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ELEONORA GRANZIERA
AND MARKUS SIHVONEN



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Bonds, Currencies and Expectational Errors

Eleonora Granziera and Markus Sihvonен*

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Abstract

We propose a model in which sticky expectations concerning short-term interest rates generate joint predictability patterns in bond and currency markets. Using our calibrated model, we quantify the effect of this channel and find that it largely explains why short rates and yield spreads predict bond and currency returns. The model also creates the downward sloping term structure of carry trade returns documented by [Lustig et al. \(2019\)](#), difficult to replicate in a rational expectations framework. Consistent with the model, we find that variables that predict bond and currency returns also predict survey-based expectational errors concerning interest and FX rates. The model explains why monetary policy induces drift patterns in bond and currency markets and predicts that long-term rates are a better gauge of market's short rate expectations than previously thought.

Keywords: Bond and currency premia, sticky expectations, interest rate forecast errors

JEL classification: E43, F31, D84

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1 Introduction

This paper presents the first unified theory of bond and currency markets based on expectational errors. According to this theory forecast errors concerning short-term interest rates give rise to joint predictability patterns in bond and currency markets. These predictability patterns nest, and can explain, many of the predictability puzzles documented in the previous literature.

Lustig et al. (2019) argue that the literature’s key findings concerning currency and bond return predictability are related: while a high short-term interest rate predicts high returns for a currency, it predicts low returns for long-term bonds denominated in this currency. Similarly, a steep slope of the yield curve predicts low returns for a currency but high returns for corresponding long-term bonds. Such negative correlation between the currency and bond premia represents a puzzle for rational expectations macrofinance models. The model presented in this paper explains this correlation.

Our model is based on the well-documented finding that forecasters update their short rate predictions sluggishly (Coibion and Gorodnichenko, 2012). We do not offer an explanation for this pattern, though we note that it can be caused indirectly due to slow updating concerning factors driving interest rates.¹ However, the key assumption of our approach is that currencies and bonds are priced consistently with such biased expectations concerning short rates.

Then, the return on a bond or currency can be decomposed into a rational risk premium, a short rate misperception effect and a risk premium misperception effect. This decomposition is an identity, it holds in all models in which subjective expectations are given by a probability measure. To the extent that subjective short rate expectations can be measured using survey data, their contribution to return variation can be identified econometrically.

We use our calibrated model to quantify the effect of the interest rate misperception channel. We find that it can account for most of the variation in bond and currency premia driven by changes in short rates and yield spreads. The channel generates coefficients in predictability regressions similar to those found in the data.

Various authors, including Gourinchas and Tornell (2004), Cieslak (2017)

¹There are various possible explanations, for example D’Acunto et al. (2019) argue that household forecast errors are related to *cognitive frictions*. Sticky expectations are also consistent with *inattention* (see e.g. Gabaix (2019)). Moreover, Ilut (2012) notes that similar effects follow from models with ambiguity averse preferences.

and Piazzesi et al. (2015), have explored the effects of expectational errors on bond and currency returns separately. However, what has heretofore been unnoticed is that expectational errors concerning short rates provide a natural candidate for a *joint* theory of bond and currency markets.

The economic intuition behind our key results is simple. The current home and foreign short-term interest rates are known but agents must forecast their future values. The value of a foreign currency is increasing in expected foreign short-term interest rates and the value of foreign long-term bond decreasing in expected (foreign) short-term interest rates. When agents underpredict the path of future foreign interest rates, the value of the foreign currency is lower than under rational expectations but the value of the foreign bond higher than under rational expectations. This implies high actual returns for the currency but low returns for the corresponding bond.

In the data this underprediction is associated with *sticky expectations*. When short-term interest rates increase, for example due to a contractionary monetary policy shock, it takes time for forecasters to revise their future short rate expectations up. This leads forecasters to underpredict the future path of short rates. As the forecasters slowly increase their expectations over future foreign short-term interest rates, the foreign currency appreciates but the value of the foreign bond falls. Before the forecasters have updated their expectations closer to rational values, the returns for a currency will be high but the returns for the bond low.

Note that sticky expectations gives rise to a relation between the *level* of short-term interest rates and the degree of underprediction concerning future interest rates. When short-term interest rates are high, they have on average increased recently. Therefore high short-term interest rates are associated with larger underprediction concerning future interest rates. This implies that a high short-term interest rate predicts high returns for a currency but low returns for the corresponding long-term bond.

We now demonstrate this intuition further with a simplified version of the model. Assume that the currencies are subject to similar perceived risk premia.² Denote the log short-term interest rate differential, also known as forward premium, between the foreign and home country by $x_t \equiv i_t^* - i_t$ and the log FX rate by s_t , where an increase in s_t implies an appreciation of the foreign currency. The logarithmic perceived uncovered interest rate parity condition is:

²Given the symmetric model of the paper, this case emerges when the time-varying parts of market prices of risk are constant: $\varphi_1 = \bar{\varphi}_1 = 0$ in the notation of section 2.1.

$$\mathbb{E}_t^S[s_{t+1}] - s_t + x_t = 0, \quad (1)$$

where S denotes the subjective probability measure of the agents. Roughly, this states that the perceived expected return from borrowing in the home currency and investing in the foreign currency is zero. For simplicity assume a stationary nominal exchange rate and a long-run expected log exchange rate of 0 (e.g. due to symmetric countries).³ From this one can solve:

$$s_t = \sum_{i=0}^{\infty} \mathbb{E}_t^S[x_{t+i}]. \quad (2)$$

Given persistent interest rates, the foreign currency is strong after shocks that raise foreign interest rates above home interest rates: $x_t > 0$. The violations of uncovered interest parity are due to the fact that now under subjective expectations the interest rate differential tends to remain lower than under rational expectations $\mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}] > 0$. This is because the forecasters are slow at increasing their interest rate forecasts after the positive interest rate shocks. On the other hand, this implies that $\mathbb{E}_t[s_{t+1}] - \mathbb{E}_t^S[s_{t+1}] > 0$. That is, the foreign currency will be stronger on average the next period than predicted by forecasters.

The relative log price of a zero coupon bond of maturity n is:

$$q_t^*(n) - q_t(n) = - \sum_{i=0}^{n-1} \mathbb{E}_t^S[x_{t+i}]. \quad (3)$$

When $x_t > 0$ the price of the foreign bond, $q_t^*(n)$, that is known by all agents, is relatively low and the yield high. However, because this is due to a recent interest rate shock the forecasters believe $\mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}] > 0$ and therefore $\mathbb{E}_t[q_{t+1}^*(n-1) - q_{t+1}(n-1)] - \mathbb{E}_t^S[q_{t+1}^*(n-1) - q_{t+1}(n-1)] < 0$. The misestimation of the interest rate process therefore creates variation in bond risk premia, measured under rational expectations, as high interest rate currencies have long-term bonds that are overpriced compared to prices under rational expectations.

Why does this type of model explain the joint behaviour of bonds and currencies? When $x_t > 0$ foreign currency short-term securities have high returns. At the same time the long-term bond of the same currency is relatively overpriced and yields low actual returns. Higher maturity increases the sensitivity of a bond to predictions about future interest rates, so this

³We discuss the role of the permanent component of the FX rate later.

effect is stronger the longer the maturity of the bond. One can see that these effects partly offset each other so that a strategy that buys a long-term bond of the foreign currency and sells a similar bond of the home currency yields small domestic currency returns. This explains why the term structure of expected carry trade returns is downward sloping.

We provide strong empirical evidence that supports the importance of short rate forecast errors for bond and currency returns. In particular, we show that the same variables that predict bond and currency returns also predict survey-based expectational errors concerning FX rates and long-term interest rates. For example, when (domestic or foreign) short-term interest rates are high, forecasters underestimate the future level of long-term interest rates and overestimate the future value of long-term bonds. Similarly, when foreign short-term interest rates are high relative to domestic interest rates, forecasters underestimate the future value of the foreign currency relative to the home currency. Moreover, we show that foreign currency returns tend to be particularly high, and bond returns low, when foreign short rates have recently increased.

Finally, we discuss the policy implications of the results. Under rational expectations, central bank policies that affect short-term rates transmit instantaneously to bond yields and FX rates. However, according to our model this transmission occurs more sluggishly, which we argue to be consistent with the data. Moreover, the model emphasizes the importance of beliefs concerning future short rates over risk effects. The implication is that for example long-term yields can be used to approximate the market's beliefs over future short rates. However, these beliefs are typically not rational, especially after recent changes in short rates.

Related Literature This paper contributes to the vast literature on markets for currencies and government bonds. Special attention is given to explaining predictability patterns in bond and currency returns. The seminal paper for currencies is [Fama \(1984\)](#) that finds that currencies with high short-term interest rates appreciate rather than depreciate as predicted by uncovered interest rate parity. On the other hand, [Fama and Bliss \(1987\)](#) and [Cochrane and Piazzesi \(2005\)](#) find that high bond yields are associated with high bond returns, a violation of the expectations hypothesis. [Lustig et al. \(2019\)](#) argue that these two findings are related as high relative bond yields predict low returns for the corresponding currency.

A large literature in the tradition of rational expectations consumption based asset pricing models has attempted to explain the predictability pat-

terns in bond and currency markets. Examples include applications of the habit model for the bond market (see e.g. Wachter (2006)) and those for the currency market (see e.g. Verdelhan (2010)). More recently some authors have proposed risk-based models that could possibly explain both the properties of bonds and currencies. Bansal and Shaliastovich (2012) apply the long-run risk model for both bonds and currencies, a related exercise with the habit model is conducted by Ermolov (2014).

A second literature in the tradition of no-arbitrage term structure models (see e.g. Duffie and Kan (1996)) has taken a more reduced form approach to modeling bonds. Similar models have been applied to currencies (see e.g. Backus et al. (2001) and Lustig et al. (2011)). We adopt this approach in this paper largely because we do not want to take a stance on the deeper sources of bond and currency risk that are not the focus of this paper. Note that Lustig et al. (2019) argue that neither the standard structural models nor these no-arbitrage models are able to replicate the term structure of carry trade returns.

A key alternative to the risk-based approach is to relax the assumption of rational expectations. This choice can be motivated for example by the systematic expectational errors documented in surveys (see e.g. Bacchetta et al. (2009), Coibion and Gorodnichenko (2012) and Greenwood and Shleifer (2014)).⁴ The idea that currency returns are driven by mispricings has been explored by Froot and Frankel (1989), McCallum (1994), Gourinchas and Tornell (2004) and Burnside et al. (2011). Similarly, the effects of belief distortions on interest rates have been studied by, for example, Froot (1989), Xiong and Yan (2010), Hong and Sraer (2013), Piazzesi et al. (2015) and Cieslak (2017). However, to our best knowledge this is the first paper that offers a joint explanation for bond and currency markets based on expectational errors.

The above mentioned risk-based models are based on the assumption of frictionless markets. Jylhä and Suominen (2010) and Gabaix and Maggiori (2015) argue that financial frictions can explain currency carry trade returns. In concurrent work Greenwood et al. (2019) posit that asset market frictions can explain both the properties of bonds and currencies, including the downward sloping term structure of carry trade returns. These effects can potentially complement those presented in this paper.

⁴Stavrakeva and Tang (2018) also describe stylized facts about survey expectations concerning exchange rates.

2 A Term Structure Model with Expectational Errors

2.1 Model Structure

We first introduce the basic model structure which is similar to that in the currency model of [Gourinchas and Tornell \(2004\)](#). There are two symmetric countries, home and foreign, where the latter variables are denoted by stars. Moreover, let there be two probability measures P and S . Here P corresponds to objective probabilities as viewed by a rational econometrician. On the other hand, S represents the (homogeneous)⁵ subjective beliefs of the agents. For simplicity we omit the P -symbol from expectations taken under rational beliefs.

Markets are complete. Under the subjective measure S , the home and foreign nominal stochastic discount factors (SDFs), $M_{t,t+1}$ and $M_{t,t+1}^*$, follow symmetric (conditionally) log-normal processes

$$\log(M_{t,t+1}) \equiv m_{t,t+1} = -\log R - \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^2}{2} - \bar{z}_t - \frac{\sigma_\epsilon^2 \varphi_t^2}{2} - z_t - \bar{\varphi}_t \bar{\epsilon}_{t+1} - \varphi_t \epsilon_{t+1} \quad (4)$$

$$\log(M_{t,t+1}^*) \equiv m_{t,t+1}^* = -\log R - \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^{*2}}{2} - \bar{z}_t - \frac{\sigma_\epsilon^2 \varphi_t^{*2}}{2} - z_t^* - \bar{\varphi}_t^* \bar{\epsilon}_{t+1} - \varphi_t^* \epsilon_{t+1}^*. \quad (5)$$

The shocks $\epsilon_t = (\epsilon_t, \epsilon_t^*, \bar{\epsilon}_t)$ are independent and follow a (joint) normal distribution with mean zero and variances⁶ σ_ϵ^2 , σ_ϵ^{*2} and $\bar{\sigma}_\epsilon^2$. z_t and z_t^* are country specific states and \bar{z}_t is a state shared by both countries. These states can represent either deep structural state variables or reduced form factors often used in term structure models.

