Estimating Inflation Expectations using Forward Starting Inflation Swaps

Simon Thinggaard Hetland*

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Abstract

We introduce a new model of inflation expectations
JEI:

1 Introduction

Inflation-linked securities are important indicators of financial market participants' expectations about future inflation. Inflation swaps, for example, provide a means of measuring inflation expectations of different maturities, and can, in combination with yields on nominal government bonds, be used to calculate ex ante real interest rates. Furthermore, forward inflation swap rates are often viewed as providing information about central bank credibility; if forward inflation swaps are close to the central bank's inflation target, the bank may be viewed as being credible and committed to its target. As such, inflation swap rates are routinely used as proxies for financial market participants expectations about future inflation.[1]

Of course, inflation swaps and other inflation-linked securities not only contain expectations about future inflation, but also risk premia. These risk premia, such as inflation risk premia and liquidity premia, complicate the interpretation of inflation swap rates. E.g. the inflation risk premium tend to correlate with contemporaneous events (**oil, covid etc, source?**), rather than reflecting changes in perception about the future state of the macro economy. As risk premia are not directly observable, they must be estimated from data in order to gauge inflation expectations. No-arbitrage term structure models have

*Danmarks Nationalbank, sthe@nationalbanken.dk. The views expressed in this paper do not necessarily reflect those of Danmarks Nationalbank.

[1] An alternative measure is the break-even inflation rate based on inflation-linked and nominal government bonds, which also conveys a (noisy) measure of inflation expectations.

been used to disentangle these premia from the inflation expectations, both in the context of inflation swaps and inflation-linked bonds. Hördahl (2008), Beechey (2008) and Garcia and Werner (2010) model inflation expectations by estimating affine term structure models on government-bond based break-even inflation rates, while Abrahams et al. (2016) and Andreasen, Christensen, and Riddell (2021) model the real and nominal government yield curves simultaneously, and explicitly model the differences in liquidity in real and nominal bonds. In inflation swap space, Camba-Méndez and Werner (2017) estimates an affine term structure model on spot starting inflation swaps.

Inflation swaps are widely used in the euro area as a surveillance and analytical tool for conducting monetary policy, see e.g. Böninghausen, Kidd, Vincent-Humphreys, et al. (2018) and Baumann et al. (2021). Euro area inflation swaps are linked to the HICP excluding tobacco for the entire monetary union, and provide a gauge of inflation expectations for the entire union, as opposed to nationally issued inflation-linked bonds, which are commonly tied to a national price index, see e.g. **find review of inflation linked bonds**. Boneva et al. (2019) document using regulatory data that euro area inflation swaps are widely traded in the 1-10 year segments.

One standing issue, however, is that inflation swaps have an "indexation lag" of three months in the euro area and the US.[2] That is, a spot starting inflation swap traded in month m is based on the underlying price index for month $m-3$. Hence, spot starting inflation swaps not only contain information about expectations and risk premia, but also prices in two-three months of realized inflation (depending on the release date of inflation prints). This indexation lag biases the estimated inflation expectations up or down, depending on sign and size of inflation prints.[3] Practitioners will, for this reason, often use forward starting swap rates instead when gauging market pricing of inflation compensation, to avoid the bias of realized inflation in the spot starting swap **(cite something here, e.g. a textbook? CFA material?).**

Camba-Méndez and Werner (2017) propose one solution to the indexation lag issue, namely by interpolating the inflation swap curve. That is, for e.g. the one-year swap, they interpolate between the most recent realized inflation in month and the two-year market quote (which matures in 21 months). Based on this interpolated curve, and adding in historical seasonal patterns, they compute the "inedxation lag adjusted" one-year swap rate. This approach makes several implicit assumptions.

- First, the forward HICPxT curve is assumed to be piecewise linear(?), or at least

[2] Inflation-indexed government bonds do not suffer from this problem / Interpolated / most uses canadian model, see some reference.

[3] The indexation lag will especially influence short term swaps and hence expectations: In a 1-year spot starting inflation swap, up to $3/12 = 0.25$ per cent of the HICP excluding tobacco developments is known, while this is only $3/60 = 0.05$ per cent in a 5-year inflation swap.

that the functional form of the inflation swap curve is known between knots

- Second, that historical seasonality patterns are representative for the inflation swap curve.

