Optimal Monetary and Fiscal Policy at the Zero Lower Bound in a Small Open Economy*

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Abstract

We study optimal monetary and fiscal policy at the zero lower bound in a small open economy model with sticky prices and a flexible exchange rate. In such a liquidity trap situation, the economy suffers from a negative output gap, producer price deflation, and an appreciated real exchange rate (compared to its efficient level). The extent of these adverse effects and the duration of the liquidity trap is higher, lower is the elasticity of substitution between domestic and foreign goods. Under discretion, compared to commitment, in addition to the usual "deflation bias" present in a closed economy, the equilibrium in a small open economy also features an "overvaluation bias": the real exchange rate is excessively appreciated compared to its efficient level. Countercyclical fiscal policy, that is, increasing government spending above the efficient level during the liquidity trap, constitutes optimal policy and helps decrease the extent of negative output gap and deflation, especially under discretion, but the extent of the increase in government spending is lower when the elasticity of substitution of between domestic and foreign goods is higher.

JEL Classification: E31, E52, E58; E61; E62; E63; F41

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1 Introduction

As a response to the recent global financial crisis and its adverse consequences on output and employment, several central banks of small open economies engaged in countercyclical policy response and lowered their conventional policy instrument, the short-term nominal interest rate. Many of these central banks, such as the Bank of England, the Bank of Canada, and the Bank of Japan, in fact, found themselves frustrated by the zero lower bound (ZLB) on the short-term nominal interest rate and thereby, unable to provide further conventional policy accommodation.

Such a development has raised several questions such as: what constitutes optimal monetary policy when a small open economy finds itself in a liquidity trap situation, where the short-term nominal interest rate has been lowered all the way to zero and cannot be lowered further; is there a significant difference in equilibrium outcomes when the central bank can commit to future actions compared to when it cannot; and, what is the role of the elasticity of substitution between domestic and foreign goods and real exchange rate dynamics in driving equilibrium outcomes? Moreover, given the constraints imposed by the zero lower bound on conventional monetary policy, another natural question that arises is whether fiscal policy, in particular, countercyclical government spending, can be optimal and welfare-improving.

Motivated by these considerations, we study optimal monetary and fiscal policy at zero nominal interest rates, both with and without commitment on the part of the government, in a small open economy with sticky prices and a flexible exchange rate. In particular, we consider a general environment without restrictions on some key model parameters which ensures that the optimal policy problem in an open economy is no longer isomorphic to the closed economy.[1] This implies for example, that the government's objective, which is to maximize the representative household's utility, cannot be summarized purely by home inflation and output gap stabilization.[2] This difference from the closed economy arises because there exists a standard motive related to optimal terms of trade manipulation as the small open economy produces a good that is an imperfect substitute of the foreign good. Overall, in this general environment, monetary policy cannot achieve the efficient outcome, even at positive interest rates, in the presence of standard technology or preference shocks and there is a role for fiscal policy, even at positive interest rates, as government spending optimally deviates from its efficient level.

We show that when the small open economy is in a liquidity trap, it experiences a negative output gap, producer price deflation, and a negative real exchange rate gap (that is, a real exchange rate that is appreciated compared to its efficient level).[3] The extent of these adverse effects and the duration of the liquidity trap is higher, lower is the elasticity of substitution between domestic and foreign goods. This is so because when the elasticity of substitution between domestic and

[1] As we discuss later, the key model parameters are the intertemporal elasticity of substitution and the elasticity of substitution between home and foreign goods.

[2] Moreover, the equilibrium does not necessarily feature balanced trade.

[3] As is standard in open economy models, an increase of the exchange rate implies a depreciation in our model and by home or producer inflation, we refer to inflation of home produced goods. Moreover, throughout the paper, we refer to a "gap"as the difference between a variable and its efficient counterpart.

foreign goods is lower, equilibrium requires a higher response of relative prices to clear the goods market. In a liquidity trap situation, since the price adjustment channel gets severely impaired, this implies that the economy suffers from a bigger output gap, producer price deflation, and a real exchange rate gap. As a result of these adverse effects, optimal monetary policy, both with and without commitment, keeps nominal interest rates at zero for longer when the elasticity of substitution between domestic and foreign goods is lower. Thus, the duration of liquidity trap depends importantly on trade elasticity in a small open economy.

We find that optimal policy under commitment, like in a closed economy, can be expressed in terms of a suitably defined time-varying price-level target that features history dependence, while optimal policy under discretion is purely forward looking. The equilibrium under commitment, compared to discretion, then features a less severe negative output gap and producer price deflation. Thus, the usual "deflation bias" of discretionary policy at the ZLB that is present in a closed economy is also a feature of the small open economy. In particular, under commitment, the central bank is able to promise low real interest rates and a higher output gap in future, which helps mitigate the extent of the negative output gap during the liquidity trap. In a small open economy environment, there is in addition, also a "overvaluation bias" associated with discretionary policy: the real exchange rate is relatively more appreciated. Thus, under optimal policy with commitment, the central bank also promises a more depreciated real exchange rate.[4]

While the commitment outcome is superior to discretion, it is well-known that it suffers from dynamic time inconsistency: the central bank has incentives to renege on its promises in future. We therefore, next analyze joint conduct of optimal monetary and fiscal policy, where the government also chooses optimally the level of (utility-yielding) government spending, an action that involves current actions. Intuitively, increasing government spending during the liquidity trap and/or promising to decrease it in future can be beneficial as it reduces the real interest rate gap, that is one of the main reasons behind adverse outcomes at the ZLB. The reason is that such a path of government spending increases the efficient real interest rate, thereby decreasing the real interest rate gap.

We indeed show that increasing government spending above the efficient level helps decrease adverse outcomes during the liquidity trap, especially under discretion. Optimal fiscal policy thus entails countercyclical government spending, as in a closed economy. In particular, the extent of negative output gap, producer price deflation, and negative real exchange rate gap gets mitigated with higher government spending. Under discretion, government spending increases during the liquidity trap by more compared to commitment while at the same time, unlike commitment, the government spending gap does not go negative in later periods. The reason is that once the government cannot commit, to achieve its goal of decreasing the real interest rate gap, it cannot promise to a negative government spending gap in future. Thus, it solely has to rely on a higher level of government spending today.

[4]Note that this does not necessarily imply that net exports is higher under commitment. This depends on the trade elasticity.

Finally, under both commitment and discretion, the extent of the optimal increase in government spending beyond the efficient level decreases when the elasticity of substitution between domestic and foreign goods is higher. This is because the increase in government spending generates real exchange rate appreciation pressures (and thus increases the government spending gap), which reduces welfare more when the trade elasticity is higher.

Our contribution is to study optimal monetary and fiscal policy, both with and without commitment, in a general small open economy environment, with the main focus on a situation where the ZLB on the short-term nominal interest rate binds. Thus, we are clearly building on a recently burgeoning literature on optimal policy at the ZLB, which is mostly based on closed economy models. In particular, our work is closely related to the set of papers that study optimal monetary and government spending policy in a liquidity trap. Important contributions on optimal monetary policy in a liquidity trap situation in a closed economy context, either under commitment or discretion or both, include Eggertsson and Woodford (2003), Jung, Teranishi, Watanabe (2005), Adam and Billi (2006 and 2007), and Werning (2013). Relatedly, Eggertsson (2006) highlights the "deflation bias" of discretionary policy at zero interest rates and suggests issuing nominal debt as a way to improve on outcomes. In studying the role for government spending in a liquidity trap situation, our work is related to that of Eggertsson (2001) and Werning (2013), who point out the efficacy of countercyclical government spending policy in closed economy models.

In the open economy literature, using two-country models, Jeanne (2009) and Cook and Devereux (2013) study optimal policy, where countries coordinate on their actions, in a global liquidity trap scenario. Our environment of a small open economy provides a different focus, as only the home country is in a liquidity trap situation and it decides on optimal policy taking the rest of the world as given (on which, it exerts a negligible effect). A separate policy relevant role for the terms of trade (or the real exchange rate) in this environment is also a new feature. Our work is closely related to that of Svensson (2003 and 2004) and Jeanne and Svensson (2007), who also use a small open economy model and provide insights related to exchange rate dynamics and how to mitigate adverse outcomes under discretion. The main difference is that we consider an explicitly welfare maximizing government in characterizing optimal monetary and fiscal policy.[5] In terms of methodology, our small open economy set-up is very similar to that of Gali and Monacelli (2005), Faia and Monacelli (2008), and De Paoli (2009), which we augment with a role for government spending when we consider optimal fiscal policy

[5] In the interest of space, we only discuss the literature that focusses on optimal policy in a liquidity trap, but there is by now also a large literature that analyzes various policy relevant issues, such as the effects of government spending, while modelling monetary policy as being governed by a Taylor type rule. Important contributions include, among others, closed economy studies by Wolman (2005), Eggertsson (2011), Christiano, Eichenbaum, and Rebelo (2011), Woodford (2011), and Ercez and Linde (2014), a two-country study by Bodenstein, Erceg, and Guerrieri (2009), and a current union model by Eggertsson, Ferrero, and Raffo (2013).

2 Model

There are two countries, home (H) and foreign (F). The home country is a small open economy. The foreign country is effectively a closed economy as the home country variables have negligible effects on foreign variables.

2.1 Private sector

We first describe the environment faced by households and firms, their optimization problem, and the associated equilibrium conditions. Our model is a standard sticky price set-up along the lines of Woodford (2003). Moreover, it is similar to the small open economy set-up in Faia and Monacelli (2008), augmented with a role for government spending as in Woodford (2003).

2.1.1 Households

A household at home maximizes expected discounted utility over the infinite horizon

$$E_t \sum_{s=0}^{\infty} \beta^s U_{t+s} = E_t \sum_{s=0}^{\infty} \beta^s \left[u\left(C_{t+s}, \xi_{t+s}\right) - \int_0^1 v\left(h_{t+s}(i), \xi_{t+s}\right) di + g\left(G_{t+s}, \xi_{t+s}\right) \right] \tag{1}$$

where β is the discount factor, C_t is household consumption of the composite final good, $h_t(i)$ is quantity suppled of labor of type i, G_t is government consumption of the composite final good, and ξ_t is a vector of aggregate exogenous (domestic) shocks. E_t is the mathematical expectation operator conditional on period-t information, $u\left(.\right)$ is concave and strictly increasing in C_t for any possible value of ξ_t, $g\left(.\right)$ is concave and strictly increasing in G_t for any possible value of ξ_t, and $v\left(.\right)$ is increasing and convex in $h_t(i)$ for any possible value of ξ_t.

The composite household final good is an aggregate of the home, $C_{H,t}$, and foreign, $C_{F,t}$, goods

$$C_t = \left[(1-\alpha)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

where the goods in turn are a standard aggregate of a continuum of varieties indexed by i, $C_{H,t} = \int_0^1 \left[c_{H,t}(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$ and $C_{F,t} = \int_0^1 \left[c_{F,t}(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$. Here, $\eta > 0$ is the elasticity of substitution between the goods, $\varepsilon > 1$ is the elasticity of substitution among the varieties, and $\alpha < 1$ denotes the weight of the foreign, imported good in the home basket and therefore, is the degree of openness.[6] For simplicity, we assume that the composite government final good is defined similarly as an

[6] Note here that for simplicity, we impose the "small open economy limit" already in defining the consumption bundles. A more general notation, following Faia and Monacelli (2008) would be to write $C_t = \left[\left(1-\alpha^{'}\right)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \alpha^{'\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$ with $\alpha^{'} = (1-n)\alpha$, where n is the size and α the trade openness of the home country. Then, $n \to 0$ would constitue the "small open economy limit."

aggregate of the home, $G_{H,t}$, and foreign, $G_{F,t}$, goods

$$G_t = \left[(1-\alpha)^{\frac{1}{\eta}} G_{H,t}^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} G_{F,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

where the goods in turn are an aggregate of a continuum of varieties indexed by i, $G_{H,t} = \int_0^1 \left[g_{H,t}(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$ and $G_{F,t} = \int_0^1 \left[g_{F,t}(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$.

The optimal price index of the composite good P_t is given by

$$P_t = \left[(1-\alpha) P_{H,t}^{1-\eta} + \alpha P_{F,t}^{1-\eta} \right]^{\frac{1}{1-\eta}}$$

where $P_{H,t}$ is the price in home currency of the home good while $P_{F,t}$ is the price in home currency of the foreign good. The demand for the aggregate goods is then given by

$$\begin{aligned} \frac{C_{H,t}}{C_t} &= (1-\alpha) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} \quad & \frac{C_{F,t}}{C_t} &= \alpha \left(\frac{P_{F,t}}{P_t} \right)^{-\eta} \\ \frac{G_{H,t}}{G_t} &= (1-\alpha) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} \quad & \frac{G_{F,t}}{G_t} &= \alpha \left(\frac{P_{F,t}}{P_t} \right)^{-\eta} . \end{aligned}$$

Similarly, the optimal price indices for the home and foreign good are given by $P_{H,t} = \left[\int_0^1 p_{H,t}(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$ and $P_{F,t} = \left[\int_0^1 p_{F,t}(i)^{1-\varepsilon} dj \right]^{\frac{1}{1-\varepsilon}}$ where $p_{H,t}(i)$ is the price in home currency of the home variety i while $p_{F,t}(i)$ is the price in home currency of the foreign variety i. The demand for the individual varieties is then given by $\frac{c_{H,t}(i)}{C_{H,t}} = \left(\frac{p_{H,t}(i)}{P_{H,t}} \right)^{-\varepsilon}$, $\frac{c_{F,t}(i)}{C_{F,t}} = \left(\frac{p_{F,t}(i)}{P_{F,t}} \right)^{-\varepsilon}$, $\frac{g_{H,t}(i)}{G_{H,t}} = \left(\frac{p_{H,t}(i)}{P_{H,t}} \right)^{-\varepsilon}$, and $\frac{g_{F,t}(i)}{G_{F,t}} = \left(\frac{p_{F,t}(i)}{P_{F,t}} \right)^{-\varepsilon}$.

The home household is subject to a sequence of flow budget constraints

$$P_t C_t + B_t + E_t\{\rho_{t,t+1} A_{t+1}\} \leq \int_0^1 n_t(i) h_t(i) di + (1 + i_{t-1}) B_{t-1} + A_t - P_t T_t + \int_0^1 Z_t(i) di \qquad (2)$$

where $n_t(i)$ is nominal wage of labor of type i, $Z_t(i)$ is nominal profit of home firm i, B_t is the household's holding of one-period risk-less nominal government bond at the beginning of period $t+1$, and T_t is (real) government taxes. A_{t+1} is the value of the complete set of state-contingent securities at the beginning of period $t+1$, denominated in home currency for simplicity. Finally, i_{t-1} is the nominal interest rate on government bond holdings at the beginning of period t (which is subject to the zero lower bound $i_t \geq 0$), and $\rho_{t,t+1}$ is the stochastic discount factor between periods t and $t+1$ that is used to value random nominal income in period $t+1$ in monetary units at date t.[7] Note that financial markets are complete both domestically and internationally.[8]

The problem of the home household is thus to choose $\{C_{t+s}, h_{t+s}(i), B_{t+s}, A_{t+s}\}$ to maximize (1) subject to a sequence of flow budget constraints given by (2), while taking as exogenously given

[7] The household is subject to a standard no-Ponzi game condition.

[8] We do not explicitly consider international trade in claims on profits of firms.

initial wealth and $\{P_{t+s}, n_{t+s}(i), i_{t+s}, \rho_{t,t+s}, \xi_{t+s}, Z_{t+s}(i), T_{t+s}\}$.

Moving on to the foreign country, in terms of notation, all foreign variables are denoted by a $*$. Since the home country is a small open economy, the home good will have a negligible weight on the foreign composite consumption good. Thus, we have

$$P_t^* = P_{F,t}^*$$

where P_t^* is the price in terms of foreign currency of the foreign composite good while $P_{F,t}^*$ is the price in terms of foreign currency of the foreign good. Moreover, from the perspective of the home country, foreign private and government consumption, C_t^* and G_t^*, will evolve exogenously. Finally, the foreign demand for the goods and the varieties are given by analogous expressions as above.

2.1.2 Firms

There is a continuum of monopolistically competitive firms indexed by i in the two countries. Each firm at home produces a variety i according to the production function

$$y_t(i) = f(h_t(i), \xi_t) \tag{3}$$

where $f(.)$ is an increasing concave function for any ξ_t, where ξ_t is again a vector of aggregate exogenous (domestic) shocks.

There is no international price discrimination and thus law of one price for each of the traded varieties holds. Thus,

$$p_{H,t}(i) = S_t p_{H,t}^*(i) \text{ and } p_{F,t}(i) = S_t p_{F,t}^*(i)$$

where S_t is the nominal exchange rate.[9] This implies

$$P_{H,t} = S_t P_{H,t}^* \text{ and } P_{F,t} = S_t P_{F,t}^*.$$

Next, define Q_t the real exchange rate as $Q_t = \frac{S_t P_t^*}{P_t}$ and ς_t the terms of trade as $\varsigma_t = \frac{P_{F,t}}{P_{H,t}} = \frac{P_{F,t}^*}{P_{H,t}^*} = \frac{P_t^*}{P_{H,t}^*}$. This implies the following relationships

$$\left[(1-\alpha) + \alpha \varsigma_t^{1-\eta}\right]^{\frac{1}{1-\eta}} = \frac{P_t}{P_{H,t}} = r(\varsigma_t) \tag{4}$$

$$Q_t = \frac{S_t P_t^*}{P_t} = \frac{S_t \frac{P_t^*}{P_{F,t}^*}}{\frac{P_t}{P_{H,t}}} \frac{P_{F,t}^*}{P_{H,t}} = \frac{1}{\left[(1-\alpha) + \alpha \varsigma_t^{1-\eta}\right]^{\frac{1}{1-\eta}}} \frac{P_{F,t}}{P_{H,t}} = \frac{\varsigma_t}{r(\varsigma_t)} = q(\varsigma_t). \tag{5}$$

As in Rotemberg (1983), firms face a cost of changing prices given by $d\left(\frac{p_{H,t}(i)}{p_{H,t-1}(i)}\right)$. We assume, as is standard that $d(1) = d'(1) = 0$ and that $d'(\Pi_H) > 0$ if $\Pi_H > 1$ and $d'(\Pi_H) < 0$ if $\Pi_H < 1$.

[9] An increase of S_t thus implies a depreciation of the home currency in our model.