Under the objective measure, the states $\mathbf{z}_t = [z_t, z_t^*, \bar{z}_t]'$ follow the process

$$\mathbf{z}_t = \Lambda \mathbf{z}_{t-1} + \epsilon_t, \quad (6)$$

where

$$\Lambda = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \bar{\lambda} \end{bmatrix}.$$

⁵Alternatively one can view S as the agents' average belief.

⁶Note that we assume countries are symmetric and the shocks ϵ_t and ϵ_t^* have the same variance σ_ϵ^2 .

Here $0 < \lambda < 1$ and $0 < \bar{\lambda} < 1$. On the other hand, the investors believe that these follow (i.e. their S -dynamics are given by)⁷

$$\mathbf{z}_t = \mathbf{1}_t + \mathbf{v}_t, \quad (7)$$

$$\mathbf{1}_t = \Lambda \mathbf{1}_{t-1} + \boldsymbol{\epsilon}_t. \quad (8)$$

Here $\mathbf{v}_t = [v_t, v_t^*, \bar{v}_t]'$ and $v_t, v_t^* \sim N(0, \sigma_v^2)$ and $\bar{v}_t \sim N(0, \bar{\sigma}_v^2)$, where each shock is independent.⁸ Note that the agents correctly observe all the state variables but misperceive their law of motion. In particular they erroneously believe that the effects of the shocks are transitory. This implies that the investors' expectations react to new information sluggishly.

Finally, the market prices of risk are given by

$$\Phi_t = [\varphi_t \ \bar{\varphi}_t \ \varphi_t^* \ \bar{\varphi}_t^*]' = \Phi_0 + \Phi_1 [z_t \ z_t^*]'$$

$$\Phi_0 = [\varphi_0 \ \bar{\varphi}_0 \ \varphi_0 \ \bar{\varphi}_0]'$$

$$\Phi_1 = \begin{bmatrix} \varphi_1 & 0 \\ \bar{\varphi}_1 & 0 \\ 0 & \varphi_1 \\ 0 & \bar{\varphi}_1 \end{bmatrix}.$$

Note that for simplicity we assume that only the local factor prices country specific and common shocks. However, in most of the empirical part we assume $\varphi_1 = \bar{\varphi}_1 = 0$ so that all of the return predictability due to local shocks will be due to expectational errors.

The model yields a simple formula for the interest rate differential given by the following lemma

Lemma 1. *The log-interest rate differential is given by $x_t \equiv i_t^* - i_t$. The true law of motion for x_t is*

$$x_t = \lambda x_{t-1} + \epsilon_t^* - \epsilon_t \equiv \lambda x_{t-1} + \tilde{\epsilon}_t. \quad (9)$$

The perceived law of motion for x_t is

$$x_t = \tilde{l}_t + \tilde{v}_t, \quad (10)$$

⁷Here the shocks are redefined that is they are different from those in the underlying AR(1) process.

⁸All parameters in the model are assumed to be known by the agents

$$\tilde{l}_t = \lambda \tilde{l}_{t-1} + \tilde{\epsilon}_t, \quad (11)$$

where $\tilde{v}_t = v_t^* - v_t$.

Proof. Note that

$$x_t = \log(\mathbb{E}_t^S [\exp(m_{t,t+1})]) - \log(\mathbb{E}_t^S [\exp(m_{t,t+1}^*)]).$$

All the states related to $m_{t,t+1}$ and $m_{t,t+1}^*$ are known. Using the mean of an exponential of a normal random variable

$$x_t = z_t^* - z_t.$$

The perceived and actual law of motion follow directly from the corresponding processes for z_t^* and z_t . □

One implication of the lemma is that the λ coefficient is also the persistence parameter of the relative short rate process x_t . This implies that the states in the model can either represent deep economic factors or a reduced form characterization of a short rate process. Note that the agents always observe the correct short-rates.

The following gives a solution to the learning problem based on the standard recursion formulas for the Kalman filter (see e.g. [Hamilton \(1994\)](#)).

Proposition 1 (Learning Problem). *Assume initial beliefs about l_1, l_1^*, \bar{l}_1 are normally distributed with l_1, l_1^* coming from the same distribution. Now the beliefs (are Gaussian and) evolve as*

$$\mathbb{E}_t^S [z_{t+1}] = \begin{bmatrix} \lambda(1-k_t) & 0 & 0 \\ 0 & \lambda(1-k_t) & 0 \\ 0 & 0 & \bar{\lambda}(1-\bar{k}_t) \end{bmatrix} \mathbb{E}_{t-1}^S [z_t] + \begin{bmatrix} \lambda k_t & 0 & 0 \\ 0 & \lambda k_t & 0 \\ 0 & 0 & \bar{\lambda} \bar{k}_t \end{bmatrix} z_t, \quad (12)$$

$$\mathbb{E}_t^S [x_{t+1}] = \lambda(1-\tilde{k}_t) \mathbb{E}_{t-1}^S [x_t] + \lambda \tilde{k}_t x_t. \quad (13)$$

The formulas for $k_t, \bar{k}_t, \tilde{k}_t$ and the volatilities of the persistent components are given in the appendix. As $t \rightarrow \infty$, these estimators converge to steady-state values $\sigma^2, \bar{\sigma}^2, \tilde{\sigma}^2, k$ and \bar{k} given in the appendix.

The learning process for the foreign country is defined analogously. For the main results of this paper for simplicity we assume the estimators have converged to their steady-state values. Note that k_t and \tilde{k}_t are generally different but converge to the same value.

To understand the key differences between subjective and objective expectations for the states, take the example of z_t . From the above proposition one has

$$\mathbb{E}_t^S[z_{t+1}] = (1 - k_t)\lambda\mathbb{E}_{t-1}^S[z_t] + k_t\lambda z_t$$

$$\mathbb{E}_t[z_{t+1}] = \lambda z_t.$$

If beliefs are rational $k_t = 1$ and the two expectations coincide. However, typically $0 < k_t < 1$ so that the subjective expectation is a weighted average of the last period expectation and the current value for the state. Effectively the biased measure underreacts to new interest rate shocks. For the rest of this paper we follow the literature and assume that these estimators have converged to their steady-state values, especially $k_t = k$.

2.2 The Yield Curve

The following proposition shows that the yield curve takes a standard affine form. However, the biased beliefs enter as state variables:

Proposition 2 (The yield curve). *Denote the state variable $\mathbf{Y}_t = [z_t, \bar{z}_t, \mathbb{E}_t^S[z_{t+1}], \mathbb{E}_t^S[\bar{z}_{t+1}]]'$. The home logarithmic prices of zero coupon bonds are affine functions of \mathbf{Y}_t and given by*

$$q_t(n) = A(n) + \mathbf{B}(n)' \mathbf{Y}_t, \tag{14}$$

where $A(n)$ and $\mathbf{B}(n)$ are given by $A(1) = -\log R$ $\mathbf{B}(1) = [-1 \ -1 \ 0 \ 0]'$ and

$$B_1(n) = -1 - \varphi_1(B_1(n-1) + k\lambda B_3(n-1))\sigma_\epsilon^2 - \bar{\varphi}_1(B_2(n-1) + \bar{k}\bar{\lambda}B_4(n-1))\bar{\sigma}_\epsilon^2,$$

$$B_2(n) = -1, \quad B_3(n) = \lambda B_3(n-1) + B_1(n-1),$$

$$B_4(n) = \bar{\lambda}B_4(n-1) + B_2(n-1).$$

Finally $A(n)$ is provided in the appendix.

By further assuming $\varphi_1 = \bar{\varphi}_1 = 0$, then $\mathbf{Y}_t = [\hat{z}_t, \mathbb{E}_t^S[z_{t+1}], \mathbb{E}_t^S[\bar{z}_{t+1}]]$, where $\hat{z}_t = z_t + \bar{z}_t$ and $q_t(n) = A(n) + \mathbf{B}(n)' \mathbf{Y}_t$. Here $A(n)$ and $\mathbf{B}(n)$ are given by $A(1) = -\log R$ $\mathbf{B}(1) = [-1 \ 0 \ 0]'$ and

$$B_1(n) = -1 \quad B_2(n) = \lambda B_2(n-1) - 1 \quad B_3(n) = \bar{\lambda} B_3(n-1) - 1.$$

Finally $A(n)$ is given in the appendix.

The interest rates are generally high when the factors are high and when subjective expectations about their future values are high. Note that in a rational model the only factors determining the home yield curve would be z_t and \bar{z}_t .

2.3 Term Structure of Expected Carry Trade Returns

We now characterize the expected returns of the two currencies. The general expression for the currency risk premium under the objective measure can be decomposed as follows:⁹

$$\underbrace{\Theta_t}_{\text{Currency premium}} = \underbrace{-\Gamma_t}_{\text{Risk premium differential}} + \underbrace{\mathbb{E}_t \left[\mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} x_{t+1+j} - \mathbb{E}_t^S \sum_{j=0}^{\infty} x_{t+1+j} \right]}_{\text{Interest rate misperception effect}} + \underbrace{\mathbb{E}_t \left[\mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} \Gamma_{t+1+j} - \mathbb{E}_t^S \sum_{j=0}^{\infty} \Gamma_{t+1+j} \right]}_{\text{Risk premium misperception effect}} + \underbrace{\mathbb{E}_t \left[\lim_{j \rightarrow \infty} \mathbb{E}_{t+1}^S [s_{t+j}] - \lim_{j \rightarrow \infty} \mathbb{E}_t^S [s_{t+j}] \right]}_{\text{Permanent component misperception effect}}.$$

We later considering predicting currency and bond returns using linear regressions. The slope coefficients in these predictability regressions decompose as follows:

⁹See the following proposition for a definition of Θ_t .

$$\begin{array}{ccccccc}
\underbrace{\beta_1} & = & \underbrace{\beta_1^{RP}} & + & \underbrace{\beta_1^{IRM}} & + & \\
\text{Predictability coefficient} & & \text{Risk premium differential effect} & & \text{Interest rate misperception effect} & & \\
\underbrace{\beta_1^{RPM}} & & + & & \underbrace{\beta_1^{PCM}} & & \\
\text{Risk premium misperception effect} & & \text{Permanent component misperception effect} & & & &
\end{array}$$

For a bond of maturity n

$$\begin{array}{ccccccc}
\underbrace{\Theta_t^B(n)} & = & \underbrace{-\Gamma_t^{B,n}} & + & \underbrace{\mathbb{E}_t \left[\mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} x_{t+1+j} - \mathbb{E}_t^S \sum_{j=0}^{\infty} x_{t+1+j} \right]} & + & \\
\text{Bond premium differential} & & \text{Risk premium differential} & & \text{Interest rate misperception effect} & & \\
+ \underbrace{\mathbb{E}_t \left[\mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} \Gamma_{t+1+j}^{B,n-j-1} - \mathbb{E}_t^S \sum_{j=0}^{\infty} \Gamma_{t+1+j}^{B,n-j-1} \right]} & & & & & & \\
& & \text{Risk premium misperception effect} & & & &
\end{array}$$

The corresponding slope coefficients decompose similarly. Moreover, a analogous decomposition can be obtained for $\Theta_t(n)$, the premium for carry-trades with long-maturity bonds. These decompositions are valid in any model in which the agents' beliefs form a valid probability measure.

Rational models only include the risk premium channel. However, here misperceptions about economic states affect the actual currency premium in three ways. First they create time-variation in (actual) expected returns due to misperceptions about future interest rates. Second, they create additional time-variation in (actual) expected returns due to expectational errors about future risk adjustments. Third, they can create time-variation in expected returns due to misperceptions concerning the long-run FX rate. The first two channels also affect the bond risk premium. The long-run component misperception effect does not affect standard bonds due to finite maturity.

For the rest of this paper for simplicity we assume $\varphi_1 = \bar{\varphi}_1 = 0$, which means that the perceived market prices of risk are constant. Given our assumptions about the learning process, the agents also hold correct long-run beliefs about the state variables. This implies:

$$\mathbb{E}_t[\lim_{j \rightarrow \infty} \mathbb{E}_{t+1}^S[s_{t+j}] - \lim_{j \rightarrow \infty} \mathbb{E}_t^S[s_{t+j}]] = 0. \quad (15)$$

i.e. agents hold correct beliefs concerning the long-run component of the FX rate.¹⁰ Therefore our assumptions mean that all time-variation in expected excess returns is due to misperceptions about future interest rates.

Note that the contribution of short rate forecast errors to bond and currency return predictability depends only on the short rate process under subjective and rational beliefs. Our assumptions that shut down the other channels do not affect this process. Therefore the predictability results derived under these assumptions still represent the contribution of short rate forecast errors to return predictability in a more general model with no such restrictive assumptions.

