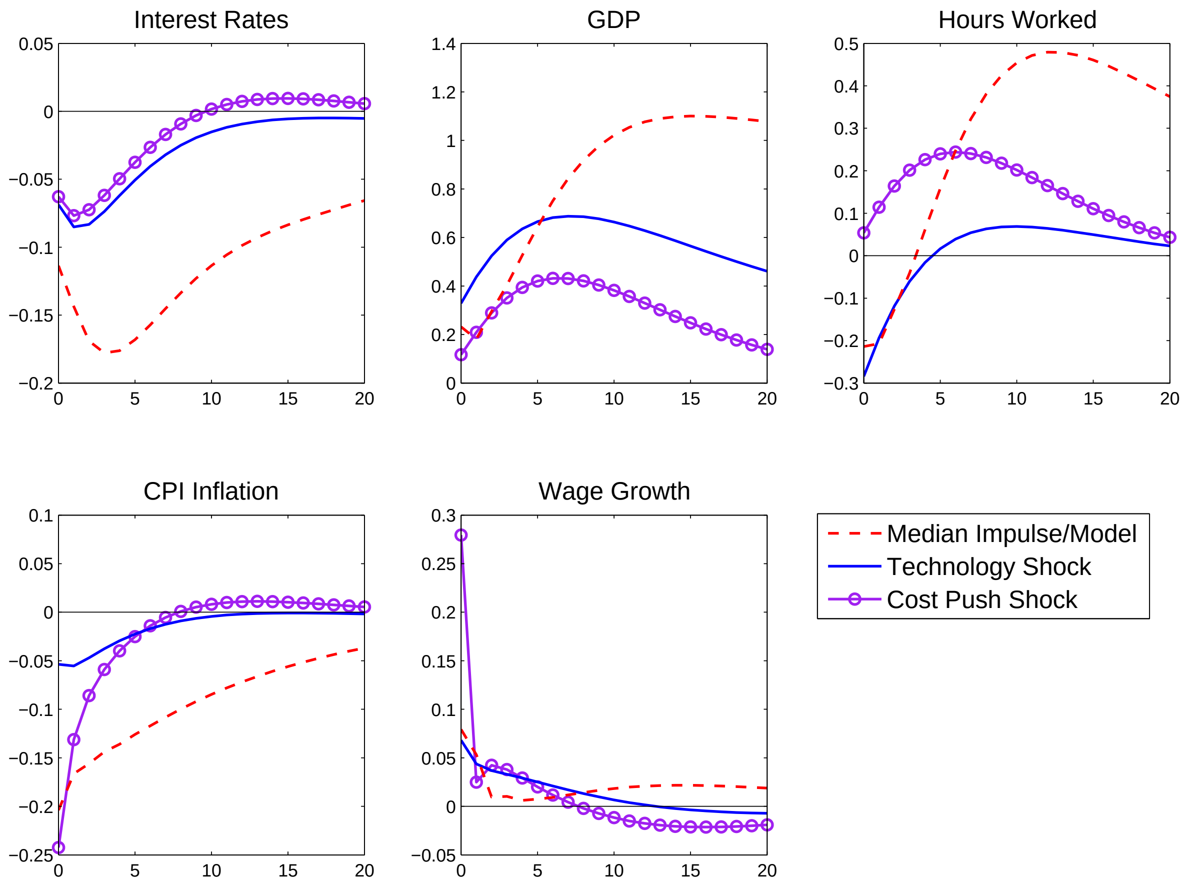
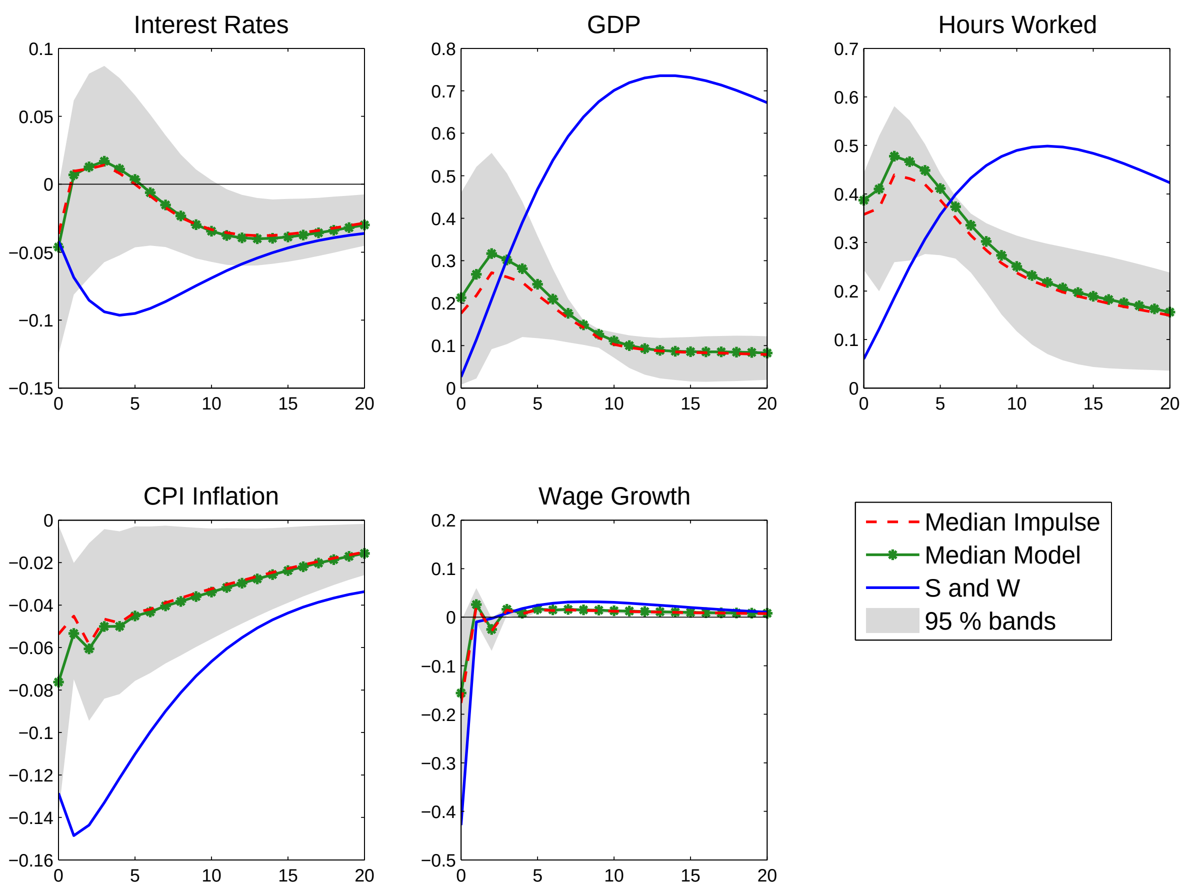


Figure B.5: Wage Mark-up Shock



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