This adjustment cost makes the firm's pricing problem dynamic. The demand function for variety i is given by

$$\frac{y_t(i)}{Y_t} = \left(\frac{p_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} \tag{6}$$

where Y_t is aggregate world demand. The firm maximizes expected discounted profits

$$E_t \sum_{s=0}^{\infty} \rho_{t,t+s} Z_{t+s}(i) \tag{7}$$

where $\rho_{t,t+s}$ is the stochastic discount factor between periods t and $t+s$. The period profits $Z_t(i)$ are given by

$$Z_t(i) = \left[(1+s)p_{H,t}(i)y_t(i) - n_t(i)h_t(i) - d\left(\frac{p_{H,t}(i)}{p_{H,t-1}(i)}\right)P_{H,t}\right]$$

where s is the production subsidy in steady-state, whose technical role we clarify later. We can re-write $Z_t(i)$ using (3) and (6) as

$$Z_t(i) = \left[(1+s)Y_t p_{H,t}(i)^{1-\varepsilon}P_{H,t}^{\varepsilon} - n_t(i)f^{-1}\left(Y_t p_{H,t}(i)^{-\varepsilon}P_{H,t}^{\varepsilon}\right) - d\left(\frac{p_{H,t}(i)}{p_{H,t-1}(i)}\right)P_{H,t}\right].$$

The problem of the home firm is thus to choose $\{p_{H,t+s}(i)\}$ to maximize (7), while taking as exogenously given $\{P_{H,t+s}, Y_{t+s}, n_{t+s}(i), \rho_{t,t+s}, \xi_{t+s}\}$.

2.1.3 Private sector equilibrium conditions

We can now derive the necessary conditions for equilibrium that arise from the maximization problems of the private sector described above. Households optimality conditions over labor supply and asset holdings are standard and given by

$$\frac{v_h\left(h_t(i), \xi_t\right)}{u_C\left(C_t, \xi_t\right)} = \frac{n_t(i)}{P_t} \tag{8}$$

$$\rho_{t,t+s} = \beta\frac{u_C\left(C_{t+s}, \xi_{t+s}\right)}{u_C\left(C_t, \xi_t\right)}\Pi_{t+s}^{-1} \tag{9}$$

$$\frac{1}{1+i_t} = E_t\left[\beta\frac{u_C\left(C_{t+1}, \xi_{t+1}\right)}{u_C\left(C_t, \xi_t\right)}\Pi_{t+1}^{-1}\right] \text{ with } i_t \geq 0. \tag{10}$$

where $\Pi_t = \frac{P_t}{P_{t-1}}$ and (9) holds for each state of nature.[10] Given the assumption of complete international financial markets, there exist a unique stochastic discount factor. Then, the standard risk-sharing condition (after assuming the same ex-ante wealth distribution), is given by

$$Q_t = q(\varsigma_t) = \frac{u_C\left(C_t^*, \xi_t^*\right)}{u_C\left(C_t, \xi_t\right)}. \tag{11}$$

[10] A standard transversality condition is also a part of these conditions.

Finally, the accounting identity given our definition of $\left[(1-\alpha)+\alpha\varsigma_t^{1-\eta}\right]^{\frac{1}{1-\eta}}=\frac{P_t}{P_{H,t}}=r(\varsigma_t)$ gives

$$\frac{\Pi_t}{\Pi_{H,t}}=\frac{r(\varsigma_t)}{r(\varsigma_{t-1})}. \tag{12}$$

The firm's optimality condition from price-setting is given by

$$\begin{aligned}
&-(1-\varepsilon)(1+s)Y_t p_{H,t}(i)^{-\varepsilon}P_{H,t}^{\varepsilon}+\varepsilon n_t(i)f_y^{-1}(y_t(i))\,Y_t p_{H,t}(i)^{-\varepsilon-1}P_{H,t}^{\varepsilon}+d'\left(\frac{p_{H,t}(i)}{p_{H,t-1}(i)}\right)\frac{P_{H,t}}{p_{H,t-1}(i)}\\
&=E_t\left[\rho_{t,t+1}d'\left(\frac{p_{H,t+1}(i)}{p_{H,t}(i)}\right)\frac{p_{H,t+1}(i)}{p_{H,t}(i)^2}P_{H,t+1}\right]
\end{aligned}$$

Next, we will focus on a symmetric equilibria where all firms charge the same price and produce the same amount of output

$$p_{H,t}(i)=P_{H,t},\ \ y_t(i)=Y_t,\ \ h_t(i)=h_t,\ \ n_t(i)=n_t.$$

Then the firm's optimality condition from price-setting can be written, after using (8) and (9), as

$$\begin{aligned}
&-(1-\varepsilon)(1+s)Y_t+\varepsilon\frac{v_h(h_t,\xi_t)}{u_C(C_t,\xi_t)}P_t f_y^{-1}(Y_t)\,Y_t P_{H,t}^{-1}+d'(\Pi_{H,t})\,\Pi_{H,t}\\
&=E_t\left[\beta\frac{u_C(C_{t+1},\xi_{t+1})}{u_C(C_t,\xi_t)}\Pi_{t+1}^{-1}d'(\Pi_{H,t+1})\,\Pi_{H,t+1}^2\right].
\end{aligned}$$

Lets manipulate further by using $\frac{P_t}{P_{H,t}}=r(\varsigma_t)$ and $\frac{\Pi_{t+1}}{\Pi_{H,t+1}}=\frac{r(\varsigma_{t+1})}{r(\varsigma_t)}$ to obtain

$$\begin{aligned}
&\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C(C_t,\xi_t)-v_h(h_t,\xi_t)\,f_y^{-1}(Y_t)\,r(\varsigma_t)\right]+u_C(C_t,\xi_t)\,d'(\Pi_{H,t})\,\Pi_{H,t}\\
&=E_t\left[\beta u_C(C_{t+1},\xi_{t+1})\,\frac{r(\varsigma_t)}{r(\varsigma_{t+1})}d'(\Pi_{H,t+1})\,\Pi_{H,t+1}\right].
\end{aligned}$$

Finally, we can replace $v_h(h_t,\xi_t)\,f_y^{-1}(Y_t)$ by $\tilde{v}_y(Y_t,\xi_t)$ where $\tilde{v}(y_t(i),\xi_t)=v\left(f^{-1}(y_t(i),\xi_t)\right)$ (Note that $y_t(i)=f(h_t(i),\xi_t)$) to get

$$\begin{aligned}
&\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C(C_t,\xi_t)-\tilde{v}_y(Y_t,\xi_t)\,r(\varsigma_t)\right]+u_C(C_t,\xi_t)\,d'(\Pi_{H,t})\,\Pi_{H,t}\\
&=E_t\left[\beta u_C(C_{t+1},\xi_{t+1})\,\frac{r(\varsigma_t)}{r(\varsigma_{t+1})}d'(\Pi_{H,t+1})\,\Pi_{H,t+1}\right].
\end{aligned} \tag{13}$$

2.2 Government

The government flow budget constraint is given by

$$B_t=(1+i_{t-1})\,B_{t-1}+P_tG_t-P_tT_t$$

where in terms of notation, for simplicity, we are assuming that all government debt is held domestically. We assume that lump-sum taxes are available and so government debt dynamics is irrelevant for the non-fiscal variables. We thus abstract from it later in the paper. We describe the objectives and the problem faced by the government in the next section.

2.3 Market clearing and net exports

Given that the law of one price holds, it is straightforward to derive an exact non-linear resource constraint

$$Y_t = (1-\alpha)\, r(\varsigma_t)^{\eta}\,(C_t + G_t) + \alpha\varsigma_t^{\eta}\,(C_t^* + G_t^*) + d\,(\Pi_{H,t})\,. \tag{14}$$

For future reference, we now derive an expression for the equilibrium trade balance or net exports (NX_t), which we define in real terms as deflated by the home price level as

$$NX_t = \frac{(Y_t P_{H,t} - C_t P_t - G_t P_t)}{P_{H,t}} = Y_t - r(\varsigma_t)\,(C_t + G_t)\,.$$

2.4 Private sector equilibrium

We are now ready to define the private sector equilibrium, that is the set of possible equilibria that are consistent with the private sector equilibrium conditions and the technological constraints on government policy. A private sector equilibrium is a collection of stochastic processes $\{Y_{t+s}, C_{t+s}, \Pi_{t+s}, \Pi_{H,t+s}, r_{t+s}(\varsigma_{t+s}), q_{t+s}(\varsigma_{t+s}), G_{t+s}, \varsigma_{t+s}, i_{t+s}\}$ for $s \geq 0$ that satisfy (3)-(5), (8)-(12), and (14), for each $s \geq 0$, given ς_{t-1} and an exogenous stochastic process for $\{\xi_{t+1}, \xi_{t+1}^*, C_t^*, G_t^*\}$.

3 Equilibrium

We now define the complete equilibrium of our model along with a detailed description of the objectives and commitment ability of the government.

3.1 Recursive representation

It is useful to first derive a recursive representation of the private sector equilibrium that we described above. Define the expectation variable

$$f_t^e = E_t\left[u_C\,(C_{t+1}, \xi_{t+1})\,\Pi_{t+1}^{-1}\right]$$

to write (10) as

$$1 + i_t = \frac{u_C\,(C_t, \xi_t)}{\beta f_t^e}.$$

Next, define another expectation variable

$$S_t^e = E_t\left[u_C\,(C_{t+1}, \xi_{t+1})\,d'\,(\Pi_{H,t+1})\,\frac{\Pi_{H,t+1}}{r(\varsigma_{t+1})}\right]$$

to write (13) as

$$\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C\left(C_t,\xi_t\right)-\tilde{v}_y\left(Y_t,\xi_t\right)r(\varsigma_t)\right]+u_C\left(C_t,\xi_t\right)d'\left(\Pi_{H,t}\right)\Pi_{H,t}=\beta r(\varsigma_t)S_t^e.$$

This means that the necessary and sufficient condition for a private sector equilibrium is that variables $\{Y_t, C_t, \Pi_t, \Pi_{H,t}, G_t, \varsigma_t, i_t\}$ satisfy: (a) the following conditions

$$1+i_t=\frac{u_C\left(C_t,\xi_t\right)}{\beta f_t^e} \tag{15}$$

$$i_t\geq 0 \tag{16}$$

$$\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C\left(C_t,\xi_t\right)-\tilde{v}_y\left(Y_t,\xi_t\right)r(\varsigma_t)\right]+u_C\left(C_t,\xi_t\right)d'\left(\Pi_{H,t}\right)\Pi_{H,t}=\beta r(\varsigma_t)S_t^e \tag{17}$$

$$Y_t=(1-\alpha)\,r(\varsigma_t)^{\eta}\left(C_t+G_t\right)+\alpha\varsigma_t^{\eta}\left(C_t^*+G_t^*\right)+d\left(\Pi_{H,t}\right) \tag{18}$$

$$\frac{\Pi_t}{\Pi_{H,t}}=\frac{r(\varsigma_t)}{r(\varsigma_{t-1})} \tag{19}$$

$$q(\varsigma_t)=\frac{u_C\left(C_t^*,\xi_t^*\right)}{u_C\left(C_t,\xi_t\right)} \tag{20}$$

given b_{t-1} and ς_{t-1} and the expectations f_t^e and S_t^e; (b) expectations are rational so that

$$f_t^e=E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\right] \tag{21}$$

$$S_t^e=E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)d'\left(\Pi_{H,t+1}\right)\frac{\Pi_{H,t+1}}{r(\varsigma_{t+1})}\right]. \tag{22}$$

Note that the possible private sector equilibrium defined above depends only on the (possibly relevant) endogenous state variable ς_{t-1}, domestic shocks ξ_t, and foreign shocks ξ_t^*, C_t^*, G_t^*. Also, note the following definitions

$$r(\varsigma_t)=\left[(1-\alpha)+\alpha\varsigma_t^{1-\eta}\right]^{\frac{1}{1-\eta}} \tag{23}$$

$$q(\varsigma_t)=\frac{\varsigma_t}{r(\varsigma_t)}=\frac{\varsigma_t}{\left[(1-\alpha)+\alpha\varsigma_t^{1-\eta}\right]^{\frac{1}{1-\eta}}}. \tag{24}$$

3.2 Efficient equilibrium

We next characterize the efficient allocation by considering the small open economy's planner's problem, which is to

$$\max\left[U\left(C_t,G_t,\xi_t\right)=u\left(C_t,\xi_t\right)-\tilde{v}\left(Y_t\right)+g\left(G_t,\xi_t\right)\right]$$

subject to the resource constraint and the international risk-sharing condition

$$Y_t = (1-\alpha)\, r(\varsigma_t)^{\eta}\,(C_t + G_t) + \alpha\varsigma_t^{\eta}\,(C_t^* + G_t^*)$$

$$q(\varsigma_t) = \frac{u_C\,(C_t^*, \xi_t^*)}{u_C\,(C_t, \xi_t)}$$

where $r(\varsigma_t) = \left[(1-\alpha) + \alpha\varsigma_t^{1-\eta}\right]^{\frac{1}{1-\eta}}$ and $q(\varsigma_t) = \frac{\varsigma_t}{r(\varsigma_t)} = \frac{\varsigma_t}{\left[(1-\alpha)+\alpha\varsigma_t^{1-\eta}\right]^{\frac{1}{1-\eta}}}$. Note here that the planner's problem is static and the details of the problem and the associated optimality conditions are in the appendix. The efficient allocation is an important benchmark and point of reference for the rest of the paper.

3.3 Commitment equilibrium

We now describe the government's problem when its objective is to maximize the representative household's utility and when it can commit at time t to a fully state-contingent path for its policy instruments i_{t+s} and G_{t+s}. This is also known as the Ramsey problem in the literature. The (Ramsey) policy problem under commitment then is to

$$\max E_t \sum_{s=0}^{\infty} \beta^s \left[U\,(C_{t+s}, G_{t+s}, \xi_{t+s}) = u\,(C_{t+s}, \xi_{t+s}) - \tilde{v}\,(Y_{t+s}) + g\,(G_{t+s}, \xi_{t+s})\right]$$

subject to the private sector equilibrium conditions (15)-(20), the rational expectations restrictions (21)-(22), and the definitions (23)-(24). The details of the problem and the associated optimality conditions are in the appendix. As is well-known, generally, the commitment equilibrium is time-inconsistent.

3.4 Discretion equilibrium

We now describe the government's problem when its objective is to maximize the representative household's utility and when it cannot commit to a fully state-contingent path for its policy instruments i_t and G_t. In particular, it acts with full discretion and chooses the values of its instruments period by period. The solution concept we use for this discretionary equilibrium is that of a Markov-perfect (time-consistent) equilibrium where the government and the private sector take actions simultaneously. The (Markov perfect) policy problem under discretion can be written recursively as

$$J\,(\varsigma_{t-1}, \xi_t, \xi_t^*, C_t^*, G_t^*) = \max\left[u\,(C_t, \xi_t) - \tilde{v}\,(Y_t) + g(G_t, \xi_t) + \beta E_t J\left(\varsigma_t, \xi_{t+1}, \xi_{t+1}^*, C_{t+1}^*, G_{t+1}^*\right)\right]$$

subject to the private sector equilibrium conditions (15)-(20), the rational expectations restrictions (21)-(22), and the definitions (23)-(24). Here, $J\,(\varsigma_{t-1}, \xi_t, \xi_t^*, C_t^*, G_t^*)$ is the value function of the government. The details of the problem and the associated optimality conditions are in the appendix. Note that while here we set up the Markov problem generally as a dynamic problem for the government, we show in the appendix that it reduces to a period-by-period maximization problem since the endogenous state variable ς_{t-1} is not relevant for policy as the constraint (19) never binds.

4 Results

We now present our results, starting with the steady-state of the model and then proceeding to linearized dynamics, both out of and in ZLB, under optimal monetary and fiscal policy. All the details of our derivations, the proofs, and the linearized equilibrium conditions are in the appendix. Throughout the paper, we refer to a "gap"as the difference between a (linearized) variable and its efficient counterpart.

We note that by starting with the non-linear original policy problem first and then linearizing the government optimality and private sector equilibrium conditions, as in Faia and Monacelli (2008) who considered optimal monetary policy with commitment out of ZLB, we can consider general values for some important preference parameters of the model.[11] This is because as shown in Gali and Monacelli (2005), in a small open economy environment, a standard linear-quadratic approach is valid only under strict restrictions on parameter values, that is one where both the intertemporal elasticity of substitution and the elasticity of substitution between domestic and foreign goods is unity.[12] These restrictive conditions negate the terms of trade manipulation motive of the small open economy policy maker and lead to balanced trade in equilibrium.[13]

4.1 Steady state

We first characterize the non-stochastic steady-state when no aggregate shocks are present.[14] Moreover, for the commitment and discretion equilibria, we focus on a positive interest rate steady-state with zero net inflation. Throughout, as is standard, we also focus on a symmetric steady-state across countries. In the proposition below, we present our first main result regarding the appropriate production subsidy that ensures that the efficient, commitment, and discretion steady-states coincide.[15] This is not very straight forward to characterize because without parameter restrictions, in steady-state, there is both the usual monopolistic competition distortion as well as the motive to manipulate the terms of trade in favor of the small open economy.

Proposition 1 *The efficient, commitment, and the discretion (non-stochastic) steady-states coin-*

[11] Khan, King, and Wolman (2003) is a pioneering study that sets-up the non-linear optimal policy problem under commitment in a closed economy model.

[12] For a similar issue in the two-country case, see Benigno and Benigno (2003). Another approach, at least under commitment from a timeless perspective, is to rely on second-order approximation of some equilibrium conditions and derive, finally, a quadratic loss function for the government. De Paoli (2009) takes this approach, following the method in Benigno and Woodford (2012).

[13] Note that the balanced trade result is specific to the case of technology shocks, which has been the focus of most of the literature.

[14] We represent variables at the non-stochastic steady-state without a t subscript.

[15] Faia and Monacelli (2008) does not feature this subsidy as they focus on the commitment solution out of ZLB.

cide when the production subsidy take the form

$$1+s=\frac{\varepsilon}{\varepsilon-1}\left[(1-\alpha)-\eta\alpha\frac{(2-\alpha)}{[1-\alpha]}(C+G)\frac{u_{CC}}{u_C}\right]^{-1}$$

where C and G are related through

$$\left[-\eta\alpha\frac{(2-\alpha)}{[1-\alpha]}(C+G)\frac{u_{CC}}{u_C}+(1-\alpha)\right]=\frac{u_C(1-\alpha)}{g_G}.$$

Proof. In appendix. ■

To get more intuition for this result, let us focus on the usual power utility functional form assumption for $u(.)$ and $g(.)$ that we use in the numerical analysis in the paper and described in detail in the appendix

$$u(C,\xi)=\xi^C\frac{C^{1-\sigma}}{1-\sigma},\ g(G,\xi)=\xi^C\lambda_G\frac{G^{1-\sigma'}}{1-\sigma'}.$$

This then gives as the subsidy

$$1+s=\frac{\varepsilon}{\varepsilon-1}\frac{\lambda_G}{(1-\alpha)}\frac{C^{\sigma}}{G^{\sigma'}}$$

where C and G are related through

$$\sigma\eta\alpha\frac{(2-\alpha)}{[1-\alpha]}+\left(\sigma\eta\alpha\frac{(2-\alpha)}{[1-\alpha]}+1-\alpha\right)\frac{C}{G}=\frac{C^{1-\sigma}}{G^{1-\sigma'}}\frac{(1-\alpha)}{\lambda_G}.$$

Proposition 1, especially the simplified version above under power utility, nests several cases in the literature. For example, it shows that in a closed-economy approximation of $\alpha=0$, we get $1+s=\frac{\epsilon}{\epsilon-1}$, as in Woodford (2003). Without government spending, and restricting to $\sigma=\eta=1$, we get $1+s=(1-\alpha)\frac{\epsilon}{\epsilon-1}$, as in Gali and Monacelli (2005), which accounts for the openness of the economy. Finally, without government spending and for general parameter values, we get as the appropriate subsidy

$$1+s=\left[\left(1-(1-\alpha)^2\right)\eta\sigma+(1-\alpha)^2\right]^{-1}(1-\alpha)\left(\frac{\varepsilon}{\varepsilon-1}\right).$$

This expression shows clearly how the subsidy balances both the motive of the policy maker to manipulate the terms of trade (which is appropriately weighted by $\left(1-(1-\alpha)^2\right)$) as well as the usual motive related to the presence of a markup due to monopolistic competition. Note in particular, that higher η and σ lead to a terms of trade appreciation motive for the policy maker as the expenditure switching effect is enhanced in this case, which means that the small open economy can buy more of the foreign good without having to expend much labor effort. In this case, since there is an incentive to generate home deflation in order to appreciate the terms of trade, the subsidy is then lower than $(1-\alpha)\left(\frac{\varepsilon}{\varepsilon-1}\right)$. On the other hand, the subsidy is higher than

$(1-\alpha)\frac{\varepsilon}{\varepsilon-1}$ when $\eta\sigma < 1$.[16] This is because in that case, there will be an incentive to generate home inflation in order to depreciate the terms of trade.