The following proposition derives results for both the relative returns on short-term bills in the two currencies as well as those for longer maturity bonds:

Proposition 3 (Term Structure of Expected Carry Trade Returns). *Let the home relative (objective) expected return of foreign currency short-term bills be $\Theta_t \equiv x_t + \mathbb{E}_t[s_{t+1}] - s_t$. Further let the home relative (objective) expected return of long-term foreign currency bonds of maturity $n \geq 2$ be $\Theta_t(n) \equiv \mathbb{E}_t[q_{t+1}^*(n-1) - q_{t+1}(n-1)] - [q_t^*(n) - q_t(n)] + \mathbb{E}_t[s_{t+1}] - s_t$. Now*

$$\Theta_t = \left[1 + \frac{\lambda k}{1 - \lambda} \right] \left[\mathbb{E}_t x_{t+1} - \mathbb{E}_t^S x_{t+1} \right] + \mathbb{E}_t \left[\mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} \Gamma_{t+1+j} - \mathbb{E}_t^S \sum_{j=0}^{\infty} \Gamma_{t+j} \right], \quad (16)$$

where $0 < k < 1$ and $0 < \lambda < 1$ and the perceived risk premium is given by

$$\Gamma_t = -\frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^2}{2} - \frac{\sigma_\epsilon^2 \varphi_t^2}{2} + \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^{*2}}{2} + \frac{\sigma_\epsilon^2 \varphi_t^{*2}}{2}. \quad (17)$$

The general expression for $\Theta_t(n)$ is given in the appendix. Assuming $\varphi_1 = \bar{\varphi}_1 = 0$,

¹⁰If we relaxed this assumption, the long-bond return parity condition would generally not hold exactly, but the term structure of carry trade returns would still be downward sloping. The determination of this permanent component is beyond the scope of this paper. For example in a model with a stationary real exchange rate, the permanent component of the exchange rate would be the permanent component of the log price differential in the two countries. If this is constant, in symmetric model the permanent component is naturally 0.

$$\Theta_t = \left[1 + \frac{\lambda k}{1 - \lambda} \right] \left[\mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}] \right] \quad (18)$$

$$\Theta_t(n) = \frac{k\lambda^n}{1 - \lambda} \left[\mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}] \right]. \quad (19)$$

Now as $n \rightarrow \infty$, $\Theta_t(n) \rightarrow 0$. The term structure of expected carry trade returns is downward sloping. The long bond parity condition holds in the limit as the maturity of the bonds increases.

Proof: see appendix.

Looking at the simplified expressions given in the proposition, one can now see that the expected relative return on foreign short-term securities is positive when $\mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}] > 0$. This tends to happen when the interest rate differential x_t is high. Effectively the high interest rate currency is undervalued because the investors do not expect the high interest rate environment to persist.

The relative returns on foreign long maturity bonds are also positive when $\mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}] > 0$. However, they are decreasing in maturity n . Moreover, the long-bond parity (LBP) condition holds exactly in the limit as $n \rightarrow \infty$. The long-term bonds of the high interest currency are overvalued because high future short rates imply low returns for long-term bonds.

Note that when $k = 1$, that is assuming rational expectations, there is no time-variation in expected currency returns. As explained before, in the simplified model all of the variation in risk premia are driven by expectational errors.

Finally, due to risk adjustments the general model does not typically satisfy the LBP condition $\Theta_t(\infty) = 0$. This is consistent with [Lustig et al. \(2019\)](#) who note that this condition does not arise naturally in many risk-based models. However, the simplified model described by (18) and (19) naturally satisfies this condition.

2.4 Model Assumptions: Further Discussion

This section includes some additional discussion concerning the model assumptions. The assumption that agents perceive changes in state variables as less persistent than actually follows [Gourinchas and Tornell \(2004\)](#). This can be seen as an admittedly simplistic way to microfound the process for

subjective expectations given in equation (16). As argued by [Coibion and Gorodnichenko \(2012\)](#) this process gives a good fit to survey data concerning short rates. It could also be justified by the assumption that each period only a fraction k of agents can update their expectation.

The assumption of symmetric pricing kernels implies that the model cannot explain persistent country level differences in returns. We leave extending the results to asymmetric pricing kernels to future work. We use the term carry trade in a loose sense to refer to a strategy that borrows in a low interest rate currency and invests the proceeds in a high interest rate currency. However, the trading opportunity implied by the model is based on exploiting time-series rather than persistent cross-country violations of the uncovered interest parity. For a careful analysis of the different types of violations of uncovered interest parity and the related trading strategies, see [Hassan and Mano \(2017\)](#).¹¹

All of the key qualitative results of this paper could be derived by assuming the investors hold rational beliefs concerning the *common* shock, that is there are misperceptions only about the *local* shocks. We could also allow for additional factors as well as distinguishing between real and nominal pricing kernels by making an assumption on the inflation process (see e.g. [Lustig et al. \(2011\)](#) and [Lustig et al. \(2014\)](#)).¹² Empirically shocks to expected inflation contribute much less to the variation in nominal yields than would be predicted by many structural models ([Duffee, 2018](#)).¹³

We assume the same shocks for the SDF and state variables. Moreover, as in [Gourinchas and Tornell \(2004\)](#), for simplicity we assume the investors do not use the information in realized SDFs to update their beliefs about the economic state. One can view this correlation between the two shocks as a reduced form way of modeling an upward sloping yield curve. In the main model of this paper, it only affects the results through the constant terms of bond prices and does not affect the results concerning return predictability and yield volatilities.

¹¹The trading opportunity implied by our model could be exploited e.g. using the forward premium trade described by [Hassan and Mano \(2017\)](#). In our sample period this trade yields an annual Sharpe ratio of 0.4.

¹²Alternatively one could formulate the theoretical predictions for real pricing kernels and use data on real interest rates and exchange rates for the empirical part. For the potential effect of inflation risk on carry trade returns see [Jylhä and Suominen \(2010\)](#).

¹³We would therefore expect our state variables to have higher correlations with real variables rather than inflation rates. However, our approach allows for different interpretations concerning these variables.

3 Empirical Predictions

The model delivers important empirical predictions concerning bond and currency returns as well as expectational errors in survey data. We now describe these predictions studied later again in the empirical section.

3.1 Return Predictability

We next consider the model predictions for bond and currency return predictability. Following [Lustig et al. \(2019\)](#), special attention is given to predicting bond and currency returns using short term rates and slope of the yield curve.

To illustrate the logic behind the results, we first show the evolution of the yield curve and exchange rate after a shock that increases the foreign interest rate. Figure 1 plots the impulse responses to an interest rate shock in the simple model in which interest rate forecast errors drive all return variation.¹⁴ When foreign interest rates increase above home interest rates, forecasters update their relative short rate forecasts upward but not as much as a rational forecaster would do. Because long term interest rates are averages over expected short rates, they increase but less than short rates, so the relative yield curve becomes downward sloping. The price of a long-term bond falls but by less than according to rational expectations. The foreign currency appreciates but by less than predicted by rational expectations. However, in the long-run expectations converge to rational values. During the interim period, a high interest rate predicts positive returns for the foreign currency but low relative returns for long-term foreign bonds.

Next, we derive predictions for the term structure of carry trade returns. A key result is given by the following proposition:

Proposition 4 (Term Structure of Carry Trade Returns and Interest Rate Differential). *Let $r_{t+1}^{FX} \equiv i_{t+1}^* - i_{t+1} + s_{t+1} - s_t$ and $r_{t+1}^{FX}(n) \equiv q_{t+1}^*(n-1) - q_{t+1}(n-1) - (q_t^*(n) - q_t(n)) + s_{t+1} - s_t$. Consider the following regressions*

$$r_{t+1}^{FX} = \beta_0^{FX} + \beta_1^{FX} x_t + \varepsilon_{t+1} \quad (20)$$

$$r_{t+1}^{FX}(n) = \beta_0^{FX}(n) + \beta_1^{FX}(n) x_t + \varepsilon_{t+1}^n \quad (21)$$

¹⁴The figure assumes the long-run log-exchange rate is 0 so here $s_t = \sum_{i=0}^{\infty} \mathbb{E}_t^S [x_{t+i}]$ and $q_t^*(n) - q_t(n) = -\sum_{i=0}^{n-1} \mathbb{E}_t^S [x_{t+i}]$. The impulse responses are computed using the benchmark calibration derived later.

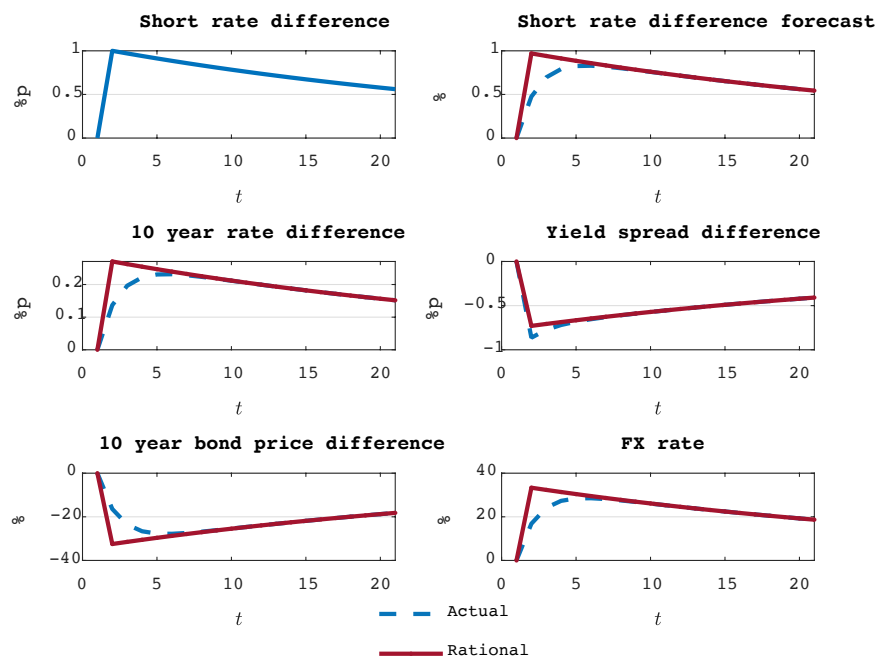


Figure 1 Impulse responses to a shock to the foreign interest rate, time is measured in months

the probability limits of the OLS estimates of β_1^{FX} , $\beta_1^{FX}(n)$ are positive and

$$\beta_1^{FX,plim} = \left[1 + \frac{\lambda k}{1 - \lambda} \right] \frac{(1 - k)(\lambda - \lambda^3)}{1 - (1 - k)\lambda^2}. \quad (22)$$

$$\beta_1(n)^{FX,plim} = \frac{k\lambda^{n-1}}{1 - \lambda} \frac{(1 - k)(\lambda - \lambda^3)}{1 - (1 - k)\lambda^2}. \quad (23)$$

$\beta_1(n)^{FX,plim}$ decays at rate λ^{n-1} and approaches zero as $n \rightarrow \infty$.

Proof: see appendix.

A positive interest rate differential predicts positive carry trade returns for any maturity bonds. However, the effect is declining in the bond maturity n and there is no predictability in the limit $n \rightarrow \infty$. Figure 2 shows the decay pattern for relative carry trade returns for different values of the persistence parameter λ ¹⁵. As explained before, the downward sloping term structure emerges because variation in expected bond returns offsets variation in expected currency returns.

Figure 3 shows the slope coefficient β_1^{FX} as a function of both k and λ . The coefficient is positive. For typical parameter values β_1^{FX} is decreasing in k and increasing in λ . The benchmark calibration used later predicts $\beta_1^{FX} \approx 0.99$

Figure 4 shows the slope coefficient of a regression of relative returns of 10 year bonds on short rate differential x_t , which is also given by $\beta_1^{FX}(n) - \beta_1^{FX}$. The coefficient is negative. For typical parameter values the slope coefficient is increasing in k and λ . The benchmark calibration discussed later predicts $\beta_1^{FX}(n) - \beta_1^{FX} \approx -0.7$. This opposite predictability in bond returns largely offsets the predictability in currency returns so that there is little predictability in the returns of carry trades implemented with 10 year bonds.

It can be shown that the model predicts the opposite patterns when relative yield spreads are used as predictors. A high slope of the yield curve predicts low currency returns but high bond returns. This occurs because the slope of the yield curve tends to be high when interest rates are low.

Finally, the model implies that foreign currency returns tend to be particularly high and bond returns low when foreign short rates have recently

¹⁵This shows the relative profitability / predictability coefficient. That is coefficient for the short maturity carry trade is normalized to 1.