We propose a different solution to the indexation lag issue. Forward starting inflation swaps are not affected by realized inflation. Consider e.g. a 1Y1Y inflation swap, which by market standards is based on the change in the price index from month $t + 9$ to $t + 21$. We derive a forward starting affine term structure model, based on the canonical representation framework of Joslin, Singleton, and Zhu (2011).[4] Under the standard pricing kernel, we derive the required no-arbitrage conditions for m over n year forward starting inflation swaps and show how to estimate the model using a simple two-step estimation procedure, based on principal components and a profiled maximum likelihood approach.

- We can both compute expected inflation for the swaps observed in the market, but for forecasting purposes, we are mostly interested in the relevant YoY expectations (e.g. for t to t+12 rather than t-3 to t-9 as priced in the market).

- No issue of seasonality, because we only deal with year over year grotwh rates, as opposed to the Camba-Méndez and Werner (2017) approach, which interpolates the *level* of the price index.

Our estimated inflation expectations line up well with the ECB's survey of professional forecasters, despite the model not containing any survey information. In our empirical exercise, covering the COVID-19 pandemic and the Russian invasion of Ukraine, we find that shorter term expectations have risen significantly, and that the inflation risk premium has changed sign, from negative to positive, indicating an increased perceived stagflation risk. Longer term expectations remain anchored around the ECB's inflation target of 2 per cent. Furthermore, we conduct a small out-of-sample forecasting exercise for three horizons, where we compare our estimated expected inflation to those of the Camba-Méndez and Werner (2017) model and market inflation swap quotes. We find that our model provides the best forecasting accuracy across all horizons, albeit the predictive power is moderate. Finally, we note that the proposed model is a simple and easy to implement and estimate, extension of classical affine term structure models[5]

[4]Affine term structure models were introduced in Duffie and Kan (1996), Dai and Singleton (2000), and Duffee (2002), see also Christensen, Diebold, and Rudebusch (2011).

[5]Matlab code is available on `https://sites.google.com/view/simonhetland/`

2 An affine term structure model in forward rates

2.1 The model

We now propose an affine term structure model for forward starting inflation swaps. The model allows us to estimate expected inflation over any given horizon, and explicitly handles the issue of the indexation lag, removing the effect of realized inflation on the estimated inflation expectations.

Consider initially the profile of a forward-starting nY$(m-n)$Y inflation swap[6].

$$(1+\pi_t^{n,m})^{(m-n)} = E_t^{\mathbb{Q}}\left[\frac{I_{t+m-3}}{I_{t+n-3}}\right], \tag{1}$$

where I_t is the relevant price index (the HICP ex. tobacco index for the Euro Area), $\pi_t^{(n,m)}$ is the (annually compounded) fixed inflation leg and $E_t^{\mathbb{Q}}[\cdot]$ is the conditional expectation under the risk-neutral probability measure. As shown, the fixing of the price indices is lagged 3 months. This fact will have a small, but noticeable effect on the asset pricing model.

By first taking the exponential, and then logs, along with utilizing that $\pi_t = I_t/I_{t-1}$, we rewrite the equation as,

$$f_t^{(n,m)} = \frac{1}{m-n}\log\left(E_t^{\mathbb{Q}}\left[\exp\left(\sum_{j=1+n}^{m}\pi_{t+j-3}\right)\right]\right), \tag{2}$$

where $f_t^{(n,m)} = \log(1+\pi_t^{(n,m)})$, i.e. the continuously compounded forward inflation compensation. Under the $\mathbb{Q}$ measure, the forward starting inflation swap quote is equal to the expected average future inflation (over the period $t+n-3$ to $t+m-3$), up to a Jensen's inequality term.