This result is very useful for us since we will focus on linearized dynamics in response to aggregate shocks. The fact that an appropriate production subsidy ensures that the steady-state is the same among the efficient, commitment, and discretion equilibria provides a very convenient point around which to linearize the non-linear equilibrium conditions. We present results based on such a linear approximation next.

4.2 Linearized private sector equilibrium

We now present the linearized conditions, around the steady-state above, that characterize the private sector equilibrium of our model.[17] For concreteness, we focus on the case of a technology shock only. The following three conditions summarize the private sector equilibrium in our model

$$\begin{aligned}\hat{x_t} = E_t \hat{x_{t+1}} - (1-\alpha)\frac{\theta_C}{\sigma}\hat{r}_t^{gap} + \frac{(2-\alpha)}{1-\alpha}\eta\alpha\left(q_t^{\hat{gap}} - E_t \hat{q_{t+1}^{gap}}\right) \\ + (1-\alpha)\left(G_t^{\hat{gap}} - E_t G_{t+1}^{\hat{gap}}\right)\end{aligned} \tag{25}$$

$$\hat{\imath}_t \geq \beta^{-1} - 1$$

$$\begin{aligned}\hat{\Pi_{H,t}} = \beta E_t \hat{\Pi_{H,t+1}} + \frac{\varepsilon}{2d_1}\frac{\lambda}{\kappa}\left[\theta_C^{\sigma}\frac{(1+\phi-\kappa)}{\kappa} + \theta_C^{\sigma-1}\frac{\sigma}{1-\alpha}\right]\hat{x}_t \\ - \frac{\varepsilon}{2d_1}\frac{\lambda}{\kappa}\sigma\theta_C^{\sigma-1}G_t^{\hat{gap}} + \frac{\varepsilon}{2d_1}\frac{\lambda}{\kappa(1-\alpha)}\left[\theta_C^{\sigma}\alpha - \theta_C^{\sigma-1}\frac{(2-\alpha)}{1-\alpha}\eta\sigma\alpha\right]\hat{q}_t^{gap} + \Psi_{22}\hat{\xi}_t^P\end{aligned} \tag{26}$$

$$\hat{x_t} = (1-\alpha)G_t^{\hat{gap}} + \frac{1}{1-\alpha}\left[(2-\alpha)\eta\alpha + (1-\alpha)^2\frac{\theta_C}{\sigma}\right]\hat{q}_t^{gap} \tag{27}$$

$$\hat{\Pi}_t = \hat{\Pi_{H,t}} + \frac{\alpha}{1-\alpha}\hat{q}_t - \frac{\alpha}{1-\alpha}\hat{q_{t-1}}. \tag{28}$$

Here, $\hat{x}_t$ is the output gap, $\hat{r}_t^{gap}$ is the real interest rate gap (where the real interest rate is $\hat{\imath}_t - E_t\hat{\pi}_{t+1}$), $\hat{q}_t^{gap}$ is the real exchange rate gap, $\hat{\Pi_{H,t}}$ is home inflation, and $G_t^{\hat{gap}}$ is the government spending gap. Moreover, θ_C is the steady-state ratio of consumption to output $\left(\frac{C}{C+G}\right)$ and Ψ_{22} is a function of the structural parameters of the model. The details of the derivations are provided in the appendix. Generally, since $\Psi_{22} \neq 0$, one sees that technology shocks in the small open economy model act like "cost-push" shocks in the closed economy model.

For more intuition on the private sector equilibrium dynamics, we can focus on the special case of $\sigma = \eta = 1$ which negates the terms of trade externality in the model (it leads to $\Psi_{22} = 0$ as

[16] Our result is similar, but not identical, to the one obtained in Farhi and Werning (2012), which features a small open economy environment without government spending. Farhi and Werning (2012) derive the appropriate subsidy that ensures that the flexible price steady-state coincides with the efficient one. The reason for differences in our results is that they consider an incomplete markets model with balanced trade as a restriction on the planner's problem, where the planner chooses pareto weights.

[17] We denote deviation of variable x from its steady-state with $\hat{x}$.

well). Then, with government spending in the model, the system simplifies to

$$\hat{x_t} = E_t \hat{x_{t+1}} - \frac{(2-\alpha)\alpha + (1-\alpha)^2 \theta_C}{(1-\alpha)} \left(\hat{r_t} - \hat{r_t^e}\right) + (1-\alpha)\left(G_t^{\hat{g}ap} - E_t G_{t+1}^{\hat{g}ap}\right) \tag{29}$$

$$\begin{aligned} \hat{\Pi_{H,t}} = {} & \beta E_t \hat{\Pi_{H,t+1}} + \frac{\varepsilon}{2d_1} \frac{\lambda}{\kappa} \theta_C^{\sigma} \left[\frac{(1+\phi-\kappa)}{\kappa} + \frac{1}{(2-\alpha)\alpha + (1-\alpha)^2 \theta_C} \right] \hat{x_t} \\ & - \frac{\varepsilon}{2d_1} \frac{\lambda}{\kappa} \theta_C^{\sigma} \frac{(1-\alpha)}{(2-\alpha)\alpha + (1-\alpha)^2 \theta_C} G_t^{\hat{g}ap} \end{aligned} \tag{30}$$

while with no government spending in the model ($\theta_C = 1,\ \lambda = 1$, and $G_t^{\hat{g}ap} = 0$) , it simplifies to

$$\hat{x_t} = E_t \hat{x_{t+1}} - \frac{1}{(1-\alpha)} \hat{r}_t^{gap} \tag{31}$$

$$\hat{\Pi_{H,t}} = \beta E_t \hat{\Pi_{H,t+1}} + \frac{\varepsilon}{2d_1} \frac{\lambda(1+\phi)}{\kappa^2} \hat{x_t}. \tag{32}$$

The last representation is the case where the monetary policy problem in the small open economy is isomorphic to the closed economy, as in Gali and Monacelli (2005).

We next move on to characterizing optimal monetary and fiscal policy in our model.

4.3 Optimal monetary policy

Before presenting results under jointly optimal monetary and fiscal policy, we first study optimal monetary policy only, both out of and in the ZLB. Thus, for now, the only policy instrument of the government is the short-term nominal interest rate and there is no government spending that yields utility to the household in the model. In terms of the derivations in the appendix of the general model, this model will simply be a specific, nested case.

Studying only optimal monetary policy at the ZLB first provides a comparison with the closed economy literature and also helps clarify the additional, complementary role of fiscal policy when we allow optimal choice of government spending. Moreover, we start with out of the ZLB case, not only because it provides a baseline to interpret the ZLB results, but also because to the best of our knowledge, a systematic comparison between commitment and discretion outcomes even at positive interest rates has not been done before for general parameter values in a small open economy environment. Our derivation of the production subsidy that ensures the coincidence of the efficient, commitment, and discretion steady-states allows us to undertake this exercise in a straight-forward manner, as is often done in the closed-economy literature.

Our approach is to linearize the non-linear government optimality conditions, as well as the private sector equilibrium conditions, around the steady-state that we compute above. We then study the dynamics of the model in the neighborhood of the steady-state when an unexpected shock hits the economy. For the commitment case, as is well-known, generally there exists a time-inconsistency feature of the equilibrium. In particular, the period 0 government optimality conditions are different from period 1 onwards. As in Khan, King, and Wolman (1999), the numerical results we present are based on setting the initial lagrange multipliers that appear in the government optimality conditions to their steady-state values.

We rely on numerical results since analytical results are not available except for the special case of $\sigma = \eta = 1$ when the ZLB does not bind.[18] Our calibration is very standard and we present the parameter values we use in Table 1. For most parts, we use the same parameter values as in Faia and Monacelli (2008), including log-utility ($\sigma = 1$), constant returns to scale in production ($\kappa = 1$) , and trade openness ($\alpha = 0.4$). To conserve space, we then mostly focus on showing results for different values of η, the elasticity of substitution between between domestic and foreign goods, since there is disagreement in the literature regarding a reasonable estimate of this parameter. We have however, undertaken robustness exercises with respect to the intertemporal elasticity of substitution, as well as, trade openness. The scale parameter in the utility function, λ, is chosen so that the steady-state is consistent with our normalization that steady-state output is 1.

Moreover, to conserve space, we focus on technology shocks since that is often the baseline case analyzed in the optimal monetary policy literature. We consider a persistent process with an AR(1) parameter (ρ) of 0.95. The ZLB case then arises when a one-time large enough technology shock hits the economy initially. The model then evolves deterministically after the shock is over and eventually, the economy exits from the liquidity trap. Our results on when we consider a large negative preference shock that drives the economy into the liquidity trap are qualitatively the same. For the computation of the ZLB case, we use the piece-wise linear algorithm developed by Jung, Teranishi, Watanabe (2005), to which we refer the reader for details. We also briefly describe the algorithm in the appendix.[19]

4.3.1 Out of ZLB

We start with the case where a positive technology shock hits the economy and the shock is not big enough to drive the economy into a liquidity trap. We first consider when the government can commit and then move on to the discretion case.

Commitment Figure 1 shows the dynamic response of various model variables under optimal monetary policy with commitment at different value of η, the elasticity of substitution between between domestic and foreign goods.[20] Panel (a) shows the responses of conventional variables. Focussing first on $\eta = 1$ (note that we have already imposed $\sigma = 1$), it is clear that the policy

[18] This is the case analyzed in Gali and Monacelli (2005), under which the open economy policy problem is isomorphic to the closed economy one; a simple linear quadratic approach to optimal policy is valid; and, finally, monetary policy achieves first best with technology shocks.

[19] In modelling the binding ZLB as arising in a perfect foresight environment due to an unexpected, one-time shock that drives the efficient real interest to a large negative value, we follow, among others, Jung, Teranishi, Watanabe (2005), Christiano, Eichenbaum, and Rebelo (2011), and Werning (2013).We use the Dynare based Occbin toolbox for our computations.

[20] Throughout, we consider a shock of size 0.1 for the out of ZLB case.

problem is isomorphic to the closed economy as it entails setting output gap and producer (home) inflation to zero. Panel (b) shows the responses of open economy variables. Again, focussing first on $\eta = 1$ and combining with results from panel (a), it is clear that optimal policy achieves the efficient outcome as the real exchange rate gap and the next exports gap are both zero. In addition, in this special case, both the efficient and the actual level of net exports is zero: the economy with and without frictions features balanced trade. These results follow the analysis of Gali and Monacelli (2005) as the models are the same substantively and while we follow a non-linear approach to optimal policy, the linear-quadratic approach of Gali and Monacelli (2005) is equivalent under $\sigma = \eta = 1$ We can formally state this result.

Proposition 2 *As in Gali and Monacelli (2005), under log-utility and unit elasticity of substitution between domestic and foreign goods* ($\sigma = \eta = 1$), *at positive interest rates, optimal monetary policy with commitment achieves the efficient outcome by setting home-inflation and output gap to zero.*

Proof. In appendix. ■

Moreover, regardless of the value of η, as to be expected, the nominal interest rate and the efficient real interest rate decline while the efficient level of the real exchange rate depreciates when a positive productivity shock perturbs the economy.[21] Generally, however, as also emphasized in Faia and Monacelli (2008), the first-best is not achieved. The central bank now faces a dynamic trade-off between its stabilization objectives. In particular, the real exchange rate now deviates from its efficient level. Moreover, note that the output gap and home inflation move in opposite direction initially, similar to the response under a "markup" shock in the closed economy case. Thus, technology shocks acts like an endogenous mark-up/trade-off shock in this general environment, as we also emphasized in discussing (26) above.[22]

As is intuitive, the real interest rate gap and the real exchange rate gap is higher, lower is the elasticity of substitution between domestic and foreign goods as now (relative) prices have to adjust more to clear the goods market. Finally, whether the output gap, home inflation, and the real exchange rate gap are affected positively or negatively depends crucially on the trade elasticity. Net exports gap however, is always positive.

Discretion Figure 2 shows the dynamic response of various model variables under optimal monetary policy without commitment at different value of η, the elasticity of substitution between domestic and foreign goods. Again, focussing first on $\eta = 1$ (note that we have already imposed $\sigma = 1$), it is clear from panel (a) that the policy problem is isomorphic to the closed economy as it entails setting output gap and producer (home) inflation to zero. Moreover, combining this with panel (b), it is clear that this also achieves the efficient outcome as the real exchange rate gap and

[21] What we mean by the efficient real interest rate is the real interest rate that is consistent with consumption being at the efficient level (and is backed out from a hypothetical consumption euler equation). This is a counterfactual notion.

[22] This is similar to the result in extended versions of the simple closed economy sticky price model, say under either wage stickiness or in a two sector sticky price model, where technology shocks greate a trade-off between several objectives and act like a markup shock in a one-sector model.

the next exports gap are both zero. Thus, in this special case, like in the closed-economy, there is no difference between the commitment and discretion outcomes. We can formally state this result.

Proposition 3 *Under log-utility and unit elasticity of substitution between domestic and foreign goods* ($\sigma = \eta = 1$), *at positive interest rates, optimal monetary policy without commitment achieves the efficient outcome by setting home-inflation and output gap to zero. There is thus no difference between the commitment and discretion outcomes.*

Proof. In appendix. ■

Generally though, the first-best is not achieved and there is a difference between when the central bank can and when it cannot commit to future actions. As is intuitive, it is clear that the outcomes are generally worse (in terms of deviations of variables from their efficient levels) under discretion compared to commitment. At the same time though, this implies that the commitment outcome is time inconsistent: the central bank has incentives to renege on its promises in future.

In order to highlight the differences between commitment and discretion, we show in Figure 3 the responses under $\eta = 0.7$ for an *iid* shock. This case is useful because it highlights how under commitment, there is "history dependence" of outcomes even without persistence of shock.[23] In particular, note that the discretion problem reduces to a period-by-period maximization in our model. As is clear, when the central bank is able to promise to a state contingent future path, it improves current outcomes as inflation, output gap, real exchange rate gap, and net exports gap get affected by less on the period when the shock hits. Thus, discretion suffers from a "stabilization bias,"a feature of forward looking models when the central bank faces a dynamic trade-off between several variables: the economy responds strongly during the current period and then reverts back to steady-state the period after.

4.3.2 In ZLB

We now move on to the case where a large enough technology shock hits the economy and drives the economy into a liquidity trap.[24] We first consider when the government can commit and then move on to the discretion case.

Commitment We first show that we can characterize optimal policy under commitment, taking into account the ZLB, by formulating it in terms of a time-varying price level target.

Proposition 4 *The targeting rule under commitment takes the form of a time-varying price level target where the central bank chooses* i_t *to achieve*

$$p^*_{H,t} = p_{H,t} + \frac{\tilde{\lambda}}{\tilde{\kappa}} \tilde{x}_t$$

[23] Technically, this history dependence arises through lagged lagrange multipliers in the commitment solution.

[24] In our calibration, we consider a shock 3.5 times that of above, which ensures that the ZLB binds across all parameter values and equilibrium concepts

if possible. Otherwise, it sets $i_t = 0$. The target for next period is determined as

$$p^*_{H,t+1} = p^*_{H,t} + \frac{1+\tilde{\kappa}\tilde{\sigma}}{\tilde{\beta}}\left(p^*_{H,t} - p_{H,t} - \frac{\tilde{\lambda}}{\tilde{\kappa}}\tilde{x}_t\right) - \frac{1}{\tilde{\beta}}\left(p^*_{H,t-1} - p_{H,t-1} - \frac{\tilde{\lambda}}{\tilde{\kappa}}\tilde{x}_{t-1}\right).$$

Here,

$$\tilde{x}_t = \Phi_1 \hat{x}_t + \Phi_2 \hat{q}_t^{gap} + \Phi_3 \hat{\xi_t^P}$$
$$\hat{x}_t = \frac{1}{1-\alpha}\left[(2-\alpha)\,\eta\alpha + \frac{(1-\alpha)^2}{\sigma}\right]\hat{q}_t^{gap}$$

where $\Phi_3 = 0$ under $\sigma = \eta = 1$ and Φ_1, Φ_2, $\tilde{\kappa}$, $\tilde{\sigma}$, $\tilde{\beta}$, and $\tilde{\lambda}$ functions of the structural parameters of the model.

Proof. In appendix. ■

Note here that generally, unlike in the closed economy case in Eggertsson and Woodford (2003) or in the special case of $\sigma = \eta = 1$, we cannot simply express $\tilde{x}_t$ as a function of deviations of variables from the efficient levels. That is, generally, $\tilde{x}_t$ also includes the shock $\hat{\xi_t^P}$.[25] In spite of this caveat however, the main insight of this proposition is that under commitment, the targeting rule takes the form of a suitably re-defined price level target that is time-varying and features history-dependence. Moreover, note that we can re-write this as a traditional targeting rule in terms of gaps and without any shocks by re-defining the target levels appropriately (rather than posit the targets as the efficient levels). With the same re-definition, we can then express private sector equilibrium conditions as a function of these gaps as well. We provide the complete derivations in that case in the appendix.[26]

Moreover, this Proposition provides another characterization of our analytical results above for $\sigma = \eta = 1$ at positive interest rates, since then, the targeting rule takes the form $p_{H,t} = -\frac{\tilde{\lambda}}{\tilde{\kappa}}\tilde{x}_t = -\frac{\tilde{\lambda}}{\tilde{\kappa}}\left(\Phi_1 \hat{x}_t + \Phi_2 \hat{q}_t^{gap}\right)$, which combined with $\hat{x}_t = \frac{1}{1-\alpha}\left[(2-\alpha)\,\eta\alpha + \frac{(1-\alpha)^2}{\sigma}\right]\hat{q}_t^{gap}$, (31), and (32), shows clearly that equilibrium implies $\Pi_{H,t} = \tilde{x}_t = \hat{q}_t^{gap}$.