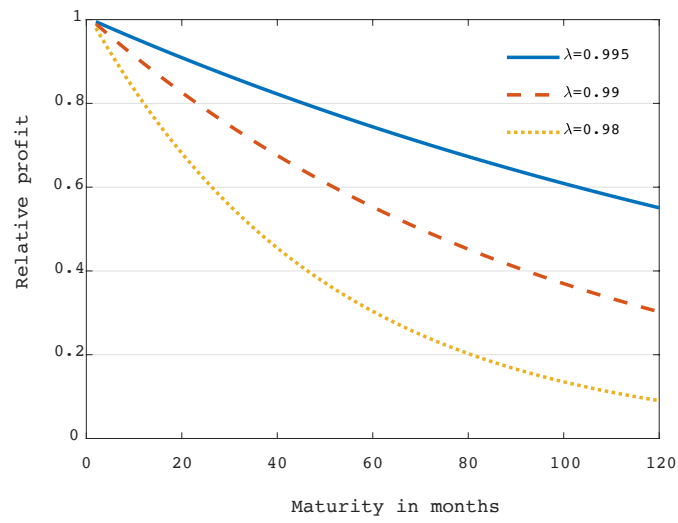


Figure 2 The term-structure of carry trade

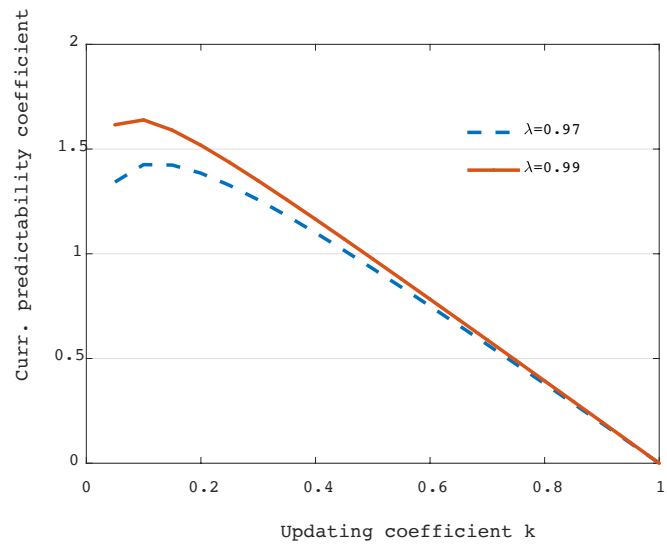


Figure 3 The currency return predictability coefficient as a function of k and λ

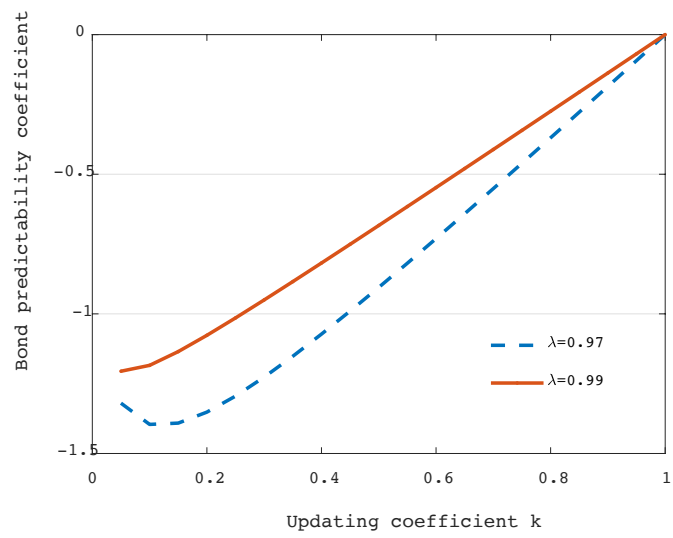


Figure 4 The bond return predictability coefficient as a function of k and λ

increased relative to past values. This is formalized in the following proposition:

Proposition 5. *Define the average past short rate difference as: $\bar{x}_t \equiv x_t + (1 - k)\lambda x_{t-1} + (1 - k)^2 \lambda^2 x_{t-2} + \dots$. Consider the regressions*

$$r_{t+1}^{FX} = \beta_0^{FX} + \beta_1^{FX} x_t + \beta_2^{FX} \bar{x}_{t-1} + \varepsilon_{t+1} \quad (24)$$

and

$$r_{t+1}^B(n) \equiv r_{t+1}^{FX}(n) - r_{t+1}^{FX} = \beta_0^B(n) + \beta_1^B(n) x_t + \beta_2^B(n) \bar{x}_{t-1} + \varepsilon_{t+1}^n \quad (25)$$

The probability limit of the OLS estimate of β_1^{FX} is positive, of β_2^{FX} is negative, of $\beta_1^B(n)$ is negative and of $\beta_2^B(n)$ is positive.

The forecast wedge $\mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}]$ is particularly wide when x_t is high relative to the past short-rate differences x_{t-1} , x_{t-2} , x_{t-3} and so on. That is, expectational errors concerning short rates are particularly large after recent short rate shocks. On the other hand, in our simplified model with $\varphi_1 = \bar{\varphi} = 0$ the rationally expected currency return is strictly increasing in this forecast wedge and the expected bond return is decreasing. This implies that high short-term interest rates relative to past short rates should predict high returns for a currency but low returns for the corresponding long-term bond. This explains why the slope coefficient on the past average short rate difference \bar{x}_{t-1} has the opposite sign than the slope coefficient on the short rate difference x_t .

3.2 Predictions for Expectational Errors in Surveys

The model bears implications for expectational errors in survey data. The following proposition shows that the model implies that forecasters underestimate the relative future strength of high interest rate currencies:

Proposition 6 (Matching Survey Data on Currencies). *Consider the following regression*

$$s_{t+j} - \mathbb{E}_t^S[s_{t+j}] = \beta_0 + \beta_1 x_t + e_{t+j}. \quad (26)$$

The probability limit of the OLS estimate of β_1 is positive.

Proof: see appendix.

This prediction emerges also in [Gourinchas and Tornell \(2004\)](#), though here we show that it carries over to long horizons as well. The intuition is

illustrated in figure 1. When foreign interest rates increase, the forecasters are sluggish at updating their predictions and the subjective interest rate forecast falls below the rational forecast. Similarly, the FX rate falls below its rational value. The gradual convergence of the FX rate and interest rate forecasts to rational values leads to an unexpected appreciation pattern in the value of the foreign currency.

Similarly, due to the negative correlation between the yield spread and short rate level, the model implies that forecasters overestimate the future strength of currencies with steep yield curves. As can be seen from figure 1, the increase in interest rates leads to a decline in yield curve slope.

Then consider the regression:

$$q_{t+j}(n)^* - \mathbb{E}_t^S[q_{t+j}(n)^*] - (q_{t+j}(n) - \mathbb{E}_t^S[q_{t+j}(n)]) = \beta_0 + \beta_1 x_t + e_{t+j}. \quad (27)$$

Using similar arguments it can be shown that the model implies that $\beta_1 < 0$, that is when short-term home interest rates are relatively high, forecasters overpredict the relative future value of foreign bonds. The opposite prediction, $\beta_1 > 0$, is obtained when long-term interest rates are used on the LHS of the equation or when the slope of the yield curve is used on the RHS of the equation.

4 Empirical Evidence

We now turn to empirically test the model predictions and quantifying the effect of interest rate misperceptions on bond and currency returns.

4.1 Data

We next briefly describe the data used. We focus on the G10 currencies of Australia, Canada, Germany, Japan, Norway, New Zealand, Sweden, Switzerland, U.K. and U.S. We utilize FRED to obtain data on end of month FX rates and interest rates on 3 month and 10 year government securities.

We calibrate the agents' expectations using survey data. Consensus economics provides a monthly report of forecasts for 3 month and 10 year interest rates as well as FX rates. Following [Coibion and Gorodnichenko \(2012\)](#), we average over the forecasts provided by different financial institutions.¹⁶ Forecasts are available for all countries except Australia and

¹⁶This approach is taken in most other papers. It is still unclear whether the results

	AUS	CAN	GER	JAP	NOR	NZ	SWE	CH	UK	US
	Bonds Index									
Start	85M1	85M1	85M1	85M1	NA	85M1	85M1	85M1	85M1	85M1
End	19M2	19M2	19M2	19M2	NA	19M2	19M2	19M2	19M2	19M2
Obs	410	410	410	410	NA	410	410	410	410	410
	Consensus									
Start	NA	91M5	91M5	91M5	98M6	NA	94M12	98M6	91M5	89M10
End	NA	19M4	19M4	19M4	19M4	NA	19M4	19M4	19M4	19M4
Obs	NA	336	336	336	251	NA	293	252	336	355

Table 1 Start and end dates for the bond index and survey data.

New Zealand. Note that the use of professional forecasts might provide a conservative estimate of the biases reflected in asset prices.

We calculate bond returns using Citigroup government bond local currency 10 year indices available for all countries except Norway.¹⁷

The start and end dates for the bond indices and survey data are given in table 1, where we also report the number of observations.

4.2 Calibration

To quantify the importance of interest rate misperceptions, we need to calibrate the model. As discussed before we set the time-varying parts of market prices of risk to zero: $\varphi_1 = \bar{\varphi}_1 = 0$. This implies that all of the time-variation in excess returns will be due to expectational errors. It turns out that such a simple model can go a surprisingly long way in explaining bond and currency returns. Because risk effects are also second order, similar predictions could be derived by considering a first-order approximation of the pricing conditions. However, this would imply that also the constant parts of risk premia are zero. As explained before using the decompositions in section 2.3, we can view the key exercise of the paper as quantifying the contribution of interest rate misperceptions to return predictability in a more general model which includes also risk effects.

would be different if considering individual forecasts. [Bordalo et al. \(2019\)](#) argue that the underreaction result is partly driven by averaging though they still find underreaction in individual short-rate forecasts. However, their analysis does not take into account that averaging might reduce measurement error (see e.g. [Juodis and Kucinskis \(2019\)](#))

¹⁷The downside of bond indices is that they are based on coupon bonds, while the theoretical predictions are for zero-coupon bonds. However, in unreported robustness checks we obtain similar results using the zero-coupon yield curve data set of [Wright \(2011\)](#), for market data see also the results in [Lustig et al. \(2019\)](#). Moreover, the difference between the predictions for coupon and zero-coupon bonds is small according to simulations. The benefit of using bond indices is that they are free from approximation error in common interpolation procedures and corresponding returns are tradable.

	$\hat{\lambda}_0$	<i>s.e.</i>	$\hat{\lambda}_1$	<i>s.e.</i>	R^2
panel	-0.011	0.020	0.987***	0.008	0.979
CAN	0.020	0.023	0.957***	0.016	0.932
GER	-0.016	0.022	0.994***	0.013	0.988
JAP	-0.032	0.017	0.989***	0.008	0.992
NOR	0.014	0.038	0.989***	0.016	0.975
SWE	-0.017	0.027	0.991***	0.011	0.980
CH	-0.014	0.021	0.988***	0.011	0.974
UK	0.005	0.022	0.981***	0.015	0.975

Table 2 shows the results from regressing monthly 3 month yield differential on its first lag. The standard errors of the panel regression are calculated using the (Driscoll and Kraay, 1998) methodology with 13 lags, which corrects for heteroskedasticity, serial correlation, and cross-equation correlation. The standard errors for individual regressions are corrected for heteroskedasticity and autocorrelation (Newey and West, 1987). *, ** and *** denote significance at 10%, 5% and 1% levels respectively.

All of the predictability results hinge only on the actual and perceived processes for the state variables, not on the other terms in the SDFs. Moreover, we focus on predictability regression slope coefficients. These coefficients do not depend on factor variances. For the main results of this paper, we therefore need to calibrate only two parameters: the persistence of local shocks λ and the corresponding underreaction coefficient k .

Calibrating λ and k We now calibrate the persistence parameter λ and the underreaction coefficient k . An estimate of the persistence parameter can be obtained by exploiting Lemma 1, which implies that the short rate differential process is given by an AR(1) process:

$$x_t = \lambda x_{t-1} + \tilde{\epsilon}_t. \quad (28)$$

We choose US as the home country and construct a monthly series of the interest rate differential with respect to the other countries (foreign - US rate). We estimate the process separately for each country as well as for a panel with all the countries. Note that taking differences largely removes the secular downward trend in interest rates. The resulting persistence parameters are given in table 2. Interest rate differentials are highly persistent with estimates ranging between 0.96 and 0.99. We choose the panel estimate $\lambda \approx 0.99$ as the baseline calibration.

We then need an estimate of the underreaction coefficient k . For this purpose we consider the following regression, similar to that in Coibion and Gorodnichenko (2012):

	$\hat{\beta}_0$	<i>s.e</i>	$\hat{\beta}_1$	<i>s.e</i>	R^2	<i>implied k</i>
panel	0.202	0.172	1.059**	0.391	0.035	0.486
CAN	0.282	0.164	-0.230	0.217	0.003	1.299
GER	0.304*	0.186	1.628**	0.684	0.074	0.380
JAP	0.331*	0.190	1.771***	0.535	0.083	0.361
NOR	0.351	0.279	1.995***	0.720	0.089	0.334
SWE	-0.319	0.238	1.461***	0.582	0.055	0.406
CH	0.305	0.185	0.972*	0.530	0.028	0.507
UK	0.200	0.171	0.564	0.373	0.008	0.639

Table 3 shows the results from regressing the difference in forecast error from forecasting spot 3 month 12 steps ahead on the difference in forecast revisions. Difference is defined as foreign minus US. The standard errors of the panel regression are calculated using the (Driscoll and Kraay, 1998) methodology with 13 lags, which corrects for heteroskedasticity, serial correlation, and cross-equation correlation. The standard errors for individual regressions are corrected for heteroskedasticity and autocorrelation (Newey and West, 1987). *, ** and *** denote significance at 10%, 5% and 1% levels respectively.

$$x_{t+12} - \mathbb{E}_t^S[x_{t+12}] = \beta_0^{FR} + \beta_1^{FR}[\mathbb{E}_t^S[x_{t+12}] - \mathbb{E}_{t-1}^S[x_{t+12}]] + e_{t+12} \quad (29)$$

Here we regress the forecast error for the short rate differential on the corresponding forecast revision. As explained in the appendix, the model implies that $\beta_0^{FR} = 0$ and $\beta_1^{FR} = \frac{1-k}{k}$. In a rational model $k = 1$, $\beta_1^{FR} = 0$ and forecast errors are not predictable. More generally, a positive (negative) coefficient for the regression indicates underreaction (overreaction).