The expression in (2) can be solved analytically using a Gaussian Affine Term Structure model, see e.g. Duffie and Kan (1996) or Dai and Singleton (2000). To arrive at the closed form solution, for the inflation expectations under the risk-neutral pricing measure $\mathbb{Q}$ we assume that instantaneous inflation is driven by k latent factors, X_t,

$$\pi_t = \rho_{X,0} + \rho_{X,1}' X_t, \tag{3}$$

where $\rho_{X,0}$ is a scalar and $\rho_{X,1}$ is a $k\times 1$ vector. The latent factors is assumed to be driven

[6] I.e., the m year inflation swap rate starting in n years from now, such that the 1Y4Y inflation swap pays out the annualized inflation over the next four years, starting one year from now.

by a VAR(1) under the risk-neutral measure

$$X_t = K_X^{\mathbb{Q}} + \Phi_X^{\mathbb{Q}} X_{t-1} + \Sigma \epsilon_t^{\mathbb{Q}}, \qquad \epsilon_t^{\mathbb{Q}} \sim N(0, I_k), \tag{4}$$

where $K_X^{\mathbb{Q}}$ is a $k \times 1$ vector and $\Phi_X^{\mathbb{Q}}, \Sigma$ are $k \times k$ matrices. Based on these assumptions, we can characterize the full cross-section of the inflation curve, and we show in the appendix that the forward-starting inflation swap rates are linear functions in the underlying factors,

$$f_t^{n,m} = a_X^{n,m} + b_X^{n,m} X_t, \tag{5}$$

such that the parameters in $a_X^{n,m}$ (1×1) and $b_X^{n,m}$ $(1 \times k)$ are

$$a_X^{n.m} = \frac{-1}{m-n}(A_X^m - A_X^n), \tag{6}$$

$$b_X^{n.m} = \frac{-1}{m-n}(B_X^m - B_X^n), \tag{7}$$

where the coefficients A_X^n and B_X^n are standard Riccati differential equations, see e.g. Joslin, Singleton, and Zhu (2011),

$$A_X^n = -\rho_{X,0} + A_X^{n-1} + B_X^{n-1\prime} K_X^{\mathbb{Q}} + \frac{1}{2} B_X^{n-1\prime} \Sigma \Sigma' B_X^{n-1}, \tag{8}$$

$$B_X^{n\prime} = -\rho_{X,1}' + B_X^{n-1\prime} \Phi^{\mathbb{Q}}. \tag{9}$$

The recursions are initiated in $A_X^0 = 0$ and $B_X^0 = 0$. **check transpose of recursion**

Having characterized how the forward starting inflation swaps are priced using no-arbitrage in the cross-section, we make an assumption of the factor dynamics under the physical measure, $\mathbb{P}$. Here, we assume that the factors, X_t, also follow a VAR(1),

$$X_t = K_X^{\mathbb{P}} + \Phi_X^{\mathbb{P}} X_{t-1} + \Sigma \epsilon_t^{\mathbb{P}}, \qquad \epsilon_t^{\mathbb{P}} \sim N(0, I_k), \tag{10}$$

where $K_X^{\mathbb{P}}$ is a $k \times 1$ vector and $\Phi_X^{\mathbb{P}}$ is a $k \times k$ matrix.

2.2 Identifying restrictions

Without parameter restrictions, the model and its parameters are not statistically identified. To overcome this, we employ the "canonical representation" introduced by Joslin, Singleton, and Zhu (2011) (JSZ henceforth). JSZ show that many existing model parameterizations in the literature have the same core, and are in fact identical up to a rotation. They present a simple identification scheme, which works extremely well numerically. In essence, JSZ impose identifying restrictions on the "canonical" (unobserved) factors, X_t, and then rotate said canonical factors into observable factors, denoted P_t. These observable factors could e.g. be principal components, and we return to rotating the factors, after having discussed the identifying restrictions.