Next, we show numerical illustrations. Figure 4 shows the dynamic response of various model variables under optimal monetary policy with commitment at the ZLB. Note first here that the shock is big enough to drive the efficient real interest rate very low, which in turn makes the ZLB bind. Unable to reduce the nominal interest rate further and decrease the real interest rate, the central bank now has to confront a quite positive real interest rate gap (the difference between the real interest rate and its efficient level). As is standard in the closed economy literature and shown in panel (a), the equilibrium thereby features a negative output gap and producer price (home) deflation initially. Moreover, now in an open economy, the ZLB also leads to a positive real exchange rate gap as shown in panel (b): the real exchange rate is appreciated compared to its efficient level.

[25] Thus it might be not entirely in the spirit of the closed economy model to call this a "targeting rule."
[26] Our redefinition of the targets and the gaps is similar in form to that of De Paoli (2009).

The extent of these adverse effects and the duration of the liquidity trap is higher, lower is the elasticity of substitution between domestic and foreign goods. This is so because when the elasticity of substitution between domestic and foreign goods is lower, equilibrium requires a higher response of relative prices to clear the goods market. In a liquidity trap situation, since the price adjustment channel gets severely impaired, this implies that the economy suffers from a bigger output gap, producer price deflation, and a real exchange rate gap. As a result of these adverse effects, optimal monetary policy, keeps nominal interest rates at zero for longer when the elasticity of substitution between domestic and foreign goods is lower.[27] Thus, the duration of liquidity trap depends importantly on the trade elasticity in a small open economy.

Discretion We next show that we can characterize optimal policy under discretion by formulating it in terms of an inflation target.

Proposition 5 *The targeting rule under discretion takes the form of an inflation target where the central bank chooses i_t to achieve*

$$\Pi^*_{H,t} = \hat{\Pi_{H,t}} + \frac{\tilde{\lambda}}{\tilde{\kappa}}\tilde{x}_t$$

if possible. Otherwise, it sets $i_t = 0$. Here,

$$\tilde{x}_t = \Phi_1 \hat{x_t} + \Phi_2 \hat{q}_t^{gap} + \Phi_3 \hat{\xi_t^P}$$
$$\hat{x_t} = \frac{1}{1-\alpha}\left[(2-\alpha)\eta\alpha + \frac{(1-\alpha)^2}{\sigma}\right]\hat{q}_t^{gap}$$

where $\Phi_3 = 0$ under $\sigma = \eta = 1$ and Φ_1 and Φ_2 are functions of the structural parameters of the model.

Proof. In appendix. ■

Thus, under discretion, optimal policy does not feature history dependence, a feature that contributes to making outcomes significantly worse compared to commitment. Moreover, this Proposition provides another characterization of our analytical results above for $\sigma = \eta = 1$ at positive interest rates, since then, the targeting rule takes the form $\hat{\Pi_{H,t}} = -\frac{\tilde{\lambda}}{\tilde{\kappa}}\tilde{x}_t = -\frac{\tilde{\lambda}}{\tilde{\kappa}}\left(\Phi_1\hat{x}_t + \Phi_2\hat{q}_t^{gap}\right),$ which combined with $\hat{x_t} = \frac{1}{1-\alpha}\left[(2-\alpha)\eta\alpha + \frac{(1-\alpha)^2}{\sigma}\right]\hat{q}_t^{gap}$, (31), and (32), shows clearly that equilibrium implies $\Pi_{H,t} = \tilde{x}_t = \hat{q}_t^{gap}$. This is another illustration of the result that in this special case, under both commitment and discretion, the first-best allocation can be achieved.

Figure 5 shows the dynamic response of various model variables under optimal monetary policy without commitment at the ZLB. First note that like under commitment, the extent of adverse effects during a liquidity trap are higher, lower is the elasticity of substitution between domestic and foreign goods. Next, several results stand out when compared to commitment. The equilibrium under commitment, compared to discretion, features a less severe negative output gap and producer price deflation, as shown clearly by comparing panel (a)s of the two Figures. Thus, the usual

[27] This is true regardless of whether the government can commit or not as is clear next.

“deflation bias”of discretionary policy at zero interest rates that is present in a closed economy is also a feature of the small open economy. In particular, under commitment, the central bank is able to promise low real interest rates and a higher output gap in future, which helps mitigate the extent of the negative output gap during the initial periods. As part of the equilibrium, optimal commitment policy involves an increase in home inflation in later periods as well

In a small open economy environment, there is in addition, also a “overvaluation bias” associated with discretionary policy: the real exchange rate is relatively more appreciated. This can be seen clearly by comparing panel (b)s of the two Figures. Thus, under optimal policy under commitment, the central bank promises a more depreciated real exchange rate. Contrary to what might appear intuitive however, whether the net exports gap is more negative under discretion compared to commitment depends on the trade elasticity. Thus, it is generally not true that if possible, a discretionary central bank at the ZLB would like to commit to a higher level of net exports in future.

In order to highlight the differences between commitment and discretion, like before, we show in Figure 6 the responses under $\eta = 0.7$ for an *iid* shock. This comparison is particularly informative since now the duration of zero nominal interest rates is simply one period for the discretion case. The usual “deflation bias” (panel (a)) and the new “overvaluation bias” (panel (b)) of discretionary monetary policy in a liquidity trap are now especially clear. In particular, under commitment, the central bank promises a future low real interest rate gap and associated with it, a less negative output gap, higher inflation, and a more depreciated real exchange rate. In this parameterization, this is achieved by keeping the nominal interest rate at zero for several additional periods after the shock is over.

4.4 Optimal monetary and fiscal policy

We now move on to considering joint conduct of optimal monetary and fiscal policy. The policy instruments of the government are now the short-term nominal interest rate and the level of public spending. As given in Table 1, we choose the steady-state government spending-to-output ratio, θ_G, of 0.2. The scale parameter in the utility function, λ_G, is then chosen so that the steady-state of the model is consistent with this choice.

One of the main motivations for this extension is that while the commitment outcome under optimal monetary policy is superior to discretion, it suffers from dynamic time inconsistency: the central bank has incentives to renege on its promises in future. This manifests itself at the ZLB via a much larger negative output gap, producer price deflation, and an appreciated real exchange rate. Thus, we allow the government to choose optimally the level of (utility-yielding) government spending, an action that involves current actions. Intuitively, increasing government spending initially and/or promising to decrease it in future can be beneficial since it reduces the real interest rate gap that is one of the main reasons behind adverse outcomes at the ZLB. The reason is that the efficient real interest rate in the model can be expressed as

$$\begin{aligned}\hat{r}_t^e &= -\frac{\sigma\theta_C^{-1}}{1-\alpha}\left[\hat{Y}_t - E_t\hat{Y_{t+1}}\right] + \sigma\theta_C^{-1}(2-\alpha)\eta\alpha\left[\hat{q}_t - E_t\hat{q}_{t+1}\right] + \sigma\theta_C^{-1}\left[\hat{G}_t - E_t\hat{G_{t+1}}\right] \\ &= -\frac{(1-\alpha)}{(2-\alpha)\eta\alpha + (1-\alpha)^2\frac{\theta_C}{\sigma}}\left[\hat{Y}_t - E_t\hat{Y_{t+1}}\right] + \frac{(1-\alpha)^2}{(2-\alpha)\eta\alpha + (1-\alpha)^2\frac{\theta_C}{\sigma}}\left[\hat{G}_t - E_t\hat{G_{t+1}}\right].\end{aligned} \tag{33}$$

Thus, when the real interest rate gap, $\hat{r}_t - \hat{r}_t^e$ is high because of the ZLB that precludes a reduction in $\hat{r}_t$, one way to reduce the gap is to increase $\hat{r}_t^e$ by increasing $\hat{G}_t - E_t\hat{G_{t+1}}$. Under commitment, both the $\hat{G}_t$ and $E_t\hat{G_{t+1}}$ channels will be used, while under discretion, since policy can only take current actions, only the $\hat{G}_t$ channel will be used.

4.4.1 Out of ZLB

We again start with the case where a positive technology shock hits the economy and the shock is not big enough to drive the economy into a liquidity trap. We first consider when the government can commit and then move on to the discretion case. Even out of ZLB, our independent contribution is to analyze how government spending responds optimally since this case has not been considered in the literature.

Commitment Figure 7 shows the dynamic response of various model variables under optimal monetary and fiscal policy with commitment at different value of η, the elasticity of substitution between between domestic and foreign goods. We focus mostly on a discussion of panel (c) since that shows the response of government spending, which is the new aspect here. Focussing first on $\eta = 1$ (note that we have already imposed $\sigma = 1$), it is clear that now, compared to when the model only featured monetary policy, optimal policy now also entails setting government spending at its efficient level (the government spending gap is thus zero). In this case, optimal monetary and fiscal policy is able to attain the first-best as all the relevant gaps are zero. This thus generalizes the result for monetary policy in Gali and Monacelli (2005) for this specific parameterization. One can also obtain intuition for this result by noticing from (29)-(30) that the private sector equilibrium is consistent with $\Pi_{H,t} = \hat{x}_t = \hat{q}_t^{gap} = G_t^{\hat{gap}} = 0$. We state this result formally below.

Proposition 6 *Under log-utility and unit elasticity of substitution between domestic and foreign goods* ($\sigma = \eta = 1$)*, at positive interest rates, optimal monetary and fiscal policy with commitment achieves the efficient outcome by setting home-inflation, output gap, and government spending gap to zero.*

Proof. In appendix. ■

Generally, for any value of η however, the first-best is not achieved. In particular, generally, it is optimal to let government spending to deviate from its efficient level. Whether government spending increases or decreases compared to the efficient level however, depends crucially on the trade elasticity. In particular, the government spending gap is positive for low values of η. Overall, regardless of a positive or negative government spending gap, optimal fiscal policy leads to a smaller real interest rate gap compared to only optimal monetary policy. This in turn lead to a lower effect on output gap, home inflation, and real exchange rate gap. There is thus an *independent* and *complementary* role for government spending even at positive interest rates in a general small open economy environment.

Discretion Figure 8 shows the dynamic response of various model variables under optimal monetary and fiscal policy without commitment at different value of η, the elasticity of substitution between domestic and foreign goods. Focussing on panel (c) and first on $\eta = 1$ (note that we have already imposed $\sigma = 1$), it is clear that now, compared to when the model only featured monetary policy, optimal policy now also entails setting government spending at its efficient level (the government spending gap is thus zero). In this case, optimal monetary and fiscal policy is again able to attain the first-best and there is no difference between the commitment and discretion outcomes. We state this result formally below.

Proposition 7 *Under log-utility and unit elasticity of substitution between domestic and foreign goods* ($\sigma = \eta = 1$), *at positive interest rates, optimal monetary and fiscal policy without commitment achieves the efficient outcome by setting home-inflation, output gap, and government spending gap to zero. There is thus no difference between the commitment and discretion outcomes.*

Proof. In appendix. ■

Generally though, the first-best is not achieved and there is a difference between when the central bank can and when it cannot commit to future actions. As is intuitive, it is clear that the outcomes are worse (in terms of deviations of variables from their efficient levels) under discretion compared to commitment. The most interesting comparison between the two cases is how the path of government spending gap is different: under commitment, there is a hump-shaped response that is absent under discretion. This is because under commitment, the government can affect the real interest rate gap by both current-period actions as well as those in future periods. In contrast, under discretion, since the maximization problem is period-by-period, the responses of government spending simply follow the persistence of the shock.

4.4.2 In ZLB

We now move on to the case where a large enough technology shock hits the economy and drives the economy into a liquidity trap. We first consider when the government can commit and then move on to the discretion case.

Commitment Figure 9 shows the dynamic response of various model variables under optimal monetary and fiscal policy with commitment at the ZLB. We focus on discussing the dynamics of government spending in panel (c) since that is new now compared to before. As is clear, optimal fiscal policy entails a positive government expenditure gap for several periods initially, which then reverses to a negative output gap eventually. Comparing with the dynamics of output gap, then it is clear that optimal fiscal policy entails countercyclical government spending. This response is optimal because by both increasing government spending initially, and then promising to decrease it in future once the economy has recovered, the government is able to reduce the real interest rate gap during the liquidity trap. This is clear from the expression for the efficient real interest rate in (33). This then improves outcomes at the ZLB with respect to negative output gap, deflation, and the real exchange rate gap.[28]

The extent of the optimal increase in government spending beyond the efficient level decreases when the elasticity of substitution between domestic and foreign goods is higher. This is because the increase in government spending generates real exchange rate appreciation, which reduces welfare more when the elasticity of substitution between domestic and foreign goods is higher. This is true both under commitment as well as discretion, which we discuss next.

Discretion Figure 10 shows the dynamic response of various model variables under optimal monetary policy without commitment at the ZLB. Again, we focus on discussing the dynamics of government spending since that is new now compared to before. Note that now under discretion, government spending increases by more initially compared to commitment. In particular, government spending gap does not go negative in later periods, unlike commitment. The reason is that now since the government cannot commit, to achieve its goal of decreasing the real interest rate gap, it cannot promise a negative government spending gap in future. Thus, it solely has to rely on a higher level of government spending today. Even then however, this additional policy tool allows it to improve outcomes on output gap, deflation, and real exchange rate gap, all of which now are affected less compared to when we considered only discretionary monetary policy.

5 Conclusion

In this paper, we studied optimal monetary and fiscal policy at zero nominal interest rates, both with and without commitment, in a small open economy with sticky prices. In such a liquidity trap situation, the economy suffers from a negative output gap, producer price deflation, and an appreciated real exchange rate (compared to its efficient level). The extent of these adverse effects and the duration of the liquidity trap is higher, lower is the elasticity of substitution of between domestic and foreign goods. Under discretion, compared to commitment, in addition to the usual "deflation bias" present in a closed economy, the equilibrium in a small open economy also features an "overvaluation bias": the real exchange rate is excessively appreciated compared to its efficient level. Countercyclical fiscal policy, that is, increasing government spending above the efficient level during the liquidity trap, constitutes optimal policy and helps decrease the extent of negative output gap and deflation, especially under discretion, but the extent of the increase in government

[28] Note however that while the government spending gap is positive initially always, whether the level of government spending increases or decreases depends on the trade elasticity.

spending is lower when the elasticity of substitution of between domestic and foreign goods is higher.

In future work, we can extend our work in both theoretical and quantitative directions. First, while we have only considered optimal government spending policy, in closed economy models, Eggertsson and Woodford (2006) and Correia et al (2013) have shown that with enough tax instruments, fiscal policy can achieve the first-best even at the ZLB. It will be interesting to do a similar exercise in our small open economy set-up and our result on the nature of the appropriate production subsidy that ensures that the steady-state is efficient can help guide us. Second, we have considered a model where there is perfect risk-sharing between the small open economy and the rest of the world. Relaxing this assumption by considering incomplete markets would provide additional role for real exchange rate and trade imbalances in driving optimal policy decisions. Finally, to evaluate more thoroughly the quantitative predictions of our model, we can consider a set-up that features more frictions such as sticky wages and pricing to market/local currency pricing.

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6 Tables and Figures

Parameter	Value	Parameter	Value
β	0.99	η	0.7, 1, 2
$\sigma = \sigma'$	1	κ	1
α	0.4	d_1	75/2
ϕ	3	ε	7.5
ρ	0.95	θ_G	0.2