Table 3 shows the results from this regression along with the implied values for k . We use the panel estimate $k \approx 0.49$ as the baseline calibration.

With a k above one, indicating overreaction, Canada seems to be an outlier but we still include it in the panel regression. Most of the country specific coefficient values are close to each other. Indeed with the exception of Canada, none of the country -level values are statistically different from the panel estimate.

Note that our assumptions imply that under correct beliefs $k = 1$. This seems also empirically reasonable. In particular using the actual short rate process we obtain a panel estimate $k \approx 0.983$.¹⁸ Moreover, $k = 1$ clearly cannot be rejected.

¹⁸This can be estimated either using the above regression procedure or maximum likelihood.

4.3 Short-rate misperceptions and bond and currency returns

We start by replicating the four key predictability regressions in [Lustig et al. \(2019\)](#). These include regressions of bond returns either on short-rate differentials or on yield spread differentials. According to Proposition 3 and the discussion in section 3, the slope coefficient in the regressions involving short term rates should be negative and the slope coefficient in the regressions involving yield spreads should be positive.

Table 4, panel A, gives the results for the bond return regressions both for individual countries as well as for the fixed effects panel regressions. The signs of the slope coefficients are as predicted by the model. The results for the panel regressions are statistically significant. The slope coefficients in the regression with short-rate differentials are similar to those reported by ([Lustig et al., 2019](#)). However, the slope coefficients in the regressions with yield spreads are smaller than in [Lustig et al. \(2019\)](#). Moreover, their predictive power is somewhat smaller.

We also consider regressing currency returns on short-rate differentials and yield spread differentials. According to Proposition 3 and the discussion in section 3, the slope coefficient in the first regression should be positive and the slope coefficient in the second regression negative. The results are given in table 4, panel B. The signs in the panel regressions are statistically significant and as predicted by the model. The absolute magnitudes of the slope coefficients are somewhat smaller than those in [Lustig et al. \(2019\)](#), especially when using yield spread differentials as predictors.

Table 5 summarizes the results for the panel regressions in Table 4. Here we also show the slope coefficient from a regression of relative bond returns, expressed in the same currency, on short-rate and yield spread differentials. This is mechanically the sum of the slope coefficients in the bond and currency regressions. The table also shows the model simulated coefficients using the calibration described earlier.

Overall, the model implied values are similar to those obtained in the panel data regressions but somewhat smaller in absolute values for the coefficients in the predictability regressions with short-rates. The model implies larger absolute values for the coefficients in the predictability regressions with yield spreads than those seen in the data. On the other hand, these estimates are somewhat noisy and for example the coefficients in the spread regressions are larger than those obtained by [Lustig et al. \(2019\)](#) for a similar sample period. However, our results suggest that forecast errors concerning short-rates can go a surprisingly long way in explaining the above predictability patterns. As explained by [Lustig et al. \(2019\)](#) standard

PANEL A: Bond Returns										
	3 month rate					yield slope				
	$\hat{\beta}_0$	<i>s.e</i>	$\hat{\beta}_1(3)$	<i>s.e</i>	R^2	$\hat{\beta}_0$	<i>s.e</i>	$\hat{\beta}_1$	<i>s.e</i>	R^2
panel	-0.000	0.000	-1.259***	0.253	0.019	0.000	0.000	1.250**	0.480	0.008
AUS	0.002**	0.001	-1.348***	0.543	0.028	0.001	0.001	2.095**	1.024	0.020
CAN	0.001*	0.000	-1.895***	0.530	0.039	0.001*	0.000	2.554***	0.722	0.034
GER	-0.000	0.000	-0.845**	0.347	0.008	-0.000	0.000	0.598	0.500	0.003
JAP	-0.002***	0.001	-1.212***	0.389	0.089	-0.000	0.000	0.405	0.478	0.001
NZL	0.002**	0.001	-1.588***	0.503	0.016	-0.001	0.001	0.948	0.720	0.001
SWE	0.001	0.000	-1.111**	0.542	0.021	0.001	0.001	1.407	1.038	0.012
CH	-0.002***	0.000	-1.045**	0.513	0.007	-0.000	0.000	1.036	0.666	0.004
UK	0.001*	0.000	-1.427***	0.481	0.021	0.000	0.000	1.072*	0.653	0.004

PANEL B: Currency Returns										
	3 month rate					yield slope				
	$\hat{\beta}_0$	<i>s.e</i>	$\hat{\beta}_1(3)$	<i>s.e</i>	R^2	$\hat{\beta}_0$	<i>s.e</i>	$\hat{\beta}_1$	<i>s.e</i>	R^2
panel	0.000	0.001	1.489**	0.476	0.015	0.000	0.001	-1.943*	1.019	0.010
AUS	-0.001	0.002	1.477**	0.581	0.014	-0.000	0.002	-2.085	1.388	0.008
CAN	0.002	0.001	1.159*	0.719	0.006	0.002**	0.001	-0.577	1.267	0.001
GER	0.005***	0.002	-.455	1.178	0.001	0.005**	0.002	0.196	1.449	0.000
JAP	0.005**	0.002	2.972***	0.933	0.030	-0.004**	0.001	-4.122***	1.153	0.029
NZL	-0.003	0.002	2.023***	0.419	0.052	-0.001	0.002	-3.467***	1.099	0.050
SWE	-0.002	0.002	0.849	0.897	0.004	-0.001	0.001	0.479	1.595	0.001
CH	0.003*	0.002	1.754*	1.046	0.008	-0.000	0.001	-2.531*	1.418	0.012
UK	-0.000	0.002	1.517*	0.944	0.010	-0.000	0.001	-1.963	1.640	0.007

Table 4 shows the results from regressing the relative bond and currency returns on short-rate and yield spread differences. The standard errors of the panel regression are calculated using the (Driscoll and Kraay, 1998) methodology with 13 lags, which corrects for heteroskedasticity, serial correlation, and cross-equation correlation. The standard errors for individual regressions are corrected for heteroskedasticity and autocorrelation (Newey and West, 1987). *, ** and *** denote significance at 10%, 5% and 1% levels respectively.

Coefficient	Data	Model ($\varphi_1 = \bar{\varphi} = 0$, plim)
β_1 , LHS: Currret, RHS: x_t	1.489	0.99
β_1 , LHS: Currret, RHS: yield spread	-1.943	-3.3
β_1 , LHS: $r_{t+1}^{FX}(120)$, RHS: x_t	0.23	0.29
β_1 , LHS: $r_{t+1}^{FX}(120)$, RHS: yield spread	-0.69	-0.98
β_1 , LHS: Bondret, RHS: x_t	-1.259	-0.7
β_1 , LHS: Bondret, RHS: yield spread	1.250	2.339
Variance ratio, 10 year rate, 3 month rate	0.67	0.57

Table 5 shows key statistics measured from the data (panel regressions) as well as those predicted by the model with $\varphi_1 = \bar{\varphi} = 0$ calibrated solely to match the biases in survey data.

rational expectations macrofinance models have trouble replicating these patterns. Note that these moments have not been targeted in any way, rather the model is calibrated directly using survey data.

The table also shows the ratio between the volatilities of 10 year rate differentials and 3 month rate differentials. The number is 0.67 in the data as compared to 0.57 predicted by the model. [Shiller \(1979\)](#) explains that empirically the volatility of long-term rates is higher than would be justified by the path of rationally expected short-rates. However, sticky expectations concerning short-rates increase the volatility of long-rates so that the model predicted value is not far from that in the data.¹⁹

Bond and Currency Returns and Past Short Rates In a sticky expectations model, short rate forecast errors tend to be particularly high after recent short rate changes. This is because it takes time for forecasters to update their predictions. This implies that high short-term interest rates relative to past short rates should predict high returns for a currency but low returns for the corresponding long-term bond, as explained in Proposition 5.

We now test this implication of the model. We construct the past average short rate difference \bar{x}_t using our estimates of k and λ ²⁰. We then regress relative bond and currency returns on x_t and \bar{x}_{t-1} . The results are given in table 6. The slope coefficients on short rate differences are as before though larger in magnitude. However, as predicted by the model the slope coefficient on the average past short rate is negative in the currency regression but positive in the bond regression. Foreign currency returns tend to be particularly high when the foreign short rate has recently increased. Similarly foreign bond returns tend to be particularly high when the foreign short rate has recently decreased.

These results further support the model mechanism depicted in figure 1 and the idea that bond and currency return predictability patterns are largely *drift* patterns. Here a positive shock to the foreign short rate leads to a slow appreciation of the foreign currency and a sluggish fall in the value of foreign bond.

The above findings are related to the delayed overshooting ([Eichenbaum and Evans, 1995](#)) and FOMC post announcement drift (see e.g. [Brooks et al.](#)

¹⁹[Gourinchas and Tornell \(2004\)](#) also show that the exchange rate process implied by our benchmark model can account for the related currency persistence and volatility puzzles (e.g. [Backus et al. \(1993\)](#), [Moore and Roche \(2002\)](#)).

²⁰Because we weight the past rates with our estimates of k and λ , this regression is generally subject to a generated regressor problem. However, alternative weighting schemes that do not depend on these estimates yield similar results.

PANEL A: Bond returns							
	$\hat{\beta}_0$	<i>s.e.</i>	$\hat{\beta}_1(x_t)$	<i>s.e.</i>	$\hat{\beta}_2(\bar{x}_{t-1})$	<i>s.e.</i>	R^2
panel	-0.001	0.000	-8.651***	1.172	3.917***	0.564	0.044
AUS	0.000	0.001	-8.962***	2.118	4.061***	1.038	0.045
CAN	0.000	0.001	-8.228**	3.678	3.560**	1.773	0.052
GER	-0.001	0.001	-9.132***	2.451	4.232***	1.185	0.038
JAP	-0.002**	0.001	-10.880***	3.171	4.975***	1.562	0.022
NZL	0.000	0.001	-7.513***	2.822	3.263**	1.401	0.032
SWE	0.000	0.001	-7.419***	1.373	3.369***	0.722	0.064
CH	-0.002***	0.001	-10.698***	3.359	4.879***	1.589	0.046
UK	-0.001***	0.001	-8.783***	2.339	4.017***	1.164	0.036

PANEL B: Currency returns							
	$\hat{\beta}_0$	<i>s.e.</i>	$\hat{\beta}_1(x_t)$	<i>s.e.</i>	$\hat{\beta}_2(\bar{x}_{t-1})$	<i>s.e.</i>	R^2
panel	0.001	0.001	4.961**	1.717	-1.838*	0.827	0.015
AUS	-0.001	0.002	7.265***	2.813	-2.797**	1.322	0.045
CAN	0.001	0.001	0.544	3.675	0.397	1.734	0.052
GER	0.005**	0.002	2.201	5.064	-1.233	2.436	0.038
JAP	0.004*	0.002	3.626	6.349	-0.417	3.159	0.022
NZL	-0.001	0.002	6.894**	2.899	-2.582*	1.481	0.032
SWE	-0.001	0.002	3.810*	2.287	-1.796	1.312	0.064
CH	0.003	0.002	7.355	5.604	-2.645	2.802	0.046
UK	0.000	0.002	2.423	3.574	-0.717	1.676	0.036

Table 6 shows the results from regressing the relative bond and currency returns on short rate differences and an average of past short rate differences. The standard errors of the panel regression are calculated using the (Driscoll and Kraay, 1998) methodology with 13 lags, which corrects for heteroskedasticity, serial correlation, and cross-equation correlation. The standard errors for individual regressions are corrected for heteroskedasticity and autocorrelation (Newey and West, 1987). *, ** and *** denote significance at 10%, 5% and 1% levels respectively.

(2019)) patterns documented in the previous literature. Delayed overshooting refers to the fact that the response of the FX rate to interest rate shocks is hump-shaped. A contractionary shock to US monetary policy induces a gradual appreciation of the US dollar followed by depreciation. The benchmark model with $\varphi = \bar{\varphi} = 0$ implies an identical exchange rate process to that in [Gourinchas and Tornell \(2004\)](#). They show that this process can account for the delayed overshooting puzzle.

A pattern similar to the delayed overshooting puzzle of currencies is the FOMC post announcement drift in bond markets. Here the yields of long-maturity treasuries respond sluggishly to changes in the Federal Funds Rate. In concurrent work, [Brooks et al. \(2019\)](#) argue that slow updating concerning short-term interest rates can explain the FOMC post announcement drift. This pattern is also generated by our model as can be seen in the FX impulse response plotted in figure 1.

4.4 Predicting expectational errors in survey based expectations

Expectational errors should be unpredictable under rational expectations. However, a key prediction of the model is that the same variables that predict bond and currency returns also predict expectational errors concerning FX rates and long-term interest rates. [Bacchetta et al. \(2009\)](#) find support for the model prediction that short-rate differentials predict expectational errors concerning FX rates. They also find that yield spreads predict expectational errors concerning long-term interest rates but do not offer an explanation for these findings.

We verify these predictions using an alternative dataset and a different sample period. Moreover, we find support for the additional model prediction that yield spread differentials explain expectational errors concerning FX rates.

The results are given in table 7. As predicted by the model a high short-rate differential between the foreign and home country predicts that the foreign currency will turn stronger than expected. The opposite prediction is obtained when using yield spread differentials as the explanatory variable. The results are given in table 7. The results for panel regressions are statistically significant and as predicted by the model.