Under the canonical representation, $\pi_{X,0}, \pi_{X,1}, K_X^{\mathbb{Q}}, \Phi_X^{\mathbb{Q}}$ and Σ are restricted to ensure statistical identification. The instantaneous inflation rate is restricted in the canonical representation to have no intercept and unit loadings on the factors X_t, i.e.,

$$\rho_X^0 = 0, \tag{11}$$

$$\rho_X^1 = \iota, \tag{12}$$

such that the factors have the same impact on the instantaneous inflation rate. Next, the constant term under the $\mathbb{Q}-$measure is restricted as follows,

$$K_X^{\mathbb{Q}} = [k_\infty^{\mathbb{Q}}, 0, \ldots, 0]', \tag{13}$$

where $k_\infty^{\mathbb{Q}}$ is the long-run mean of the instantaneous inflation rate. Furthermore, the autoregressive matrix, $\Phi_X^{\mathbb{Q}}$, is assumed to be diagonal, with ordered eigenvalues, i.e.[7]

$$\Phi_X^{\mathbb{Q}} = \text{diag}(\lambda_1, \ldots, \lambda_k), \tag{14}$$

Finally, to identify the covariance matrix of the states, we assume that Σ is lower triangular.

2.3 Rotating and estimating the model

With these identifying restrictions in hand, we could proceed to estimating the model using a standard Kalman filtering approach. However, estimation using Kalman filters can be time consuming and numerically challenging. Rather, we propose a rotation of the unobserved states, X_t. This rotation is based on JSZ, which employs the first k principal components as factors. JSZ show that if we are willing to assume that we observe the principal components of the yields without measurement error, then this rotation of factors is possible without loss of generality. Numerical issues relating to estimation are greatly reduced, as we can estimate the $\mathbb{P}$-measure dynamics in a separate step. The assumption on no measurement errors is slightly unrealistic, but it turns out that whether we use Kalman filtering techniques (allowing for measurement errors) or base the estimation on principal components, the model output differ very little. **maybe as a robustness check, estimate expected inflation using KF and identification?**

That is, assuming that the factors are linear combinations (such as eigenvectors) of the observed forward starting inflation swaps, $P_t = W f_t^{(n,m)}$, then the canonical factors, X_t,

[7]In the case of repeated eigenvalues, $\Phi_X^{\mathbb{Q}}$ is in its Jordan form, see Joslin, Singleton, and Zhu (2011).

can be rewritten using (5), where we isolate the canonical factors as a function of the principal components, P_t,

$$P_t = W(a_X^{n,m} + b_X^{n,m})X_t \leftrightarrow$$
$$X_t = (Wb_X^{n,m})^{-1}(P_t - Wa_X^{n,m}). \tag{15}$$

Hence, the forward rates in (5) can be rewritten in terms of the principal components, P_t,

$$f_t^{n,m} = a_X^{n,m} + b_X^{n,m}(Wb_X^{n,m})^{-1}(P_t - Wa_X^{n,m})$$
$$= a^{n,m} + b^{n,m}P_t, \tag{16}$$

where,

$$a^{n,m} = (I_k - b_X^{n,m}(Wb_X^{n,m})^{-1}W)a_X^{n,m} \tag{17}$$
$$b^{n,m} = b_X^{n,m}(Wb_X^{n,m})^{-1}, \tag{18}$$

where $a_x^{n,m}$ and $b_x^{n,m}$ are defined in (6)-(9). In a similar manner, we can determine the instantaneous inflation dynamics in terms of the principal component factors, from (3)

$$\pi_t = \rho_X^0 + \rho_X^{1\prime}(Wb_X^{n,m})^{-1}(P_t - Wa_X^{n,m}) \tag{19}$$
$$= \rho_0 + \rho_1' P_t, \tag{20}$$

where,

$$\rho_0 = \rho_X^0 - \rho_X^{1\prime}(Wb_X^{n,m})^{-1}Wa_X^{n,m} \tag{21}$$
$$\rho_1 = \rho_X^{1\prime}(Wb_X^{n,m})^{-1}. \tag{22}$$

Finally, the $\mathbb{Q}$-measure dynamics of the principal components. From (4)

$$(Wb_X^{n,m})^{-1}(P_t - Wa_X^{n,m}) = k_X^{\mathbb{Q}} + \Phi^{\mathbb{Q}}(Wb_X^{n,m})^{-1}(P_{t-1} - Wa_X^{n,m}) \leftrightarrow$$
$$P_t = K^{\mathbb{Q}} + \Phi^{\mathbb{Q}}P_{t-1}, \tag{23}$$

with parameters,

$$K^{\mathbb{Q}} = Wb_X^{n,m}k_X^{\mathbb{Q}} + (I_k - Wb_X^{n,m}\Phi_X^{\mathbb{Q}}(Wb_X^{n,m})^{-1})Wa_X^{n,m} \tag{24}$$
$$\Phi^{\mathbb{Q}} = Wb_X^{n,m}\Phi_X^{\mathbb{Q}}(Wb_X^{n,m})^{-1}. \tag{25}$$

Having achieved statistical identification and rotated the model into principal component space, we are now ready to discuss estimation.