Table 1: General calibration and ZLB related parameters

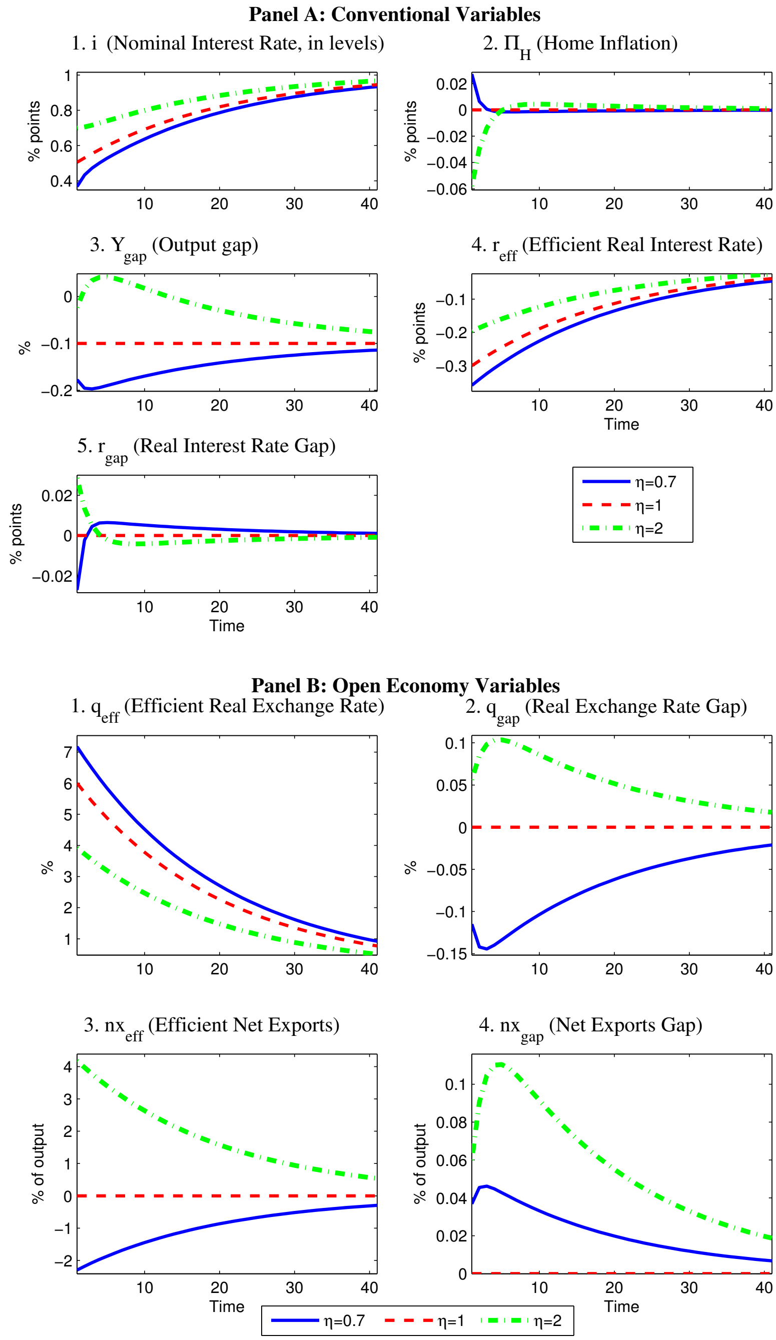

Figure 1: Response under optimal monetary policy (commitment) without ZLB

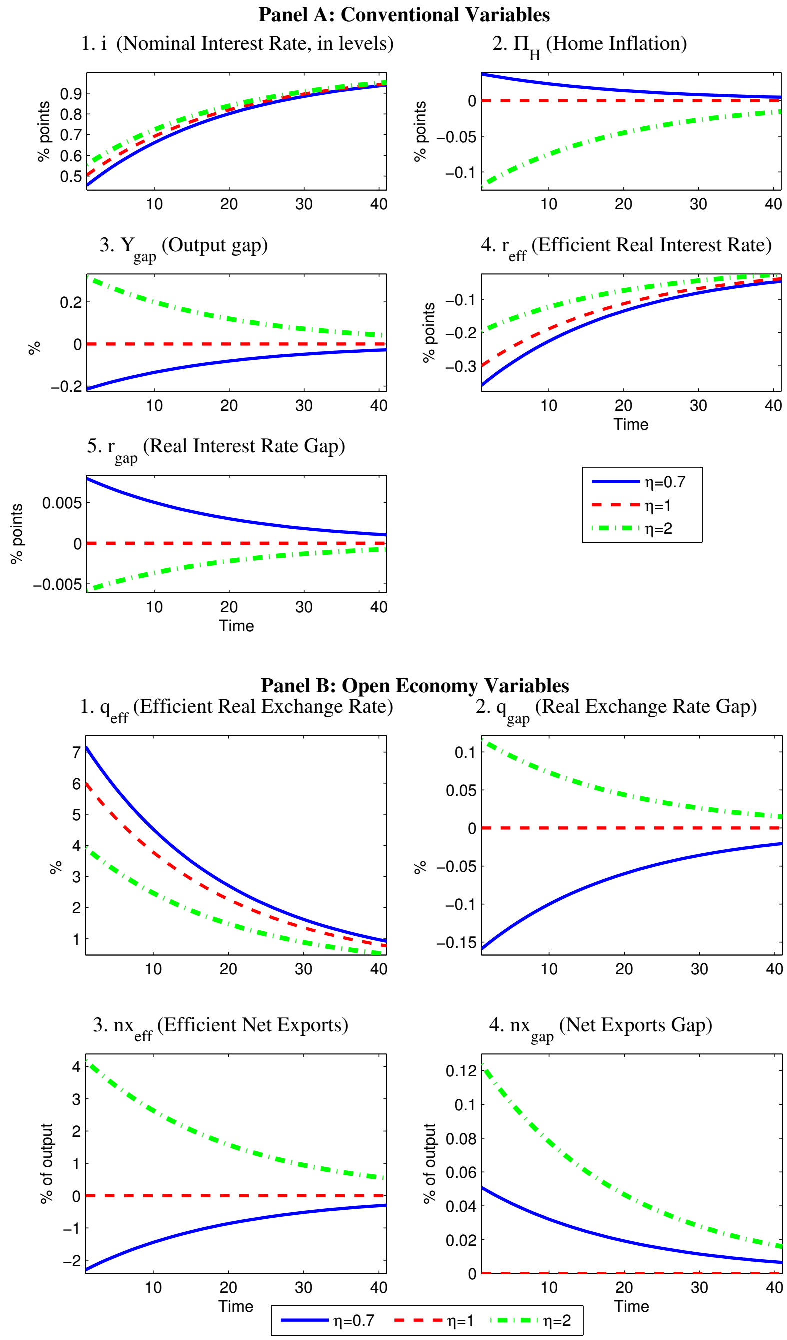

Figure 2: Response under optimal monetary policy (discretion) without ZLB

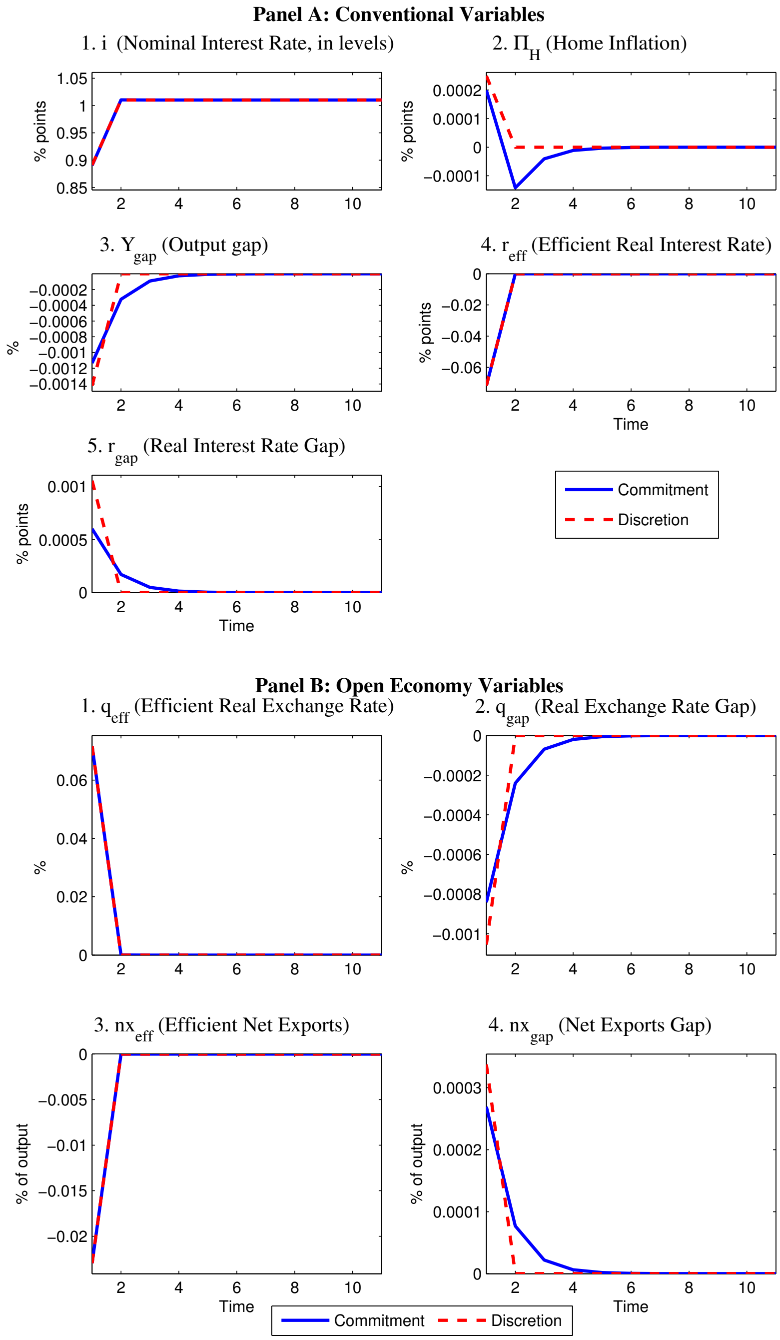

Figure 3: Response under optimal monetary policy without ZLB (iid shock)

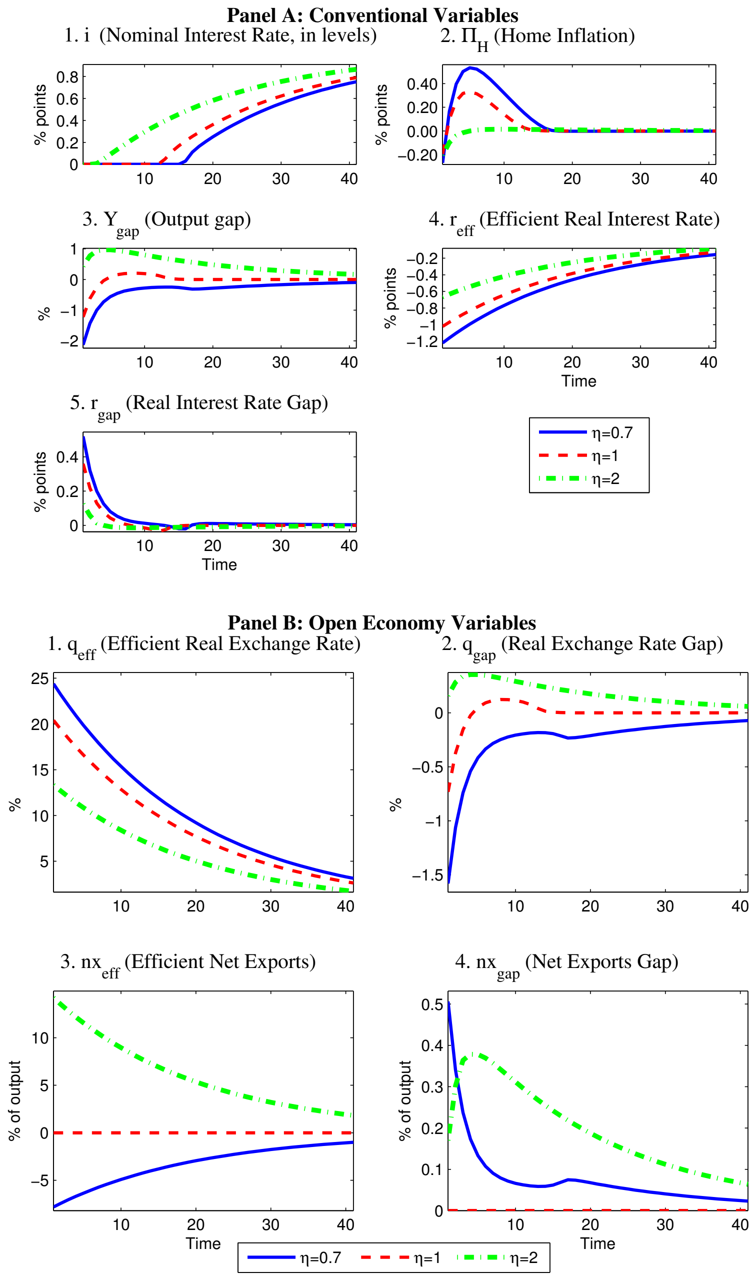

Figure 4: Response under optimal monetary policy (commitment) with ZLB

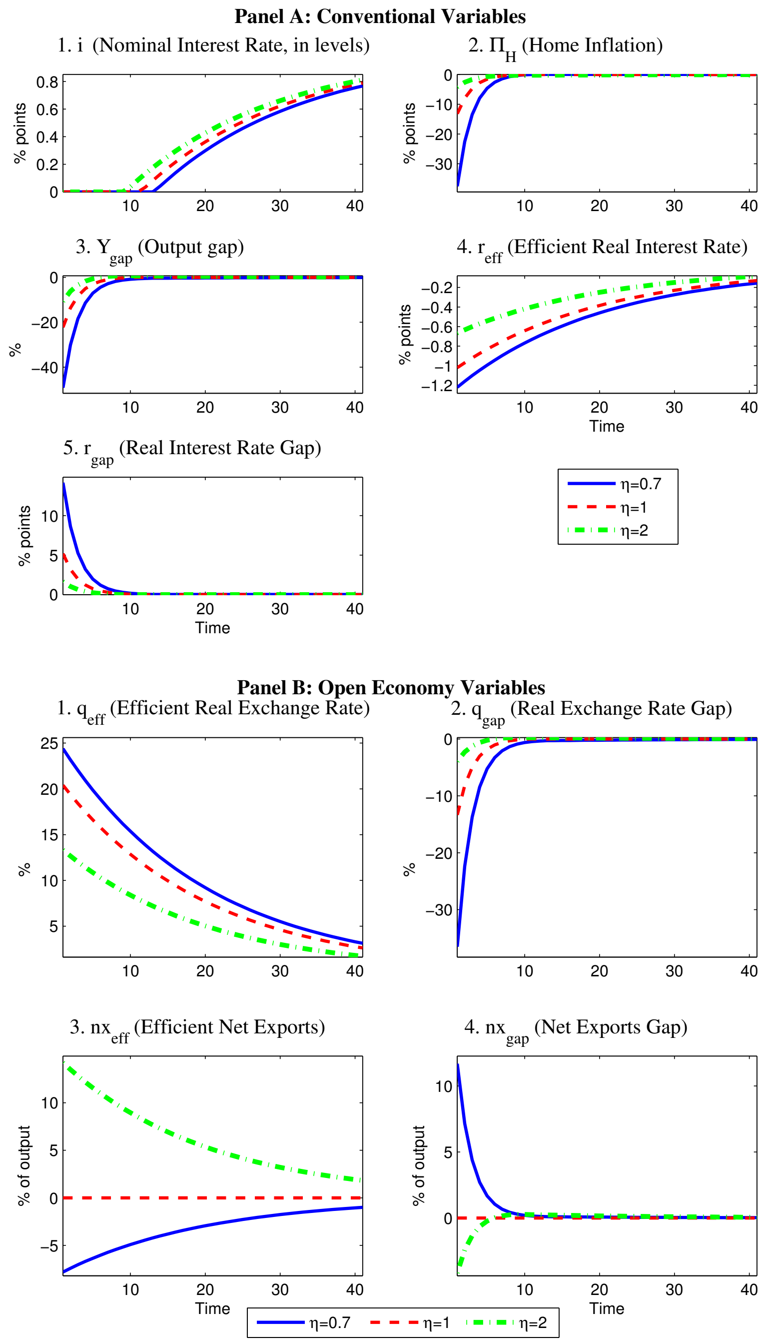

Figure 5: Response under optimal monetary policy (discretion) with ZLB

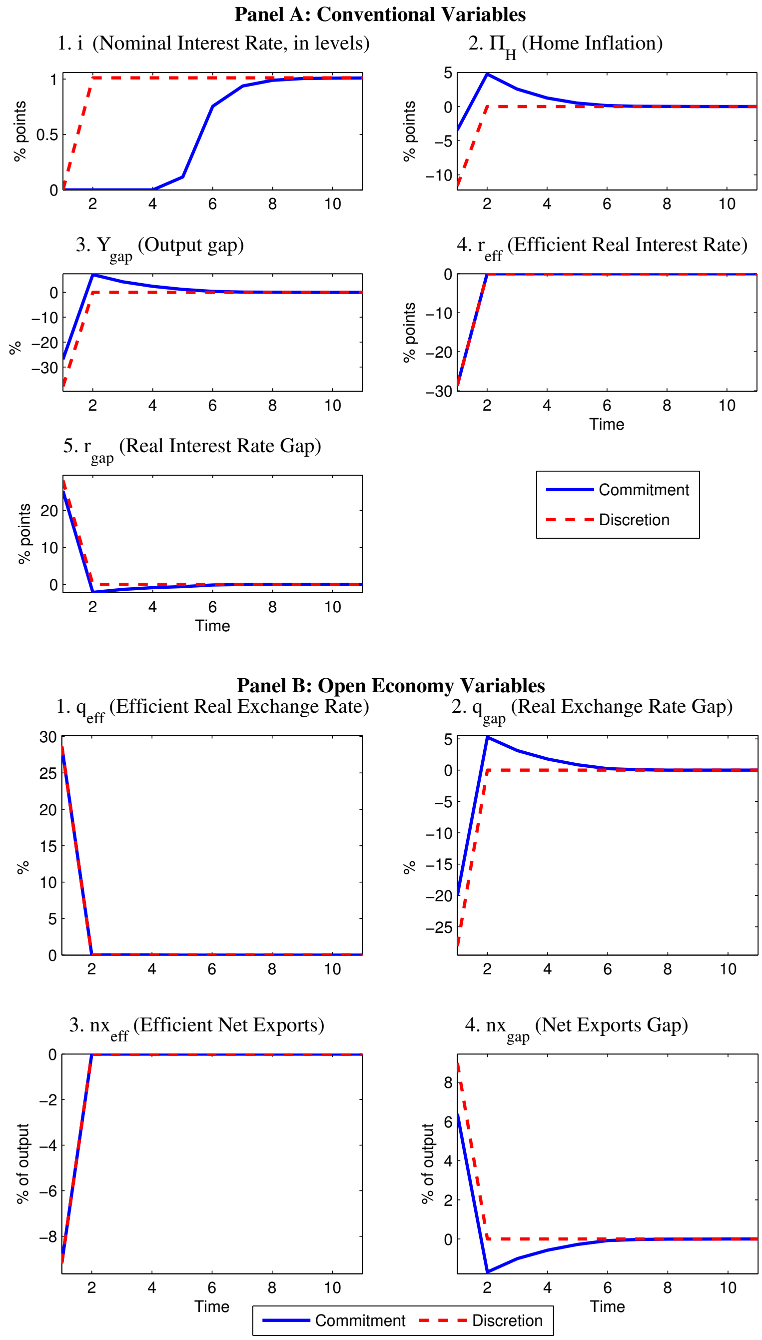

Figure 6: Response under optimal monetary policy with ZLB (iid shock)

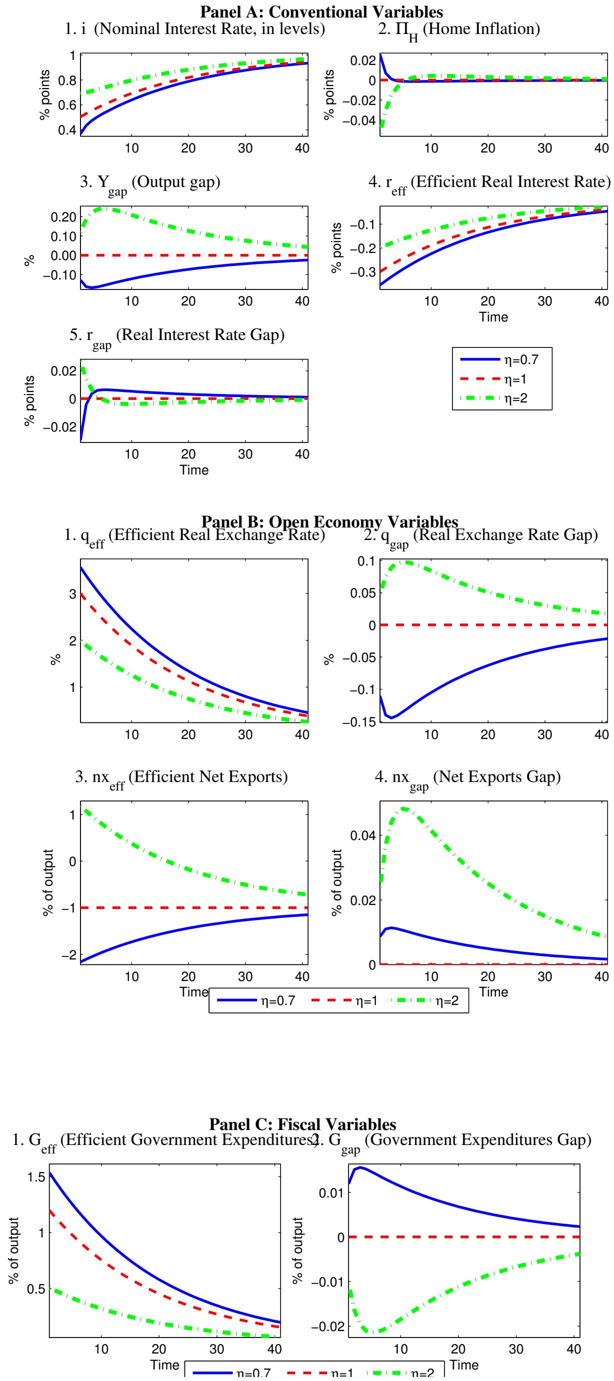

Figure 7: Response under optimal monetary and fiscal policy (commitment) without ZLB