The model also predicts that when short-rates are high, long-term interest rates (bond prices) will turn higher (lower) than predicted. Moreover, according to the model a high yield spread predicts that long-term interest

Forecast Errors FX: 12 Steps Ahead										
	3 month rate differential					term spread differential				
	$\hat{\beta}_0$	<i>s.e</i>	$\hat{\beta}_1(3)$	<i>s.e</i>	R^2	$\hat{\beta}_0$	<i>s.e</i>	$\hat{\beta}_1$	<i>s.e</i>	R^2
panel	0.012*	0.007	0.012*	0.006	0.040	0.021**	0.007	-0.016*	0.008	0.037
CAD	0.021*	0.012	0.003	0.006	0.003	0.020*	0.011	-0.003	0.008	0.002
CHF	-0.047***	0.013	0.016***	0.006	0.133	-0.013*	0.008	-0.023***	0.008	0.206
EUR	0.001	0.016	0.023**	0.012	0.094	-0.004	0.017	-0.035**	0.018	0.088
GBP	0.012	0.015	-0.004	0.008	0.006	0.022*	0.013	0.013	0.010	0.030
JPY	-0.051***	0.019	0.024***	0.007	0.171	0.029*	0.017	-0.043***	0.009	0.238
NOK	0.131***	0.045	0.033**	0.016	0.120	0.127***	0.044	-0.040**	0.019	0.107
SEK	0.029***	0.009	-0.004	0.008	0.014	0.026**	0.009	0.014	0.009	0.081

Table 7 shows the results from regressing the difference in forecast error from forecasting 3 month spot FX rates 12 steps ahead on short-rate and yield spread differentials. Difference is defined as foreign minus US. The standard errors of the panel regression are calculated using the (Driscoll and Kraay, 1998) methodology with 13 lags, which corrects for heteroskedasticity, serial correlation, and cross-equation correlation. The standard errors for individual regressions are corrected for heteroskedasticity and autocorrelation (Newey and West, 1987). *, ** and *** denote significance at 10%, 5% and 1% levels respectively.

rates will turn lower than expected. The results for this regression are given in table 8. Using a panel regression, we find positive support for the latter prediction. The slope coefficient in the panel regression with short rate is insignificant. However, it becomes significant if we exclude Japan from the sample. Here, the fact that Japan has experienced close to zero interest rates for most of the sample might complicate the relationship between short-rates and expectational errors. Moreover, evidence that short-rates predict expectational errors in long-term rates is provided by Bacchetta et al. (2009).

4.5 Robustness Checks

We conduct several robustness checks for these results. First, some authors such as Engel (2016) voluntarily leave the period after the financial crisis out from the sample due to possible changes in the driving forces of currencies. We obtain very similar results if we omit this sample period. Similarly, we obtain similar results if we leave the periods with close to zero interest rates out from the sample.²¹ Unfortunately, we do not have enough data to robustly identify if the phenomena studied in this paper are different during different sample periods such as when interest rates are near an effective lower bound. However, as mentioned before, many of our results become

²¹A lower bound is often taken into account in term-structure models through a shadow rate specification (see e.g. Kortela (2016)).

Forecast Errors Bonds: 10Y Bond, 12 Steps Ahead										
	3 month rate					yield slope				
	$\hat{\beta}_0$	<i>s.e</i>	$\hat{\beta}_1(3)$	<i>s.e</i>	R^2	$\hat{\beta}_0$	<i>s.e</i>	$\hat{\beta}_1$	<i>s.e</i>	R^2
panel	-0.632***	0.063	0.010	0.017	0.019	-0.421***	0.098	-0.166**	0.053	0.049
CAN	-0.706***	0.097	0.031	0.026	0.004	-0.494***	0.135	-0.083	0.105	0.009
GER	-0.800***	0.072	0.046**	0.020	0.022	-0.478***	0.137	-0.193**	0.081	0.080
JAP	-0.367***	0.036	-0.124***	0.023	0.109	-0.213*	0.129	-0.240**	0.113	0.102
NOR	-0.798***	0.123	0.070**	0.034	0.036	-0.473***	0.127	-0.255***	0.094	0.136
SWE	-0.713***	0.080	-0.072***	0.024	0.035	-0.385*	0.221	-0.352***	0.122	0.107
CH	-0.554***	0.058	0.059*	0.033	0.011	-0.357***	0.133	-0.136	0.117	0.019
UK	-0.502***	0.094	-0.005	0.019	0.000	-0.486***	0.111	-0.054	0.075	0.005
USA	-0.754***	0.084	0.040*	.023	0.001	-0.297	0.192	-0.186**	0.090	0.058

Table 8 shows the results from regressing the difference in forecast error from forecasting 10 year interest rates 12 steps ahead on the short-rates and yield slopes. The standard errors of the panel regression are calculated using the (Driscoll and Kraay, 1998) methodology with 13 lags, which corrects for heteroskedasticity, serial correlation, and cross-equation correlation. The standard errors for individual regressions are corrected for heteroskedasticity and autocorrelation (Newey and West, 1987). *, ** and *** denote significance at 10%, 5% and 1% levels respectively.

stronger if we omit Japan, where interest rates have been very low during most of the sample period.

4.6 Policy Implications

The analysis bears important implications for monetary policy. The short end of the nominal yield curve is typically thought to be tightly controlled by a central bank. Here the short rate part of the stochastic discount factor characterizes the monetary policy rule of the central bank. ²²

Now assume the central bank would unexpectedly increase the domestic short term rate. Assuming rational expectations, this would lead to an immediate increase in the long-term bond yield and the value of the domestic currency. However, with sticky short-term expectations these effects occur with a lag. Put differently, monetary policy induces drift patterns in bond and currency markets. Therefore, one should not rely on the initial market reaction, when interpreting the effects of monetary policy on bond yields and FX rates. Evidence for such drifts effects are given by the empirical findings related to the FOMC post announcement drift and delayed overshooting discussed in the previous section.

The second key policy implication is about the interpretation of the shape of the yield curve. Standard models emphasize the importance of risk effects

²²Here the policy rule is simply $\log R + z_t + \bar{z}_t$

in explaining bond return predictability. Our analysis instead argues that short-rate forecast errors are significant predictors of bond returns and, as a consequence, risk effects might play a smaller role. Our model and empirical evidence suggests that long-term yields are therefore a better measure of market's subjective short-rate expectations than previously thought. However, we have shown that these expectations are not rational, especially in the short run and after an unexpected change in the interest rate.

Term structure models are often fitted to match the dynamics of the yield curve. Such models are then used to decompose a long-term yield into a path of short-rate expectations and a term premium. However, without correctly specifying the subjective beliefs over short-rates, the results from these exercises can be misleading. Our analysis is a step forward in extending these models to better match the expectational errors observed in survey data.

5 Conclusion

We show that well-documented sluggish updating concerning short rates creates joint predictability patterns in bond and currency markets. These predictability patterns explain most of the variation in expected bond and currency returns driven by variation in short rates and yield spreads.

Importantly the biases work in opposite directions for bonds and currencies. The relative prices of currencies are increasing and the relative prices of long-term bonds decreasing in expected short rates. Therefore high interest rate currencies tend to be underpriced but the long-term bonds of these same currencies overpriced. This provides a novel explanation for the fact that the term structure of expected carry trade returns is downward sloping.

We can view the exercise of this paper in two ways. The first is that we have provided results in a model in which all time-variation in expected bond and currency returns is due to misperceptions concerning short-rates, disregarding alternative explanations, such as time-varying risk premium. However, we can view our results also as a quantification of the contribution of short-rate forecast errors to this variation in a more general model with a time-varying risk premium. Because short rate forecast errors are unlikely to drive all of return variation, future work should attempt to carefully combine models with risk premium effects and models with short rate forecast errors.

The analysis bears important policy implications. Monetary policy that affects short rates transmits to bond yields and FX rates at a lag. Commonly used term structure models have trouble separating short-rate expectations

and risk effects.

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6 Appendix

6.1 Formulas Left Out in Proposition 1

The Kalman gains k_t , \bar{k}_t and \tilde{k}_t are given by

$$k_t = \frac{\lambda^2 \sigma_t^2 + \sigma_\epsilon^2}{\lambda^2 \sigma_t^2 + \sigma_\epsilon^2 + \sigma_v^2} \quad \bar{k}_t = \frac{\bar{\lambda}^2 \bar{\sigma}_t^2 + \bar{\sigma}_\epsilon^2}{\bar{\lambda}^2 \bar{\sigma}_t^2 + \bar{\sigma}_\epsilon^2 + \bar{\sigma}_v^2} \quad \tilde{k}_t = \frac{\lambda^2 \sigma_t^2 + 2\sigma_\epsilon^2}{\lambda^2 \sigma_t^2 + 2\sigma_\epsilon^2 + 2\sigma_v^2}.$$

The conditional volatilities of the persistent components are

$$\sigma_{t+1}^2 = (1 - k_t)(\lambda^2 \sigma_t^2 + \sigma_\epsilon^2) \quad \bar{\sigma}_{t+1}^2 = (1 - \bar{k}_t)(\bar{\lambda}^2 \bar{\sigma}_t^2 + \bar{\sigma}_\epsilon^2)$$

for the first two states and the common state and

$$\tilde{\sigma}_{t+1}^2 = (1 - \tilde{k}_t)(\lambda^2 \sigma_t^2 + 2\sigma_\epsilon^2)$$

for the interest rate differential. The steady-state estimators are

$$\sigma^2 = \frac{1-k}{1-(1-k)\lambda^2} \sigma_\epsilon^2 \quad \bar{\sigma}^2 = \frac{1-\bar{k}}{1-(1-\bar{k})\bar{\lambda}^2} \bar{\sigma}_\epsilon^2 \quad \tilde{\sigma}^2 = 2 \frac{1-k}{1-(1-k)\lambda^2} \sigma_\epsilon^2$$

$$k = \tilde{k} = \frac{1 + \Delta - \eta(1 + \lambda^2)}{1 + \Delta + \eta(1 + \lambda^2)} \quad \bar{k} = \frac{1 + \bar{\Delta} - \bar{\eta}(1 + \bar{\lambda}^2)}{1 + \bar{\Delta} + \bar{\eta}(1 + \bar{\lambda}^2)}.$$

Here

$$\Delta^2 = [\eta(1 - \lambda^2) + 1]^2 + 4\eta\lambda^2 \quad \bar{\Delta}^2 = [\bar{\eta}(1 - \bar{\lambda}^2) + 1]^2 + 4\bar{\eta}\bar{\lambda}^2$$

and

$$\eta = \frac{\sigma_v^2}{\sigma_\epsilon^2} \quad \bar{\eta} = \frac{\bar{\sigma}_v^2}{\bar{\sigma}_\epsilon^2}.$$

6.2 Proof of Proposition 2

The expression for $A(1)$ and $\mathbf{B}(1)$ follow immediately. Given the conjecture, log bond prices and SDF are (conditionally) jointly normal. To see the recursion formulas note

$$q_t(n) = \mathbb{E}_t^S[m_{t+1} + q_{t+1}(n-1)] + \frac{1}{2} \text{Var}_t^S[m_{t+1} + q_{t+1}(n-1)]$$

or

$$A(n) + \mathbf{B}(n)' \mathbf{Y}_t = \mathbb{E}_t^S[m_{t+1} + A(n-1) + \mathbf{B}(n-1)' \mathbf{Y}_{t+1}] + \frac{1}{2} \text{Var}_t^S[m_{t+1} + A(n-1) + \mathbf{B}(n-1)' \mathbf{Y}_{t+1}].$$

Note that

$$\mathbb{E}_t^S[m_{t+1} + A(n-1) + \mathbf{B}(n-1)' \mathbf{Y}_{t+1}] = -\log R - \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^2}{2} - \frac{\sigma_\epsilon^2 \varphi_t^2}{2} - z_t - \bar{z}_t + A(n-1) + \mathbf{B}(n-1)' [\mathbb{E}_t^S[z_{t+1}], \mathbb{E}_t^S[\bar{z}_{t+1}], \mathbb{E}_t^S[z_{t+2}], \mathbb{E}_t^S[\bar{z}_{t+2}]].$$

Also

$$\mathbb{E}_t^S[z_{t+2}] = \lambda \mathbb{E}_t^S[z_{t+1}] \quad \mathbb{E}_t^S[\bar{z}_{t+2}] = \bar{\lambda} \mathbb{E}_t^S[\bar{z}_{t+1}].$$

Therefore after some algebra

$$\begin{aligned}\mathbb{E}_{t+1}^S[m_{t+1} + A(n-1) + \mathbf{B}(n-1)' \mathbf{Y}_{t+1}] &= -\log R - \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^2}{2} - \frac{\sigma_\epsilon^2 \varphi_t^2}{2} - z_t - \bar{z}_t \\ &\quad + A(n-1) + \mathbb{E}_t^S[z_{t+1}](B_1(n-1) + \lambda B_3(n-1)) + \\ &\quad \mathbb{E}_t^S[\bar{z}_{t+1}](B_2(n-1) + \bar{\lambda} B_4(n-1)).\end{aligned}$$