2.4 Estimation

We have now rewritten the (identified) model in terms of easily estimated principal components, our state space system to be estimated has the form

$$P_t = K^{\mathbb{P}} + \Phi^{\mathbb{P}} P_{t-1} + \Sigma \epsilon_t^{\mathbb{P}}, \quad \epsilon_t^{\mathbb{P}} \sim N(0, I_k) \tag{26}$$

$$f_t = a + b' P_t + \Omega \nu_t, \quad \nu_t \sim N(0, I_p) \tag{27}$$

where f_t is $p \times 1$ where p is the number of forward starting inflation swaps included in the estimation. E.g. in the case of three inflation swaps, with $n = [12, 12, 12]$ and $m = 24, 36, 48$, then $a = [a^{12,24}, a^{12,36}, a^{12,48}]'$ and $b = [b^{12,24}, b^{12,36}, b^{12,48}]$, which are $p \times 1$ and $p \times k$ respectively.

In the first step of the estimation procedure, we estimate the first k principal components of the yield data, denoted $\hat{P}_t$, and fit a VAR(1) to $\hat{P}_t$ using OLS, recovering $\hat{K}^{\mathbb{P}}$ and $\hat{\Phi}^{\mathbb{P}}$. Typically, we use the first three principal components, corresponding to the level, slope and curvature of the forward starting inflation curve.

In the second step, we condition on the estimated principal components, $\hat{P}_t$ and the parameters governing the physical measure, $\hat{\theta}^{\mathbb{P}} = [\hat{K}^{\mathbb{P}\prime}, vec(\hat{\Phi}^{\mathbb{P}})']'$, and estimate the parameter vector governing the Q-measure parameters, $\theta^{\mathbb{Q}} = [k_\infty^{\mathbb{Q}}, \lambda_1, \ldots, \lambda_k, vech(\Sigma)', \Omega]$'. The joint log-likelihood function is,

$$L_T(\theta^{\mathbb{Q}}, \hat{\theta}^{\mathbb{P}} | \hat{P}_t, f_t) = -\sum_{t=1}^{T} \Big(L_t(\hat{\theta}^{\mathbb{P}} | \hat{P}_t) + L_t(\theta^{\mathbb{Q}} | \hat{\theta}^{\mathbb{P}}, \hat{P}_t, f_t) \Big) \tag{28}$$

where $L_t(\hat{\theta}^{\mathbb{P}} | \hat{P}_t)$ is the marginal log-likelihood contribution from the first step estimation and $L_t(\theta^{\mathbb{Q}} | \hat{\theta}^{\mathbb{P}}, \hat{P}_t, f_t)$ is the conditional log-likelihood contribution for the forward starting inflation rates under the no-arbitrage restrictions,

$$L_t(\hat{\theta}^{\mathbb{P}} | \hat{P}_t) = \frac{k}{2} \log(2\pi) + \log \det(\Sigma\Sigma') + \frac{1}{2} \epsilon_t^{\mathbb{P}} (\Sigma\Sigma')^{-1} \epsilon_t^{\mathbb{P}\prime}, \tag{29}$$

$$L_t(\theta^{\mathbb{Q}} | \hat{\theta}^{\mathbb{P}}, \hat{P}_t, f_t) = \frac{p-k}{2} \log(2\pi) + \frac{1}{2\Omega^2} \nu_t \nu_t' + \frac{p-k}{2} \log \det(I_p \Omega^2), \tag{30}$$

where $\epsilon_t^{\mathbb{P}}$ and υ_t are defined in (26) and (27).