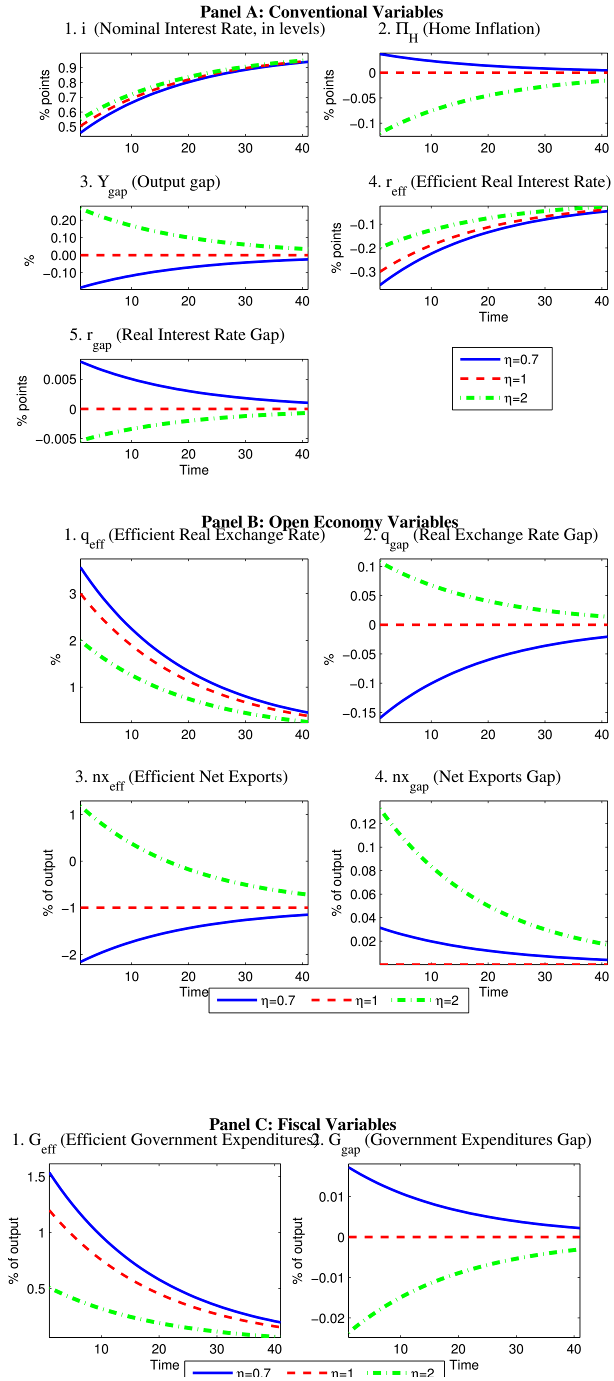

Figure 8: Response under optimal monetary and fiscal policy (discretion) without ZLB

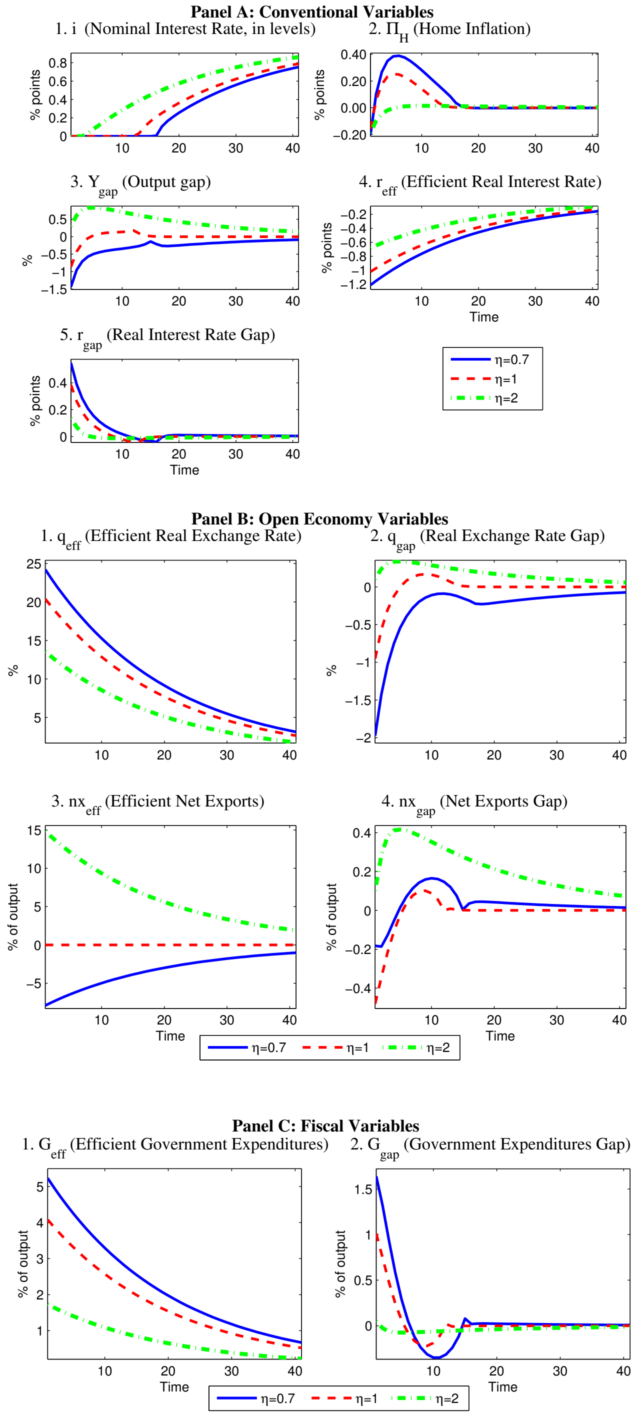

Figure 9: Response under optimal monetary and fiscal policy (commitment) with ZLB

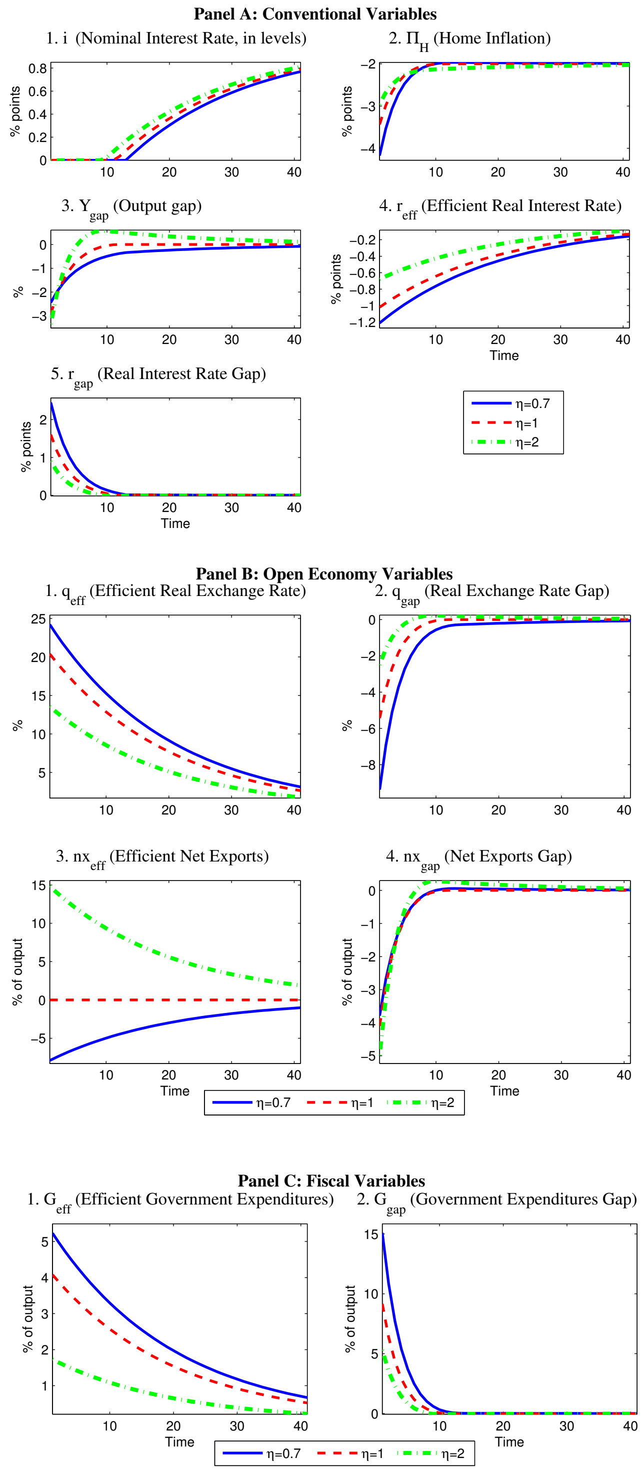

Figure 10: Response under optimal monetary and fiscal policy (discretion) with ZLB

7 Appendix

7.1 Functional forms

Throughout the paper let us assume the following functional forms (note that ξ is a vector containing ξ^C andξ^P)

$$u\left(C,\xi\right)=\xi^C\frac{C^{1-\sigma}}{1-\sigma}$$

$$v\left(h(i),\xi\right)=\lambda\xi^C\frac{h(i)^{1+\phi}}{1+\phi}$$

$$y(i)=\xi^P h(i)^{\kappa}$$

$$d(\Pi)=d_1\left(\Pi-1\right)^2$$

$$g\left(G,\xi\right)=\xi^C\lambda_G\frac{G^{1-\sigma'}}{1-\sigma'}$$

and thus

$$u_C=\xi^C C^{-\sigma},\quad u_{CC}=-\sigma\xi^C C^{-1-\sigma},\quad u_{CCC}=\left(1+\sigma\right)\sigma\xi^C C^{-2-\sigma}$$

$$u_{C\xi}=C^{-\sigma},\quad u_{CC\xi}=-\sigma C^{-1-\sigma}$$

$$d'(\Pi)=2d_1\left(\Pi-1\right),\quad d''(\Pi)=2d_1,\quad d'''(\Pi)=0$$

$$g_G=\xi^C\lambda_G G^{-\sigma'},\quad g_{GG}=-\sigma'\xi^C\lambda_G G^{-\sigma'-1},\quad g_{G\xi}=\lambda_G G^{-\sigma'}$$

$$\tilde{v}\left(y(i),\xi\right)=\frac{1}{1+\phi}\lambda\xi^C\frac{y(i)^{\frac{1+\phi}{\kappa}}}{\xi^{P\frac{1+\phi}{\kappa}}}$$

$$\tilde{v}_Y=\frac{1}{\kappa}\lambda\xi^C\frac{y^{\frac{1+\phi-\kappa}{\kappa}}}{\xi^{P\frac{1+\phi}{\kappa}}},\quad \tilde{v}_{YY}=\frac{1+\phi-\kappa}{\kappa^2}\lambda\xi^C\frac{y^{\frac{1+\phi-2\kappa}{\kappa}}}{\xi^{P\frac{1+\phi}{\kappa}}},\quad \tilde{v}_{YYY}=\frac{1+\phi-2\kappa}{\kappa}\frac{1+\phi-\kappa}{\kappa^2}\lambda\xi^C\frac{y^{\frac{1+\phi-3\kappa}{\kappa}}}{\xi^{P\frac{1+\phi}{\kappa}}}$$

$$\tilde{v}_{Y\xi^P}=-\frac{1+\phi}{\kappa^2}\lambda\xi^C\frac{y^{\frac{1+\phi-\kappa}{\kappa}}}{\xi^{P\frac{1+\phi+\kappa}{\kappa}}},\quad \tilde{v}_{YY\xi^P}=-\frac{1+\phi}{\kappa}\frac{1+\phi-\kappa}{\kappa^2}\lambda\xi^C\frac{y^{\frac{1+\phi-2\kappa}{\kappa}}}{\xi^{P\frac{1+\phi+\kappa}{\kappa}}}$$

$$\tilde{v}_{Y\xi^C}=\frac{1}{\kappa}\lambda\frac{y^{\frac{1+\phi-\kappa}{\kappa}}}{\xi^{P\frac{1+\phi}{\kappa}}},\quad \tilde{v}_{YY\xi^C}=\frac{1+\phi-\kappa}{\kappa^2}\lambda\frac{y^{\frac{1+\phi-2\kappa}{\kappa}}}{\xi^{P\frac{1+\phi}{\kappa}}}$$

7.2 Efficient equilibrium

We first characterize the efficient allocation by considering the SOE planner's problem, which is to

$$\max u\left(C_t,\xi_t\right)-\tilde{v}\left(Y_t\right)+g\left(G_t,\xi_t\right)$$

st

$$Y_t=\left(1-\alpha\right)r(\varsigma_t)^{\eta}\left(C_t+G_t\right)+\alpha\varsigma_t^{\eta}\left(C_t^*+G_t^*\right)$$

$$q(\varsigma_t)=\frac{u_C\left(C_t^*,\xi_t^*\right)}{u_C\left(C_t,\xi_t\right)}.$$

Formulate the Lagrangian

$$
\begin{aligned}
L_t &= u\left(C_t, \xi_t\right) - \tilde{v}\left(Y_t\right) + g\left(G_t\right) \\
&\quad + \phi_{1t}\left(q(\varsigma_t) - \frac{u_C\left(C_t^*, \xi_t^*\right)}{u_C\left(C_t, \xi_t\right)}\right) \\
&\quad + \phi_{2t}\left(Y_t - (1-\alpha)\, r(\varsigma_t)^{\eta}\left(C_t + G_t\right) - \alpha\varsigma_t^{\eta}\left(C_t^* + G_t^*\right)\right)
\end{aligned}
$$

FOCs (where all the derivatives are to be equated to zero)

$$
\begin{aligned}
\frac{\partial L_t}{\partial Y_t} &= -\tilde{v}_Y + \phi_{2t} \\
\frac{\partial L_t}{\partial C_t} &= u_C + \phi_{1t}\left[\frac{u_c^*}{\left(u_C\right)^2} u_{CC}\right] + \phi_{2t}\left[-\left(1-\alpha\right) r(\varsigma_t)^{\eta}\right] \\
\frac{\partial L_t}{\partial \varsigma_t} &= \phi_{1t}\left[q'(\varsigma_t)\right] + \phi_{2t}\left[-\left(1-\alpha\right)\eta r(\varsigma_t)^{\eta-1} r'\left(\varsigma_t\right)\left(C_t + G_t\right) - \alpha\eta\varsigma_t^{\eta-1}\left(C_t^* + G_t^*\right)\right] \\
\frac{\partial L_t}{\partial G_t} &= g_G + \phi_{2t}\left[-\left(1-\alpha\right) r(\varsigma_t)^{\eta}\right]
\end{aligned}
$$

7.2.1 Steady-state

We focus on symmetric steady-state where

$$C = C^*$$

$$G = G^*$$

$$Y = C + G$$

$$q(\varsigma) = \varsigma = 1.$$

Steady-state is characterized by set of FOCs and constraints without time subscripts. After simplifying we get

$$
1 + \tilde{v}_Y\left[\frac{1-(1-\alpha)^2}{(1-\alpha)}\right]\eta\left[\frac{(C+G)\, u_{CC}}{\left(u_C\right)^2}\right] + \frac{\tilde{v}_Y}{u_C}\left[-\left(1-\alpha\right)\right] = 0
$$

$$
g_G + \tilde{v}_Y\left[-\left(1-\alpha\right)\right] = 0.
$$

This two equations together with $Y = C + G$ gives us solution for steady-state.

Moreover, in what follows we adopt several normalizations in the steady state. That is we set $Y = C + G = 1$ and we target some value of the ratio of government expenditures to GDP $\theta_G \equiv \frac{G}{Y}$ (and accordingly $1 - \theta_G \equiv \theta_C$). We do this by choosing appropriate values of labor supply parameter λ and utility parameter λ_G. We give more details in section 7.3.2.

7.2.2 Linear approximation

Linearizing the system of FOCs and constraints gives us

$$
\begin{aligned}
0 &= \hat{C}_t (1-\alpha)\left[\frac{\kappa}{\lambda}\frac{\sigma-1}{\sigma}\theta_C^{-\sigma} + (1-\alpha)\left(\frac{1}{\sigma}-\eta\right)\right] - \hat{\xi}_t^P \frac{(1-\alpha)}{\sigma\lambda}(1+\phi)\,\theta_C^{1-\sigma} \\
&- \hat{G}_t (1-\alpha)^2 \eta + \hat{\varsigma}_t (1-\alpha)\left[\eta\alpha(2\eta\alpha-\alpha-\eta) - \eta(\eta\alpha-1) - \frac{\eta}{(1-\alpha)}\right. \\
&\left. + \frac{(1-\alpha)}{\sigma}(2\eta\alpha-\alpha-1)\,\theta_C + \frac{\kappa}{\sigma\lambda}(\alpha+1-\eta\alpha)\,\theta_C^{1-\sigma}\right] \\
&+ \hat{\xi}_t^C \frac{(1-\alpha)}{\sigma}\left[(1-\alpha)\,\theta_C - \frac{\kappa}{\lambda}\theta_C^{1-\sigma}\right] + \hat{Y}_t\left[\eta + \frac{(1-\alpha)}{\sigma\lambda}(1+\phi-\kappa)\,\theta_C^{1-\sigma}\right] \\
0 &= -\sigma'\lambda_G\theta_G^{-\sigma'-1}\hat{G}_t + \lambda_G\theta_G^{-\sigma'}\hat{\xi_t^C} - (1-\alpha)\frac{1+\phi-\kappa}{\kappa^2}\lambda\hat{Y}_t \\
&+ (1-\alpha)\frac{1+\phi}{\kappa^2}\lambda\hat{\xi}_t^P - \frac{\lambda}{\kappa}(1-\alpha)\,\alpha\eta\hat{\varsigma}_t
\end{aligned}
$$

$$
\hat{Y}_t = (2-\alpha)\,\eta\alpha\hat{\varsigma}_t + (1-\alpha)\left(\hat{C}_t+\hat{G}_t\right) + \alpha\left(\hat{C}_t^* + \hat{G}_t^*\right)
$$

$$
(1-\alpha)\hat{\varsigma}_t + \left(\hat{\xi_t^C} - \sigma\theta_C^{-1}\hat{C}_t\right) = -\sigma\theta_C^{-1}\hat{C}_t^*
$$

where $\hat{x_t} = dx_t$ denotes deviation from steady state value for any variable x_t.

Note that from here on we do not consider foreign shock $\hat{\xi}_t^*$ explicitly in order to keep notation simple.