The variance is given by (leaving out some algebra)

$$\begin{aligned}\mathbb{V}ar_t^S[m_{t+1} + A(n-1) + \mathbf{B}(n-1)' \mathbf{Y}_{t+1}] &= \\ [\varphi_t^2 - 2\varphi_t(B_1(n-1) + k\lambda B_3(n-1))] \sigma_\epsilon^2 &+ [\bar{\varphi}_t^2 - 2\bar{\varphi}_t(B_2(n-1) + \bar{k}\bar{\lambda} B_4(n-1))] \bar{\sigma}_\epsilon^2 \\ + (\sigma^2 + \sigma_v^2)(B_1(n-1) + k\lambda B_3(n-1))^2 &+ (\bar{\sigma}^2 + \bar{\sigma}_v^2)(B_2(n-1) + \bar{k}\bar{\lambda} B_4(n-1))^2.\end{aligned}$$

Therefore we obtain the equation

$$\begin{aligned}A(n) + B_1(n)z_t + B_2(n)\bar{z}_t + B_3(n)\mathbb{E}_t^S[z_{t+1}] + B_4(n)\mathbb{E}_t^S[\bar{z}_{t+1}] &= \\ -\log R - \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^2}{2} - \frac{\sigma_\epsilon^2 \varphi_t^2}{2} - z_t - \bar{z}_t + A(n-1) + \mathbb{E}_t^S[z_{t+1}](B_1(n-1) + \lambda B_3(n-1)) &+ \\ \mathbb{E}_t^S[\bar{z}_{t+1}](B_2(n-1) + \bar{\lambda} B_4(n-1)) &+ \\ \frac{1}{2}[\varphi_t^2 - 2\varphi_t(B_1(n-1) + k\lambda B_3(n-1))] \sigma_\epsilon^2 + \frac{1}{2}[\bar{\varphi}_t^2 - 2\bar{\varphi}_t(B_2(n-1) + \bar{k}\bar{\lambda} B_4(n-1))] \bar{\sigma}_\epsilon^2 &+ \\ + \frac{1}{2}(\sigma^2 + \sigma_v^2)(B_1(n-1) + k\lambda B_3(n-1))^2 + \frac{1}{2}(\bar{\sigma}^2 + \bar{\sigma}_v^2)(B_2(n-1) + \bar{k}\bar{\lambda} B_4(n-1))^2.\end{aligned}$$

From this one can solve

$$\begin{aligned}A(n) &= -\log R + A(n-1) + \\ -\varphi_0(B_1(n-1) + k\lambda B_3(n-1)) \sigma_\epsilon^2 - \bar{\varphi}_0(B_2(n-1) + \bar{k}\bar{\lambda} B_4(n-1)) \bar{\sigma}_\epsilon^2 &+ \\ + \frac{1}{2}(\sigma^2 + \sigma_v^2)(B_1(n-1) + k\lambda B_3(n-1))^2 + \frac{1}{2}(\bar{\sigma}^2 + \bar{\sigma}_v^2)(B_2(n-1) + \bar{k}\bar{\lambda} B_4(n-1))^2\end{aligned}$$

$$B_1(n) = -1 - \varphi_1(B_1(n-1) + k\lambda B_3(n-1)) \sigma_\epsilon^2 - \bar{\varphi}_1(B_2(n-1) + \bar{k}\bar{\lambda} B_4(n-1)) \bar{\sigma}_\epsilon^2$$

$$B_2(n) = -1$$

$$B_3(n) = \lambda B_3(n-1) + B_1(n-1) \quad B_4(n) = \bar{\lambda} B_4(n-1) + B_2(n-1).$$

The coefficient values in the simplified case with non-varying market prices of risk follow easily. Here one has

$$\begin{aligned} A(n) = & -\log R + A(n-1) + \\ & -\varphi_0(B_1(n-1) + k\lambda B_3(n-1))\sigma_\epsilon^2 - \bar{\varphi}_0(B_2(n-1) + \bar{k}\bar{\lambda}B_4(n-1))\bar{\sigma}_\epsilon^2 \\ & + \frac{1}{2}(\sigma^2 + \sigma_v^2)(B_1(n-1) + k\lambda B_3(n-1))^2 + \frac{1}{2}(\bar{\sigma}^2 + \bar{\sigma}_v^2)(B_2(n-1) + \bar{k}\bar{\lambda}B_4(n-1))^2. \end{aligned}$$

6.3 Proof of Proposition 3

The standard complete market condition in logs is

$$m_{t+1} + s_{t+1} - s_t = m_{t+1}^*$$

Taking expectations

$$s_t = \mathbb{E}_t^S[m_{t+1} - m_{t+1}^*] + \mathbb{E}_t^S[s_{t+1}]$$

or

$$s_t = x_t - \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^2}{2} - \frac{\sigma_\epsilon^2 \varphi_t^2}{2} + \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^{*2}}{2} + \frac{\sigma_\epsilon^2 \varphi_t^{*2}}{2} + \mathbb{E}_t^S[s_{t+1}].$$

Iterating forward one obtains

$$s_t = \mathbb{E}_t^S \sum_{j=0}^{\infty} x_{t+j} + \mathbb{E}_t^S \sum_{j=0}^{\infty} \Gamma_{t+j} + \bar{s}_t,$$

where

$$\Gamma_t = -\frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^2}{2} - \frac{\sigma_\epsilon^2 \varphi_t^2}{2} + \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^{*2}}{2} + \frac{\sigma_\epsilon^2 \varphi_t^{*2}}{2}$$

and $\bar{s}_t = \lim_{j \rightarrow \infty} \mathbb{E}_t^S[s_{t+j}]$ is assumed to be well-defined. The sums converge.

Furthermore

$$\mathbb{E}_t^S \sum_{j=0}^{\infty} x_{t+j} = x_t + \frac{1}{1-\lambda} \mathbb{E}_t^S [x_{t+1}].$$

Therefore (leaving out some algebra)

$$\mathbb{E}_t \left[\mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} x_{t+1+j} - \mathbb{E}_t^S \sum_{j=0}^{\infty} x_{t+j} \right] = -x_t + \left[1 + \frac{\lambda k}{1-\lambda} \right] \left[\mathbb{E}_t x_{t+1} - \mathbb{E}_t^S x_{t+1} \right].$$

Furthermore

$$\begin{aligned} \Gamma_t &= -\frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^2}{2} - \frac{\sigma_\epsilon^2 \varphi_t^2}{2} + \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_t^{*2}}{2} + \frac{\sigma_\epsilon^2 \varphi_t^{*2}}{2} = \\ &= -\frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_1^2}{2} z_t^2 - \bar{\sigma}_\epsilon^2 \bar{\varphi}_0 \bar{\varphi}_1 z_t - \frac{\sigma_\epsilon^2 \varphi_1^2}{2} z_t^2 - \sigma_\epsilon^2 \varphi_0 \varphi_1 z_t + \\ &= \frac{\bar{\sigma}_\epsilon^2 \bar{\varphi}_1^2}{2} z_t^{*2} + \bar{\sigma}_\epsilon^2 \bar{\varphi}_0 \bar{\varphi}_1 z_t^* + \frac{\sigma_\epsilon^2 \varphi_1^2}{2} z_t^{*2} + \sigma_\epsilon^2 \varphi_0 \varphi_1 z_t^*. \end{aligned}$$

Typically the squared terms are small relative to the other terms. One can write

$$\Theta_t = \left[1 + \frac{\lambda k}{1-\lambda} \right] \left[\mathbb{E}_t x_{t+1} - \mathbb{E}_t^S x_{t+1} \right] + \mathbb{E}_t \left[\mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} \Gamma_{t+1+j} - \mathbb{E}_t^S \sum_{j=0}^{\infty} \Gamma_{t+j} \right].$$

Note that assuming $\varphi_1 = \bar{\varphi}_1 = 0$ means the second term equals zero. Next one needs to derive an expression for the relative long-term bond price. Note

$$\begin{aligned} q_t^*(n) - q_t(n) &= B_1(n)(z_t^* - z_t) + B_3(n)(\mathbb{E}_t^S [z_{t+1}^*] - \mathbb{E}_t^S [z_{t+1}]) = \\ &= B_1(n)x_t + B_3(n)(\mathbb{E}_t^S [z_{t+1}^*] - \mathbb{E}_t^S [z_{t+1}]) \end{aligned}$$

and hence

$$\begin{aligned} & \mathbb{E}_t[q_t^*(n-1) - q_t(n-1)] - [q_t^*(n) - q_t(n)] = \\ & B_1(n-1)\lambda x_t - B_1(n)x_t + B_3(n-1)\mathbb{E}_t[\mathbb{E}_{t+1}^S[z_{t+2}] - \mathbb{E}_t^S[z_{t+2}]] \\ & - B_3(n)[\mathbb{E}_t^S[z_{t+1}] - \mathbb{E}_t^S[z_{t+1}^*]]. \end{aligned}$$

Therefore

$$\begin{aligned} \Theta_t(n) &= \mathbb{E}_t[q_t^*(n-1) - q_t(n-1) + s_{t+1}] - [q_t^*(n) - q_t(n)] - s_t = \\ & B_1(n-1)\lambda x_t - B_1(n)x_t + B_3(n-1)\mathbb{E}_t[\mathbb{E}_{t+1}^S[z_{t+2}] - \mathbb{E}_t^S[z_{t+2}]] \\ & - B_3(n)[\mathbb{E}_t^S[z_{t+1}] - \mathbb{E}_t^S[z_{t+1}^*]] - x_t + \left[1 + \frac{\lambda k}{1-\lambda}\right] [\mathbb{E}_t x_{t+1} - \mathbb{E}_t^S x_{t+1}]. \end{aligned}$$

Finally we derive a simple expression for the above expected return assuming $\varphi_1 = \bar{\varphi}_1 = 0$. Due to log-normality

$$q_t(n) = \log(P_t^n) = \mathbb{E}_t^S \left[\sum_{i=1}^n m_{t+i-1,t+i} \right] + \frac{1}{2} \text{Var}_t^S \left[\sum_{i=1}^n m_{t+i-1,t+i} \right].$$

Variance terms for the two countries are now equal and therefore

$$\mathbb{E}_t^S \left[\sum_{i=1}^n (m_{t+i-1,t+i}^* - m_{t+i-1,t+i}) \right] = -\mathbb{E}_t^S \sum_{s=t}^n x_s.$$

Now one can solve

$$\mathbb{E}_t^S \left[\sum_{i=1}^n m_{t+i-1,t+i} \right] = -x_t - \frac{1-\lambda^{n-1}}{1-\lambda} \mathbb{E}_t^S [x_{t+1}].$$

Therefore (leaving out some algebra)

$$\begin{aligned} & \mathbb{E}_t[q_{t+1}^*(n-1) - q_{t+1}(n-1)] - [q_t^*(n) - q_t(n)] = \\ & x_t - \mathbb{E}_t[x_{t+1}] \left[1 + k\lambda \frac{1-\lambda^{n-2}}{1-\lambda} \right] + \mathbb{E}_t^S[x_{t+1}] \left[\frac{1-\lambda^{n-1}}{1-\lambda} - (1-k)\lambda \frac{1-\lambda^{n-2}}{1-\lambda} \right] \\ & = x_t - [\mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}]] \left[1 + k \frac{\lambda - \lambda^{n-1}}{1-\lambda} \right] \end{aligned}$$

We obtain (leaving out some algebra)

$$\Theta_t(n) = [\mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}]] \frac{k\lambda^{n-1}}{1-\lambda}.$$

6.4 Proof of Proposition 4

We look at the covariance with the conditional expectation function

$$\frac{cov(x_t, \Theta_t)}{Var(x_t)}.$$

Here

$$\begin{aligned} cov(x_t, \Theta_t) &= \left[1 + \frac{\lambda k}{1 - \lambda}\right] cov(x_t, \mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}]) = \\ &Var(x_t) \lambda \left[1 + \frac{\lambda k}{1 - \lambda}\right] - \left[1 + \frac{\lambda k}{1 - \lambda}\right] cov(x_t, \mathbb{E}_t^S[x_{t+1}]). \end{aligned}$$

Also

$$\mathbb{E}_t^S[x_{t+1}] = k\lambda x_t + k(1-k)\lambda^2 x_{t-1} + k(1-k)^2 \lambda^3 x_{t-2} + \dots$$

$$\begin{aligned} cov(x_t, \mathbb{E}_t^S[x_{t+1}]) &= Var(x_t) [\lambda k + k(1-k)\lambda^3 + k(1-k)^2 \lambda^5 + \dots] \\ &= \frac{\lambda k}{1 - (1-k)\lambda^2} Var(x_t). \end{aligned}$$

Hence

$$cov(x_t, \Theta_t) = -Var(x_t) \left[1 + \frac{\lambda k}{1 - \lambda}\right] \left[\frac{\lambda k - \lambda + (1-k)\lambda^3}{1 - (1-k)\lambda^2}\right]$$

and

$$\beta_1^{FX, plim} = - \left[1 + \frac{\lambda k}{1 - \lambda}\right] \left[\frac{\lambda k - \lambda + (1-k)\lambda^3}{1 - (1-k)\lambda^2}\right].$$

Also

$$\beta_1^{FX, plim}(n) = - \frac{k\lambda^{n-1}}{1 - \lambda} \left[\frac{\lambda k - \lambda + (1-k)\lambda^3}{1 - (1-k)\lambda^2}\right].$$

One can see that the expression is positive. The argument that the OLS estimate converges to these values is standard given the assumptions made in the model. The other results follow easily. □

6.5 Proof of Proposition 5

The currency risk premium is given by

$$\Theta_t = \left[1 + \frac{\lambda k}{1 - \lambda} \right] \left[\mathbb{E}_t[x_{t+1}] - \mathbb{E}_t^S[x_{t+1}] \right]$$

Here

$$\mathbb{E}_t[x_{t+1}] = \lambda x_t$$

and

$$\mathbb{E}_t^S[x_{t+1}] = k\lambda x_t + k(1-k)\lambda^2 x_{t-1} + k(1-k)^2 \lambda^3 x_{t-2} + \dots$$

Therefore

$$\begin{aligned} \Theta_t &= \left[1 + \frac{\lambda k}{1 - \lambda} \right] \left[\lambda(1-k)x_t - k(1-k)\lambda^2 x_{t-1} - k(1-k)^2 \lambda^3 x_{t-2} - \dots \right] = \\ &\quad \left[1 + \frac{\lambda k}{1 - \lambda} \right] \lambda(1-k) [x_t - k\lambda \bar{x}_{t-1}] \end{aligned}$$

Hence

$$r_{t+1}^{FX} = \left[1 + \frac{\lambda k}{1 - \lambda} \right] \lambda(1-k) [x_t - k\lambda \bar{x}_{t-1}] + \epsilon_{t+1}$$

This implies

$$\beta_1^{FX,plim} = \left[1 + \frac{\lambda k}{1 - \lambda} \right] \lambda(1-k).$$

and

$$\beta_2^{FX,plim} = - \left[1 + \frac{\lambda k}{1 - \lambda} \right] \lambda(1-k)k\lambda.$$

The signs are as predicted by the proposition. The proof for the bond regression is similar.