The estimator is then,

$$\hat{\theta}^{\mathbb{Q}} = \arg \min_{\theta^{\mathbb{Q}} \in \Theta^{\mathbb{Q}}} L_T(\theta^{\mathbb{Q}}, \hat{\theta}^{\mathbb{P}} | \hat{P}_t, f_t), \tag{31}$$

where $\Theta^{\mathbb{Q}}$ is a suitable parameter space, ensuring that the diagonal of Σ and Ω is strictly positive.[8]

3 Expected Euro Area inflation during the Covid-19 pandemic

This section explores how Euro Area marked-based inflation expectations evolved during the Covid-19 pandemic. We find that while shorter maturity expectations have been highly volatile, the longer end of the forward curve (i.e. the 5Y5Y point), has had remarkably stable inflation expectations. Furthermore, across all maturities, the inflation risk premium has turned positive for the first time since the sovereign debt crisis in 2013.

In general, inflation swaps are traded with annual maturities, with the 1, 2, 5, 7 and 10 year points being the most liquid (ref til ECB). (Seasonal effects)

3.1 Data and model fit

We use monthly data on forward starting swaps, specifically the $1Y1Y$, $1Y2Y$, $1Y3Y$, $1Y4Y$, $1Y6Y$ and $1Y9Y$ knots from XX-2004 to YY-2022.[9] After having estimated the model, we compute the expected inflation and inflation risk premium for the $1Y1Y$, $1Y3Y$ and $5Y5Y$ points, i.e. the very short, short, and medium end of the inflation forward curve.

3.2 Expected inflation and inflation risk premia

3.3 Forecasting euro area inflation

4 Concluding remarks

Throw in reference to Hetland, Pedersen, and Rahbek (2021) somewhere?

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[8]The parameter space can also be restricted to ensure that the factors are strictly stationary under the $\mathbb{Q}$-measure, i.e. $|\lambda_i| < 1$ for all $i = 1, \ldots, k$. Something about Ω being scalar (pileup problem).

[9]tickers XXX in the Bloomberg Terminal

Baumann, Ursel, Matthieu Darracq Paries, Thomas Westermann, Marianna Riggi, Elena Bobeica, Aidan Meyler, Benjamin Böninghausen, Friedrich Fritzer, Riccardo Trezzi, Jana Jonckheere, et al. (2021). "Inflation expectations and their role in Eurosystem forecasting". *ECB Occasional Paper Series.*

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A Solution for Equation (5)

For a standard Gaussian Affine Term Structure model of the form, based on the assumptions for the short rate in (3) and risk-neutral dynamics in (4), the no-arbitrage solution is,

$$y_t^n = -\frac{1}{n}(A_X^n + B_X^{n\prime} X_t). \tag{32}$$

the coefficients A_X^n and B_X^n are,

$$A_X^n = -\rho_{X,0} + A_X^{n-1} + B_X^{n-1\prime} K_X^{\mathbb{Q}} + \frac{1}{2} B_X^{n-1\prime} \Sigma\Sigma' B_X^{n-1} \tag{33}$$

$$B_X^{n\prime} = -\rho_{X,1}' + B_X^{n-1\prime} \Phi^{\mathbb{Q}}, \tag{34}$$

as shown in e.g. Appendix A of Joslin, Singleton, and Zhu (2011), The recursions are initiated in $A_X^0 = 0$ and $B_X^0 = 0$.

In order to arrive derive the parameters in the forward rate model outlined in (5), we compute the forward rate for an inflation swap starting in n years, maturing in m years,

$$\begin{aligned} f_t^{n,m} &= \frac{1}{m-n}(m y_t^m - n y_t^n) \\ &= \frac{-1}{m-n}(A_X^m - A_X^n + (B_X^{m\prime} - B_X^{n\prime})X_t) \\ &= a_X^{n,m} + b_X^{n,m} X_t, \end{aligned} \tag{35}$$

where $a_X^{n,m}$ and $b_X^{n,m}$ are functions of the well-known recursions from the term structure literature,

$$a_X^{n.m} = \frac{-1}{m-n}(A_X^m - A_X^n), \tag{36}$$

$$b_X^{n.m} = \frac{-1}{m-n}(B_X^m - B_X^n). \tag{37}$$

- Snak med Ken
- Thomas Werner
- Wolfgang Lemke
- Andrea Vladu