Solving this linear system of equations provide the following closed form solutions

$$
\hat{Y}_t = \Psi_7\hat{\xi}_t^P + \Psi_8\hat{\xi_t^C} + \frac{\Psi_5+\alpha\Psi_1}{\Psi_6}\hat{C}_t^* + \alpha\frac{\Psi_5+\Psi_1}{\Psi_6}\hat{G}_t^*
$$

$$
\hat{G}_t = -\frac{\lambda}{\kappa}\frac{\theta_G^{\sigma'+1}}{\sigma'\lambda_G}(1-\alpha)\left[\Psi_{10}\hat{\xi}_t^P + \Psi_{11}\hat{\xi_t^C} + \Psi_{12}\hat{C}_t^* + \alpha\Psi_{13}\hat{G}_t^*\right]
$$

$$
\begin{aligned}
\hat{r_t} &= \Psi_{14}\left(\hat{\xi_t^C} - E_t\hat{\xi_{t+1}^C}\right) + \Psi_{15}\left(\hat{\xi_t^P} - E_t\hat{\xi_{t+1}^P}\right) \\
&+ \Psi_{16}\left(\hat{C_t^*} - E_t\hat{C_{t+1}^*}\right) + \Psi_{17}\left(\hat{G_t^*} - E_t\hat{G_{t+1}^*}\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\varsigma}_t &= \Psi_4\left[\Psi_7 + (1-\alpha)^2\frac{\theta_G^{\sigma'+1}}{\sigma'\lambda_G}\lambda\left[\frac{1+\phi-\kappa}{\kappa^2}\Psi_7 - \frac{1+\phi}{\kappa^2}\right]\right]\hat{\xi}_t^P \\
&+ \Psi_4\left[\Psi_8 + \Psi_8(1-\alpha)^2\frac{\theta_G^{\sigma'+1}}{\sigma'\lambda_G}\frac{1+\phi-\kappa}{\kappa^2}\lambda - (1-\alpha)\left[\frac{\theta_G}{\sigma'}+\frac{\theta_C}{\sigma}\right]\right]\hat{\xi_t^C} \\
&+ \Psi_4\left\{\Psi_9\frac{\Psi_5+\alpha\Psi_1}{\Psi_6} - 1\right\}\hat{C}_t^* \\
&+ \alpha\Psi_4\left\{\Psi_9\frac{\Psi_5+\Psi_1}{\Psi_6} - 1\right\}\hat{G}_t^*
\end{aligned}
$$

where $\{\Psi_i\}_i$ are just functions of model parameters.

7.3 Private sector equilibrium

Linearized private sector equilibrium conditions, say, from 7.4.3 takes the from

$$\beta\hat{i_t} - \hat{\xi^C_t} + \sigma\theta_C^{-1}\hat{C}_t = E_t\left[-\hat{\xi^C_{t+1}} + \sigma\theta_C^{-1}\hat{C_{t+1}} + \hat{\Pi_{H,t+1}} + \alpha\hat{\varsigma_{t+1}}\right] - \alpha\hat{\varsigma_t}$$

$$\begin{aligned}0 = &-\varepsilon s\frac{\varepsilon-1}{\varepsilon}\sigma\theta_C^{-1-\sigma}\hat{C}_t - \varepsilon\frac{1}{\kappa^2}\left((1+\phi-\kappa)\lambda\hat{Y}_t - (1+\phi)\lambda\hat{\xi^P_t}\right)\\ &-\varepsilon\frac{\lambda}{\kappa}\alpha\hat{\varsigma_t} + 2d_1\theta_C^{-\sigma}\hat{\Pi_{H,t}} - \beta E_t\left[2d_1\theta_C^{-\sigma}\hat{\Pi_{H,t+1}}\right]\end{aligned}$$

$$\hat{Y}_t = (2-\alpha)\eta\alpha\hat{\varsigma_t} + (1-\alpha)\left(\hat{C}_t + \hat{G}_t\right) + \alpha\left(\hat{C^*_t} + \hat{G^*_t}\right)$$

$$(1-\alpha)\hat{\varsigma_t} + \left(\hat{\xi^C_t} - \sigma\theta_C^{-1}\hat{C}_t\right) = -\sigma\theta_C^{-1}\hat{C^*_t}$$

After combining together market clearing and risk sharing conditions (last two equations), we can rewrite this system in terms of gaps as

$$\begin{aligned}\hat{x_t} = &E_t\hat{x_{t+1}} - (1-\alpha)\frac{\theta_C}{\sigma}\left(\hat{r}_t - \hat{r^e_t}\right)\\ &+ (2-\alpha)\eta\alpha\left(\hat{\varsigma^{gap}_t} - E_t\hat{\varsigma^{gap}_{t+1}}\right) + (1-\alpha)\left(\hat{G^{gap}_t} - E_t\hat{G^{gap}_{t+1}}\right)\\ &+\alpha\left(\hat{C^*_t} - E_t\hat{C^*_{t+1}}\right) + \alpha\left(\hat{G^*_t} - E_t\hat{G^*_{t+1}}\right) + (1-\alpha)\frac{\theta_C}{\sigma}\left(\hat{\xi^C_t} - E_t\hat{\xi^C_{t+1}}\right)\\ &+E_t\hat{Y^e_{t+1}} - \hat{Y^e_t} - (1-\alpha)\frac{\theta_C}{\sigma}\hat{r^e_t} + (1-\alpha)\left(\hat{G^e_t} - E_t\hat{G^e_{t+1}}\right)\\ &+(2-\alpha)\eta\alpha\left(\hat{\varsigma^e_t} - E_t\hat{\varsigma^e_{t+1}}\right)\end{aligned}$$

$$\begin{aligned}\hat{\Pi_{H,t}} = &\beta E_t\hat{\Pi_{H,t+1}} - s(\varepsilon-1)\sigma\frac{\theta_C^{-1}}{2d_1}\hat{G^{gap}_t}\\ &+\left[\frac{\varepsilon}{\kappa^2}\frac{\theta_C^{\sigma}}{2d_1}(1+\phi-\kappa)\lambda + s\frac{\varepsilon-1}{1-\alpha}\sigma\frac{\theta_C^{-1}}{2d_1}\right]\hat{x_t}\\ &+\left[\varepsilon\frac{\theta_C^{\sigma}}{2d_1}\frac{\lambda}{\kappa}\alpha - s\frac{\varepsilon-1}{1-\alpha}\sigma\frac{\theta_C^{-1}}{2d_1}(2-\alpha)\eta\alpha\right]\hat{\varsigma^{gap}_t}\\ &-\frac{\varepsilon}{\kappa^2}\frac{\theta_C^{\sigma}}{2d_1}(1+\phi)\lambda\hat{\xi^P_t} - s\frac{\varepsilon-1}{1-\alpha}\sigma\frac{\theta_C^{-1}}{2d_1}\alpha\left(\hat{C^*_t} + \hat{G^*_t}\right)\\ &-s(\varepsilon-1)\sigma\frac{\theta_C^{-1}}{2d_1}\hat{G^e_t} + \left[\frac{\varepsilon}{\kappa^2}\frac{\theta_C^{\sigma}}{2d_1}(1+\phi-\kappa)\lambda + s\frac{\varepsilon-1}{1-\alpha}\sigma\frac{\theta_C^{-1}}{2d_1}\right]\hat{Y^e_t}\\ &+\left[\varepsilon\frac{\theta_C^{\sigma}}{2d_1}\frac{\lambda}{\kappa}\alpha - s\frac{\varepsilon-1}{1-\alpha}\sigma\frac{\theta_C^{-1}}{2d_1}(2-\alpha)\eta\alpha\right]\hat{\varsigma^e_t}\end{aligned}$$

$$0 = \hat{x_t} - (1-\alpha)\hat{G^{gap}_t} - \left[(2-\alpha)\eta\alpha + (1-\alpha)^2\frac{\theta_C}{\sigma}\right]\hat{\varsigma^{gap}_t}$$

Finally, we can plug solutions from 7.2.2 instead of efficient variables and get

$$\begin{aligned}\hat{x_t} = &E_t\hat{x_{t+1}} - (1-\alpha)\frac{\theta_C}{\sigma}\left(\hat{r}_t - \hat{r^e_t}\right)\\ &+(2-\alpha)\eta\alpha\left(\hat{\varsigma^{gap}_t} - E_t\hat{\varsigma^{gap}_{t+1}}\right) + (1-\alpha)\left(\hat{G^{gap}_t} - E_t\hat{G^{gap}_{t+1}}\right)\end{aligned}$$

$$\hat{\Pi_{H,t}} = \beta E_t \hat{\Pi_{H,t+1}} - s\left(\varepsilon-1\right)\sigma \frac{\theta_C^{-1}}{2d_1} G_t^{\hat{gap}}$$
$$+ \frac{\varepsilon}{2d_1}\frac{\lambda}{\kappa}\left[\theta_C^{\sigma}\frac{(1+\phi-\kappa)}{\kappa} + \theta_C^{\sigma-1}\frac{\sigma}{1-\alpha}\right]\hat{x_t}$$
$$+ \frac{\varepsilon}{2d_1}\frac{\lambda}{\kappa}\left[\theta_C^{\sigma}\alpha - \theta_C^{\sigma-1}\frac{(2-\alpha)}{1-\alpha}\eta\sigma\alpha\right]\varsigma_t^{\hat{gap}}$$
$$+ \Psi_{22}\hat{\xi}_t^P + \Psi_{23}\hat{\xi_t^C} + \Psi_{24}\hat{C_t^*} + \alpha\Psi_{25}\hat{G_t^*}$$

$$0 = \hat{x_t} - (1-\alpha)\,G_t^{\hat{gap}} - \left[(2-\alpha)\,\eta\alpha + (1-\alpha)^2\,\frac{\theta_C}{\sigma}\right]\varsigma_t^{\hat{gap}}$$

where $\{\Psi_i\}_i$ are just functions of model parameters.

7.4 Commitment (Ramsey) equilibrium

7.4.1 Definition and optimality conditions

The policy problem is

$$E_t \sum_{s=0}^{\infty} U\left(C_{t+s}, G_{t+s}, \xi_{t+s}\right)$$

st

$$1 + i_t = \frac{u_C\left(C_t, \xi_t\right)}{\beta f_t^e}$$
$$i_t \geq 0$$
$$\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C\left(C_t,\xi_t\right) - \tilde{v}_y\left(Y_t,\xi_t\right)r(\varsigma_t)\right] + u_C\left(C_t,\xi_t\right)d'\left(\Pi_{H,t}\right)\Pi_{H,t} = \beta r(\varsigma_t)S_t^e$$
$$Y_t = (1-\alpha)\,r(\varsigma_t)^{\eta}\left(C_t + G_t\right) + \alpha\varsigma_t^{\eta}\left(C_t^* + G_t^*\right) + d\left(\Pi_{H,t}\right)$$
$$\frac{\Pi_t}{\Pi_{H,t}} = \frac{r(\varsigma_t)}{r(\varsigma_{t-1})}$$
$$q(\varsigma_t) = \frac{u_C\left(C_t^*, \xi_t^*\right)}{u_C\left(C_t, \xi_t\right)}$$
$$f_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\right] = \bar{f}^e\left(\varsigma_t, \xi_t\right)$$
$$S_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)d'\left(\Pi_{H,t+1}\right)\frac{\Pi_{H,t+1}}{r(\varsigma_{t+1})}\right] = \bar{S}^e\left(\varsigma_t,\xi_t\right)$$

Formulate the Lagrangian

$$
\begin{aligned}
L_0 = {} & E_0 \sum_{t=0}^{\infty} u\left(C_t, \xi_t\right) - \tilde{v}\left(Y_t\right) + g\left(G_t, \xi_t\right) \\
& + \phi_{2t}\left(\beta f_t^e - \frac{u_C\left(C_t, \xi_t\right)}{1+i_t}\right) \\
& + \phi_{3t}\left(\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C\left(C_t, \xi_t\right) - \tilde{v}_y\left(Y_t, \xi_t\right) r(\varsigma_t)\right] + u_C\left(C_t, \xi_t\right) d'\left(\Pi_{H,t}\right)\Pi_{H,t} - \beta r(\varsigma_t) S_t^e\right) \\
& + \phi_{4t}\left(\frac{\Pi_t}{\Pi_{H,t}} - \frac{r(\varsigma_t)}{r(\varsigma_{t-1})}\right) \\
& + \phi_{5t}\left(q(\varsigma_t) - \frac{u_C\left(C_t^*, \xi_t^*\right)}{u_C\left(C_t, \xi_t\right)}\right) \\
& + \phi_{6t}\left(Y_t - (1-\alpha) r(\varsigma_t)^{\eta}\left(C_t + G_t\right) - \alpha\varsigma_t^{\eta}\left(C_t^* + G_t^*\right) - d\left(\Pi_{H,t}\right)\right) \\
& + \psi_{1t}\left(f_t^e - u_C\left(C_{t+1}, \xi_{t+1}\right)\Pi_{t+1}^{-1}\right) \\
& + \psi_{2t}\left(S_t^e - u_C\left(C_{t+1}, \xi_{t+1}\right) d'\left(\Pi_{H,t+1}\right)\frac{\Pi_{H,t+1}}{r(\varsigma_{t+1})}\right) \\
& + \gamma_{1t}\left(i_t - 0\right)
\end{aligned}
$$

FOCs (where all the derivatives are to be equated to zero)

$$
\begin{aligned}
\frac{\partial L_s}{\partial \Pi_t} &= \phi_{4t}\Pi_{H,t}^{-1} + \beta^{-1}\psi_{1t-1}u_C\Pi_t^{-2} \\
\frac{\partial L_s}{\partial Y_t} &= -\tilde{v}_Y + \phi_{3t}\left[\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}(1+s)u_C - \tilde{v}_y r(\varsigma_t)\right) - \varepsilon Y_t \tilde{v}_{yy} r(\varsigma_t)\right] + \phi_{6t} \\
\frac{\partial L_s}{\partial i_t} &= \phi_{2t}\left[u_C\left(1+i_t\right)^{-2}\right] + \gamma_{1t} \\
\frac{\partial L_s}{\partial C_t} &= u_C + \phi_{2t}\left[-u_{CC}\left(1+i_t\right)^{-1}\right] + \phi_{3t}\left[\varepsilon Y_t\frac{\varepsilon-1}{\varepsilon}(1+s)u_{CC} + u_{CC}d'\Pi_{H,t}\right] + \phi_{5t}\left[u_{C^*}u_C^{-2}u_{CC}\right] + \phi_{6t}\left[-(1-\alpha) r(\varsigma_t)^{\eta}\right] \\
&\quad + \beta^{-1}\psi_{1t-1}\left(-u_{CC}\Pi_t^{-1}\right) + \beta^{-1}\psi_{2t-1}\left[-u_{CC}d'\left(\Pi_{H,t}\right)\frac{\Pi_{H,t}}{r\left(\varsigma_t\right)}\right] \\
\frac{\partial L_s}{\partial G_t} &= g_G + \phi_{6t}\left[-(1-\alpha) r(\varsigma_t)^{\eta}\right] \\
\frac{\partial L_s}{\partial \Pi_{H,t}} &= \phi_{3t}\left[u_C d''\Pi_{H,t} + u_C d'\right] + \phi_{4t}\left(-\Pi_t\Pi_{H,t}^{-2}\right) + \phi_{6t}\left[-d'\right] - \beta^{-1}\psi_{2t-1}u_C d'\frac{1}{r(\varsigma_t)} - \beta^{-1}\psi_{2t-1}u_C\frac{\Pi_{H,t}}{r(\varsigma_t)}d'' \\
\frac{\partial L_s}{\partial \varsigma_t} &= \phi_{3t}\left[-\varepsilon Y_t\tilde{v}_y r'(\varsigma_t) - \beta S_t^e r'(\varsigma_t)\right] + \phi_{4t}\left[-\frac{r'(\varsigma_t)}{r(\varsigma_{t-1})}\right] + \beta E_t\phi_{4t+1}r'(\varsigma_t)\left[\frac{r(\varsigma_{t+1})}{r(\varsigma_t)^2}\right] + \phi_{5t}\left[q'(\varsigma_t)\right] \\
&\quad + \phi_{6t}\left[-(1-\alpha)\eta r(\varsigma_t)^{\eta-1}r'(\varsigma_t)\left(C_t+G_t\right) - \alpha\eta\varsigma_t^{\eta-1}\left(C_t^*+G_t^*\right)\right] \\
&\quad + \beta^{-1}\psi_{2t-1}\left[u_C d'\frac{\Pi_{H,t}}{r(\varsigma_t)^2}\right]r'(\varsigma_t) \\
\frac{\partial L_s}{\partial f_t^e} &= \beta\phi_{2t} + \psi_{1t} \\
\frac{\partial L_s}{\partial S_t^e} &= \phi_{3t}\left[-\beta r(\varsigma_t)\right] + \psi_{2t}
\end{aligned}
$$

$$
\gamma_{1t} \geq 0,\ i_t \geq 0,\ \gamma_{1t}i_t = 0
$$

7.4.2 Steady-state

We will denote variables without a time-subscript as steady-state values of the variables. After simplifying the system of FOCs and constraints become

$$1+i=\frac{1}{\beta\Pi_H^{-1}}$$

$$\varepsilon Y\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C-\tilde{v}_y r\left(\varsigma\right)\right]+u_C d'\left(\Pi_H\right)\Pi_H=\beta r\left(\varsigma\right)\left[u_C d'\left(\Pi_H\right)\frac{\Pi_H}{r\left(\varsigma\right)}\right]$$

$$Y=\left(1-\alpha\right)r(\varsigma)^{\eta}\left(C+G\right)+\alpha\varsigma^{\eta}\left(C^*+G^*\right)+d\left(\Pi_H\right)$$

$$q(\varsigma)=\frac{u_C^*}{u_C}=\frac{\varsigma}{r(\varsigma)}$$

$$r(\varsigma)=\left[(1-\alpha)+\alpha\varsigma^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

$$\begin{aligned}
0&=-\tilde{v}_Y+\phi_3\left[\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}(1+s)u_C-\tilde{v}_y r(\varsigma)\right)-\varepsilon Y\tilde{v}_{yy}r(\varsigma)\right]+\phi_6\\
0&=u_C+\phi_3\left[\varepsilon Y\frac{\varepsilon-1}{\varepsilon}(1+s)u_{CC}\right]+\phi_5\left[u_{C^*}u_C^{-2}u_{CC}\right]+\phi_6\left[-\left(1-\alpha\right)r(\varsigma)^{\eta}\right]\\
0&=g_G+\phi_6\left[-\left(1-\alpha\right)r(\varsigma)^{\eta}\right]\\
0&=\phi_6\left[-d'\right]\\
0&=\phi_3\left[-\varepsilon Y\tilde{v}_y r'(\varsigma)-\beta u_C d'\frac{\Pi_H}{r(\varsigma)}r'(\varsigma)\right]+\phi_5\left[q'(\varsigma)\right]+\phi_6\left[-\left(1-\alpha\right)\eta r(\varsigma)^{\eta-1}r'(\varsigma)\left(C+G\right)-\alpha\eta\varsigma^{\eta-1}\left(C^*+G^*\right)\right]\\
&+\phi_3\left[u_C d'\frac{\Pi_H}{r(\varsigma)}\right]r'(\varsigma)
\end{aligned}$$

Now we will find such value of production subsidy $(1+s)$ so that this steady state coincides with efficient one. So we plug in some steady state relationships like $\Pi_H=1$ or $\varsigma=1$, and ultimately we get

$$0=g_G+\tilde{v}_Y\left[-\left(1-\alpha\right)\right]$$

$$\frac{\varepsilon-1}{\varepsilon}(1+s)u_C\eta\alpha\frac{(2-\alpha)}{[1-\alpha]}\left(C+G\right)=\frac{\frac{\varepsilon-1}{\varepsilon}(1+s)\left(1-\alpha\right)-1}{u_C^{-2}u_{CC}}$$

from where we arrive at

$$g_G=\left(1-\alpha\right)\tilde{v}_Y=\frac{\varepsilon-1}{\varepsilon}(1+s)u_C\left(1-\alpha\right)$$

$$(1+s)=\frac{\varepsilon}{\varepsilon-1}\left[-\eta\alpha\frac{(2-\alpha)}{[1-\alpha]}\left(C+G\right)\frac{u_{CC}}{u_C}+(1-\alpha)\right]^{-1}$$

where second equation basically defines the subsidy.