6.6 Proof of Proposition 6

Consider the regression

$$s_{t+j} - \mathbb{E}_t^S[s_{t+j}] = \beta_0 + \beta_1 x_t + e_{t+j}.$$

One can write

$$s_{t+j} - \mathbb{E}_t^S[s_{t+j}] = \sum_{k=0}^{\infty} \mathbb{E}_{t+j}^S[x_{t+j+k}] - \sum_{k=0}^{\infty} \mathbb{E}_t^S[x_{t+j+k}] + \bar{s}_{t+j} - \mathbb{E}_t^S[\bar{s}_{t+j}].$$

Because we assumed $\mathbb{E}_t^S[\bar{s}_{t+j}] = \mathbb{E}_t[\bar{s}_{t+j}]$ the permanent component does not affect the results. One needs to evaluate

$$\begin{aligned} & Cov(\mathbb{E}_{t+j}^S[x_{t+j+k}] - \mathbb{E}_t^S[x_{t+j+k}], x_t) = \\ & \lambda^{k-1} Cov(\mathbb{E}_{t+j}^S[x_{t+j+1}] - \lambda^j \mathbb{E}_t^S[x_{t+1}], x_t). \end{aligned}$$

Recall that

$$\begin{aligned} \mathbb{E}_{t+j}^S[x_{t+j+1}] &= (1-k)\lambda \mathbb{E}_{t+j-1}^S[x_{t+j}] + k\lambda x_{t+j} = \\ (1-k)^j \lambda^j \mathbb{E}_t^S[x_{t+1}] &+ k\lambda[x_{t+j} + \lambda(1-k)x_{t+j-1} + \dots + \lambda^j(1-k)^j x_t]. \end{aligned}$$

Hence after some algebra

$$\begin{aligned} & Cov(\mathbb{E}_{t+j}^S[x_{t+j+1}] - \lambda^j \mathbb{E}_t^S[x_{t+1}], x_t) = \\ & [(1-k)^j - 1] \lambda^j Cov(\mathbb{E}_t^S[x_{t+1}], x_t) + \lambda^{j+1} [1 - (1-k)^{j+1}] Var(x_t). \end{aligned}$$

On the other hand (see the proof of proposition 4)

$$Cov(\mathbb{E}_t^S[x_{t+1}], x_t) = \frac{\lambda k}{1 - (1-k)\lambda^2} Var(x_t).$$

Therefore

$$Cov(\mathbb{E}_{t+j}^S[x_{t+j+1}] - \lambda^j \mathbb{E}_t^S[x_{t+1}], x_t) = \lambda^{j+1} \left[\frac{k(1-k)^j - k}{1 - (1-k)\lambda^2} + 1 - (1-k)^{j+1} \right] Var(x_t).$$

The term

$$\left[\frac{k(1-k)^j - k}{1 - (1-k)\lambda^2} + 1 - (1-k)^{j+1} \right]$$

governs the sign of β_1 . Now

$$\left[\frac{k(1-k)^j - k}{1 - (1-k)\lambda^2} + 1 - (1-k)^{j+1} \right] > (1-k)^j - (1-k)^{j+1} > 0.$$

Hence $\beta_1^{plim} > 0$. The argument that the OLS estimate converges to this value is straightforward.

□

6.7 The Stochastic Discount Factor Under the Objective Measure

This section shows how to rewrite the home stochastic discount factor under the objective measure. It is seen that the stochastic discount factor is strictly positive under the objective measure. This implies that the economy does not allow for arbitrage opportunities under either the subjective nor objective measure. Put alternatively, the objective and subjective probability measures are equivalent.

The time t price of a payoff X_{t+1} is given by

$$P_t = \mathbb{E}_t^S[M_{t,t+1}X_{t+1}].$$

This can be rewritten under the objective measure as

$$P_t = \mathbb{E}_t[M_{t,t+1}\xi_{t,t+1}X_{t+1}],$$

where $\xi_{t,t+1} = \frac{dS}{dP}$ is the Radon-Nikodym derivative and $M_{t,t+1}\xi_{t,t+1}$ is the SDF under the objective measure. Here one can see that the objective stochastic discount factor is obtained by perturbing the subjective discount factor by the Radon-Nikodym derivative. This puts more weight on events that are relatively more probable under the subjective than the objective measure.

A natural state variable in the economy is $(z_t, z_t^*, \bar{z}_t, \mathbb{E}_t^S[z_{t+1}], \mathbb{E}_t^S[z_{t+1}^*], \mathbb{E}_t^S[\bar{z}_{t+1}], t) \equiv (\mathbf{Z}_t, t)$, where t affects the economy through the updating coefficients and variances of persistent components. Let us therefore assume that X_{t+1} depends only on these state variables as well as the shocks, that is, $X_{t+1} \equiv$

$X_{t+1}(\mathbf{Z}_{t+1}, \boldsymbol{\epsilon}_{t+1})$. Let us define $\mathcal{H}_{t+1} \equiv (\boldsymbol{\epsilon}_{t+1}, \mathbf{Z}_{t+1})$. \mathcal{H}_{t+1} is Gaussian both under the subjective and objective measure.

Now one can write

$$P_t(\mathbf{Z}_t) = \mathbb{E}[M_{t,t+1} \xi_{t,t+1}(\mathcal{H}_{t+1}|\mathbf{Z}_t) X_{t+1}(\mathcal{H}_{t+1})|\mathbf{Z}_t, t].$$

Because of the Gaussian nature of the state variables and shocks

$$\xi_{t,t+1}(\mathcal{H}_{t+1}|\mathbf{Z}_t) = \frac{\sqrt{|\Sigma_t|}}{\sqrt{|\Sigma_t^S|}} \exp\left(-\frac{1}{2} \phi_{t,t+1}^S(\mathcal{H}_{t+1}|\mathbf{Z}_t) + \frac{1}{2} \phi_{t,t+1}(\mathcal{H}_{t+1}|\mathbf{Z}_t)\right),$$

where the terms inside the brackets are

$$\phi_{t,t+1}^S(\mathcal{H}_{t+1}|\mathbf{Z}_t) = (\mathcal{H}_{t+1} - \mathbb{E}^S[\mathcal{H}_{t+1}|\mathbf{Z}_t])' (\Sigma_t^S)^{-1} (\mathcal{H}_{t+1} - \mathbb{E}^S[\mathcal{H}_{t+1}|\mathbf{Z}_t]).$$

and

$$\phi_{t,t+1}(\mathcal{H}_{t+1}|\mathbf{Z}_t) = (\mathcal{H}_{t+1} - \mathbb{E}[\mathcal{H}_{t+1}|\mathbf{Z}_t])' (\Sigma_t)^{-1} (\mathcal{H}_{t+1} - \mathbb{E}[\mathcal{H}_{t+1}|\mathbf{Z}_t]).$$

Moreover, the covariances matrices are given by

$$\Sigma_t^S = \text{Var}_t^S(\mathcal{H}_{t+1}|\mathbf{Z}_t)$$

$$\Sigma_t = \text{Var}_t(\mathcal{H}_{t+1}|\mathbf{Z}_t).$$

These could be calculated similarly to the variance and covariance terms for the yield volatility calculations. Furthermore, the expectation terms can be solved using the updating formulas for the Kalman filter and the correct processes for the state variables (the shocks have zero expectation under both measures). Note that because $\Sigma_t^S \neq \Sigma_t$, the Radon-Nikodym derivative between the subjective and objective measure is not of the form assumed for example by [Piazzesi et al. \(2015\)](#).

By computing Σ_t^S and Σ_t one can see that these covariance matrices are not degenerate in that no state variable or shock would be perfectly correlated with another state variable or shock or that some shock or state variable had zero variance. This implies that the determinants are non-zero as well as that the inverse covariance matrices are well-defined. Therefore the Radon-Nikodym derivative $\xi_{t,t+1}$ is well defined and strictly positive. Therefore the SDF is strictly positive under the objective measure.

A well-known result in asset pricing is that the existence of a strictly positive stochastic discount factor (or equivalent martingale measure) is equivalent to no-arbitrage [Harrison and Kreps \(1979\)](#). Therefore the economy does not allow for arbitrage opportunities under the objective (or subjective) measure. An analogous derivation could be performed for the foreign stochastic discount factor.

6.8 Decomposing Currency and Bond Premia

The currency premium under the objective measure is given by $\Theta_t = x_t + \mathbb{E}_t[s_{t+1}] - s_t$. On the other hand:

$$s_t = \mathbb{E}_t^S \sum_{j=0}^{\infty} x_{t+j} + \mathbb{E}_t^S \sum_{j=0}^{\infty} \Gamma_{t+j} + \lim_{j \rightarrow \infty} \mathbb{E}_t^S [s_{t+j}],$$

and

$$s_{t+1} = \mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} x_{t+1+j} + \mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} \Gamma_{t+1+j} + \lim_{j \rightarrow \infty} \mathbb{E}_{t+1}^S [s_{t+1+j}],$$

Therefore we can solve

$$\begin{aligned} \Theta_t = & -\Gamma_t + \mathbb{E}_t \left[\mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} x_{t+1+j} - \mathbb{E}_t^S \sum_{j=0}^{\infty} x_{t+1+j} \right] + \\ & + \mathbb{E}_t \left[\mathbb{E}_{t+1}^S \sum_{j=0}^{\infty} \Gamma_{t+1+j} - \mathbb{E}_t^S \sum_{j=0}^{\infty} \Gamma_{t+1+j} \right] + \mathbb{E}_t \left[\lim_{j \rightarrow \infty} \mathbb{E}_{t+1}^S [s_{t+j}] - \lim_{j \rightarrow \infty} \mathbb{E}_t^S [s_{t+j}] \right]. \end{aligned}$$

The decomposition for any predictability coefficient δ related to predictor p_t follows from the fact that

$$\delta_p = \frac{\text{Cov}(\Theta_t, p_t)}{\text{Var}(p_t)}$$

and the linearity of covariance. The decomposition for the bond risk premium follows similarly.

6.9 On Estimating k

This section derives the slope coefficient in the regression where forecast errors are explained by forecast revisions. Similarly to [Coibion and Gorodnichenko \(2012\)](#) we have

$$\mathbb{E}_t^S[x_{t+1}] = (1 - k)\lambda\mathbb{E}_{t-1}^S[x_t] + k\lambda x_t$$

$$\mathbb{E}_t[x_{t+1}] = \lambda x_t.$$

Multiplying the first expression by λ^{j-1} :

$$\mathbb{E}_t^S[x_{t+j}] = (1 - k)\mathbb{E}_{t-1}^S[x_{t+j}] + k\mathbb{E}[x_{t+j}],$$

where we used the property $\mathbb{E}_t^S[x_{t+j}] = \lambda^{j-1}\mathbb{E}_t^S[x_{t+1}]$. From this it follows that

$$k(\mathbb{E}[x_{t+j}] - \mathbb{E}^S[x_{t+j}]) = (1 - k)(\mathbb{E}_t^S[x_{t+j}] - \mathbb{E}_{t-1}^S[x_{t+j}]).$$

Hence

$$x_{t+j} - \mathbb{E}^S[x_{t+j}] = \frac{1 - k}{k}(\mathbb{E}_t^S[x_{t+j}] - \mathbb{E}_{t-1}^S[x_{t+j}]) + u_{t+j},$$

where u_{t+j} is zero mean and orthogonal to time t information. Hence $\beta_1^{FR} = \frac{1-k}{k}$ and $\beta_0^{FR} = 0$.