Now we need to show only that at this value of production subsidy conditions of efficient steady state are satisfied. Combining the last two expressions yields

$$\left[-\eta\alpha\frac{(2-\alpha)}{[1-\alpha]}\left(C+G\right)\frac{u_{CC}}{u_C}+(1-\alpha)\right]=\frac{u_C\left(1-\alpha\right)}{g_G}$$

which is exactly the same expression as we get combining the last two expressions in section 7.2.1.

This proved that with appropriate value of production subsidy efficient steady state is achieved under commitment. Now let's see how our normalizations set in section 7.2.1 could be achieved in this environment.

One of the optimality conditions in the steady state is

$$(1+s)\frac{\varepsilon-1}{\varepsilon}u_C=\tilde{v}_y$$

which under our functional form transforms to

$$(1+s)\frac{\varepsilon-1}{\varepsilon}C^{-\sigma}=\frac{\lambda}{\kappa}Y^{\frac{1+\phi-\kappa}{\kappa}}$$

so we set $\lambda=\kappa s\frac{\varepsilon-1}{\varepsilon}C^{-\sigma}$ in order to achieve $Y=1$.

Also in steady state the following condition should be satisfied

$$\left[-\eta\alpha\frac{(2-\alpha)}{[1-\alpha]}(C+G)\frac{u_{CC}}{u_C}+(1-\alpha)\right]=\frac{u_C(1-\alpha)}{g_G}$$

which becomes

$$\sigma\eta\alpha\frac{(2-\alpha)}{[1-\alpha]}+\left(\sigma\eta\alpha\frac{(2-\alpha)}{[1-\alpha]}+1-\alpha\right)\frac{1-\theta_G}{\theta_G}=\frac{(1-\theta_G)^{1-\sigma}}{\theta_G^{1-\sigma'}}\frac{(1-\alpha)}{\lambda_G}$$

so after we choose value of θ_G we set λ_G to the value implied by this relationship.

7.4.3 Linear approximation (without ZLB)

Linearizing the system of FOCs and constraints around the steady state under positive nominal interest rate gives us

$$\begin{aligned}
0 &= -\hat{Y}_t\frac{1+\phi-\kappa}{\kappa^2}\lambda+\hat{\xi_t^P}\frac{1+\phi}{\kappa^2}\lambda-\hat{\xi_t^C}\frac{\lambda}{\kappa}+\hat{\phi_{6t}}-\hat{\phi_{3t}}\varepsilon\frac{1+\phi-\kappa}{\kappa^2}\lambda\\
0 &= -\hat{\phi_{3t}}\varepsilon s\frac{\varepsilon-1}{\varepsilon}\sigma\theta_C^{-1-\sigma}+\hat{C}_t^*\sigma^2\overline{\phi_5}\theta_C^{-2-\sigma}-\hat{\varsigma}_t\frac{\lambda}{\kappa}(1-\alpha)\alpha\eta\\
&\quad+\hat{C}_t\sigma\left[-\theta_C^{-1-\sigma}+\left(1+\sigma-2\sigma\theta_C^{-\sigma}\right)\theta_C^{-2}\overline{\phi_{5t}}\right]\\
&\quad+\hat{\xi_t^C}\left[\theta_C^{-\sigma}+\theta_C^{-1}\sigma\overline{\phi_5}\right]-\sigma\theta_C^{-1}\hat{\phi_{5t}}-\hat{\phi_{6t}}(1-\alpha)\\
0 &= -\sigma'\lambda_G\theta_G^{-\sigma'-1}\hat{G}_t+\lambda_G\theta_G^{-\sigma'}\hat{\xi_t^C}-(1-\alpha)\hat{\phi_{6t}}-(1-\alpha)\alpha\eta\frac{\lambda}{\kappa}\hat{\varsigma}_t\\
0 &= \hat{\phi_{3t}}-\hat{\Pi_{H,t}}\theta_C^{\sigma}\frac{\lambda}{\kappa}-\hat{\phi_{3t-1}}\\
0 &= -\hat{C}_t\eta\alpha(1-\alpha)\frac{\lambda}{\kappa}-\hat{\phi_{3t}}\frac{\alpha\varepsilon}{\kappa}\lambda-\hat{C}_t^*\eta\alpha\frac{\lambda}{\kappa}+\hat{\phi_{5t}}(1-\alpha)-\hat{\phi_{6t}}\eta\alpha(2-\alpha)\\
&\quad+\hat{\varsigma}_t(\alpha-1)\alpha\left[(2-\eta)\overline{\phi_5}+\eta\frac{\lambda}{\kappa}\left(2\alpha\eta-\alpha-\eta+\frac{\eta-1}{1-\alpha}\right)\right]\\
&\quad-\hat{G}_t\eta\alpha(1-\alpha)\frac{\lambda}{\kappa}-\hat{G}_t^*\eta\alpha\frac{\lambda}{\kappa}
\end{aligned}$$

$$\beta\hat{i_t}-\hat{\xi_t^C}+\sigma\theta_C^{-1}\hat{C}_t=E_t\left[-\hat{\xi_{t+1}^C}+\sigma\theta_C^{-1}\hat{C_{t+1}}+\hat{\Pi_{H,t+1}}+\alpha\hat{\varsigma_{t+1}}\right]-\alpha\hat{\varsigma}_t$$

$$\begin{aligned}
0 &= -\varepsilon s\frac{\varepsilon-1}{\varepsilon}\sigma\theta_C^{-1-\sigma}\hat{C}_t-\varepsilon\frac{1}{\kappa^2}\left((1+\phi-\kappa)\lambda\hat{Y}_t-(1+\phi)\lambda\hat{\xi_t^P}\right)\\
&\quad-\varepsilon\frac{\lambda}{\kappa}\alpha\hat{\varsigma}_t+2d_1\theta_C^{-\sigma}\hat{\Pi_{H,t}}-\beta E_t\left[2d_1\theta_C^{-\sigma}\hat{\Pi_{H,t+1}}\right]
\end{aligned}$$

$$\hat{Y}_t=(2-\alpha)\eta\alpha\hat{\varsigma}_t+(1-\alpha)\left(\hat{C}_t+\hat{G}_t\right)+\alpha\left(\hat{C}_t^*+\hat{G}_t^*\right)$$

$$(1-\alpha)\hat{\varsigma}_t+\left(\hat{\xi_t^C}-\sigma\theta_C^{-1}\hat{C}_t\right)=-\sigma\theta_C^{-1}\hat{C}_t^*$$

where

$$\overline{\phi_5}=\eta\alpha\lambda_G\theta_G^{-\sigma'}\frac{(2-\alpha)}{(1-\alpha)^2}$$

7.4.4 Linear approximation (at ZLB)

The same system linearized around the same steady state but under zero nominal interest rate is

$$
\begin{aligned}
0 &= -\hat{Y}_t \frac{1+\phi-\kappa}{\kappa^2}\lambda + \hat{\xi_t^P}\frac{1+\phi}{\kappa^2}\lambda - \hat{\xi_t^C}\frac{\lambda}{\kappa} + \hat{\phi_{6t}} - \hat{\phi_{3t}}\varepsilon\frac{1+\phi-\kappa}{\kappa^2}\lambda \\
0 &= -\hat{\phi_{3t}}\varepsilon s\frac{\varepsilon-1}{\varepsilon}\sigma\theta_C^{-1-\sigma} + \hat{C}_t^*\sigma^2\overline{\phi_5}\theta_C^{-2-\sigma} - \hat{\varsigma}_t\frac{\lambda}{\kappa}(1-\alpha)\alpha\eta \\
&\quad + \hat{C}_t\sigma\left[-\theta_C^{-1-\sigma} + \left(1+\sigma-2\sigma\theta_C^{-\sigma}\right)\theta_C^{-2}\overline{\phi_{5t}}\right] \\
&\quad + \hat{\xi_t^C}\left[\theta_C^{-\sigma} + \theta_C^{-1}\sigma\overline{\phi_5}\right] - \sigma\theta_C^{-1}\hat{\phi_{5t}} - \hat{\phi_{6t}}(1-\alpha) + \sigma\theta_C^{-1}\hat{\gamma}_{1t-1} - \sigma\theta_C^{-1}\hat{\gamma}_{1t} \\
0 &= -\sigma'\lambda_G\theta_G^{-\sigma'-1}\hat{G}_t + \lambda_G\theta_G^{-\sigma'}\hat{\xi_t^C} - (1-\alpha)\hat{\phi_{6t}} - (1-\alpha)\alpha\eta\frac{\lambda}{\kappa}\hat{\varsigma}_t \\
0 &= \hat{\phi_{3t}} + \frac{\theta_C^\sigma}{2d_1}\hat{\gamma}_{1t-1} - \hat{\Pi_{H,t}}\theta_C^\sigma\frac{\lambda}{\kappa} - \hat{\phi_{3t-1}} \\
0 &= -\hat{C}_t\eta\alpha(1-\alpha)\frac{\lambda}{\kappa} - \hat{\phi_{3t}}\frac{\alpha\varepsilon}{\kappa}\lambda - \hat{C}_t^*\eta\alpha\frac{\lambda}{\kappa} + \hat{\phi_{5t}}(1-\alpha) - \hat{\phi_{6t}}\eta\alpha(2-\alpha) \\
&\quad + \hat{\varsigma}_t(\alpha-1)\alpha\left[(2-\eta)\overline{\phi_5} + \eta\frac{\lambda}{\kappa}\left(2\alpha\eta - \alpha - \eta + \frac{\eta-1}{1-\alpha}\right)\right] \\
&\quad - \hat{G}_t\eta\alpha(1-\alpha)\frac{\lambda}{\kappa} - \hat{G}_t^*\eta\alpha\frac{\lambda}{\kappa} + \hat{\gamma}_{1t-1}\alpha - \hat{\gamma}_{1t}\beta\alpha
\end{aligned}
$$

$$
\beta\left(\frac{1}{\beta}-1\right) - \hat{\xi_t^C} + \sigma\theta_C^{-1}\hat{C}_t = E_t\left[-\hat{\xi_{t+1}^C} + \sigma\theta_C^{-1}\hat{C_{t+1}} + \hat{\Pi_{H,t+1}} + \alpha\hat{\varsigma_{t+1}}\right] - \alpha\hat{\varsigma}_t
$$

$$
\begin{aligned}
0 &= -\varepsilon s\frac{\varepsilon-1}{\varepsilon}\sigma\theta_C^{-1-\sigma}\hat{C}_t - \varepsilon\frac{1}{\kappa^2}\left((1+\phi-\kappa)\lambda\hat{Y}_t - (1+\phi)\lambda\hat{\xi_t^P}\right) \\
&\quad - \varepsilon\frac{\lambda}{\kappa}\alpha\hat{\varsigma}_t + 2d_1\theta_C^{-\sigma}\hat{\Pi_{H,t}} - \beta E_t\left[2d_1\theta_C^{-\sigma}\hat{\Pi_{H,t+1}}\right]
\end{aligned}
$$

$$
\hat{Y}_t = (2-\alpha)\eta\alpha\hat{\varsigma}_t + (1-\alpha)\left(\hat{C}_t + \hat{G}_t\right) + \alpha\left(\hat{C}_t^* + \hat{G}_t^*\right)
$$

$$
(1-\alpha)\hat{\varsigma}_t + \left(\hat{\xi_t^C} - \sigma\theta_C^{-1}\hat{C}_t\right) = -\sigma\theta_C^{-1}\hat{C}_t^*
$$

7.5 Discretion (Markov perfect) equilibrium

7.5.1 Definition and optimality conditions

The policy problem can be written as

$$
J(\varsigma_{t-1}, \xi_t, C_t^*, G_t^*, \xi_t^*) = \max\left[U(C_t, \xi_t) - \tilde{v}(Y_t) + g(G_t, \xi_t) + \beta E_t J(\varsigma_t, \xi_{t+1}, C_{t+1}^*, G_{t+1}^*, \xi_{t+1}^*)\right]
$$

st

$$
\begin{gathered}
1 + i_t = \frac{u_C(C_t, \xi_t)}{\beta f_t^e} \\
i_t \geq 0 \\
\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C(C_t, \xi_t) - \tilde{v}_y(Y_t, \xi_t)\, r(\varsigma_t)\right] + u_C(C_t, \xi_t)\, d'(\Pi_{H,t})\,\Pi_{H,t} = \beta r(\varsigma_t) S_t^e \\
Y_t = (1-\alpha)\, r(\varsigma_t)^\eta (C_t + G_t) + \alpha\varsigma_t^\eta (C_t^* + G_t^*) + d(\Pi_{H,t}) \\
\frac{\Pi_t}{\Pi_{H,t}} = \frac{r(\varsigma_t)}{r(\varsigma_{t-1})} \\
q(\varsigma_t) = \frac{u_C(C_t^*, \xi_t^*)}{u_C(C_t, \xi_t)}
\end{gathered}
$$

$$f_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\right] = \bar{f}^e\left(\varsigma_t,\xi_t,C_t^*,G_t^*,\xi_t^*\right)$$

$$S_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)d'\left(\Pi_{H,t+1}\right)\frac{\Pi_{H,t+1}}{r(\varsigma_{t+1})}\right] = \bar{S}^e\left(\varsigma_t,\xi_t,C_t^*,G_t^*,\xi_t^*\right)$$

From here on we will omit foreign shocks (C_t^*, G_t^*, ξ_t^*) in the state vector to keep notation simple.

Formulate the Lagrangian

$$\begin{aligned}
L_t &= u\left(C_t,\xi_t\right) - \tilde{v}\left(Y_t\right) + g(G_t,\xi_t) + \beta E_t J\left(\varsigma_t,\xi_{t+1}\right) \\
&+ \phi_{2t}\left(\beta f_t^e - \frac{u_C\left(C_t,\xi_t\right)}{1+i_t}\right) \\
&+ \phi_{3t}\left(\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}(1+s)u_C\left(C_t,\xi_t\right) - \tilde{v}_y\left(Y_t,\xi_t\right)r(\varsigma_t)\right] + u_C\left(C_t,\xi_t\right)d'\left(\Pi_{H,t}\right)\Pi_{H,t} - \beta r(\varsigma_t)S_t^e\right) \\
&+ \phi_{4t}\left(\frac{\Pi_t}{\Pi_{H,t}} - \frac{r(\varsigma_t)}{r(\varsigma_{t-1})}\right) \\
&+ \phi_{5t}\left(q(\varsigma_t) - \frac{u_C\left(C_t^*,\xi_t^*\right)}{u_C\left(C_t,\xi_t\right)}\right) \\
&+ \phi_{6t}\left(Y_t - (1-\alpha)\,r(\varsigma_t)^\eta\left(C_t+G_t\right) - \alpha\varsigma_t^\eta\left(C_t^*+G_t^*\right) - d\left(\Pi_{H,t}\right)\right) \\
&+ \psi_{1t}\left(f_t^e - \bar{f}^e\left(b_t,\varsigma_t,\xi_t\right)\right) \\
&+ \psi_{2t}\left(S_t^e - \bar{S}^e\left(b_t,\varsigma_t,\xi_t\right)\right) \\
&+ \gamma_{1t}\left(i_t - 0\right)
\end{aligned}$$

FOCs (where all the derivatives are to be equated to zero) with substituted envelope condition are

$$\begin{aligned}
\frac{\partial L_s}{\partial \Pi_t} &= \phi_{4t}\left[\Pi_{H,t}^{-1}\right] \\
\frac{\partial L_s}{\partial Y_t} &= -\tilde{v}_Y + \phi_{3t}\left[\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}(1+s)u_C - \tilde{v}_y r(\varsigma_t)\right) - \varepsilon Y_t\tilde{v}_{yy}r(\varsigma_t)\right] + \phi_{6t} \\
\frac{\partial L_s}{\partial i_t} &= \phi_{2t}\left[u_C\left(1+i_t\right)^{-2}\right] + \gamma_{1t} \\
\frac{\partial L_s}{\partial C_t} &= u_C + \phi_{2t}\left[-u_{CC}\left(1+i_t\right)^{-1}\right] + \phi_{3t}\left[\varepsilon Y_t\frac{\varepsilon-1}{\varepsilon}(1+s)u_{CC} + u_{CC}d'\Pi_{H,t}\right] + \phi_{5t}\left[u_{C^*}u_C^{-2}u_{CC}\right] + \phi_{6t}\left[-\left(1-\alpha\right)r(\varsigma_t)^\eta\right] \\
\frac{\partial L_s}{\partial G_t} &= g_G + \phi_{6t}\left[-\left(1-\alpha\right)r(\varsigma_t)^\eta\right] \\
\frac{\partial L_s}{\partial \Pi_{H,t}} &= \phi_{3t}\left[u_C d''\Pi_{H,t} + u_C d'\right] + \phi_{4t}\left(-\Pi_t\Pi_{H,t}^{-2}\right) + \phi_{6t}\left[-d'\right] \\
\frac{\partial L_s}{\partial \varsigma_t} &= \beta E_t\left[\phi_{4t+1}\left(\frac{r(\varsigma_{t+1})}{\left[r(\varsigma_t)\right]^2}r'\left(\varsigma_t\right)\right)\right] + \phi_{3t}\left[-\varepsilon Y_t\tilde{v}_y r'\left(\varsigma_t\right) - \beta S_t^e r'(\varsigma_t)\right] + \phi_{4t}\left[-\frac{r'\left(\varsigma_t\right)}{r(\varsigma_{t-1})}\right] + \phi_{5t}\left[q'(\varsigma_t)\right] \\
&+ \phi_{6t}\left[-\left(1-\alpha\right)\eta r(\varsigma_t)^{\eta-1}r'\left(\varsigma_t\right)\left(C_t+G_t\right) - \alpha\eta\varsigma_t^{\eta-1}\left(C_t^*+G_t^*\right)\right] \\
&+ \psi_{1t}\left[-\bar{f}_\varsigma^e\right] + \psi_{2t}\left[-\bar{S}_\varsigma^e\right] \\
\frac{\partial L_s}{\partial f_t^e} &= \beta\phi_{2t} + \psi_{1t} \\
\frac{\partial L_s}{\partial S_t^e} &= \phi_{3t}\left[-\beta r(\varsigma_t)\right] + \psi_{2t}
\end{aligned}$$

Note that from the first condition it follows that $\phi_{4t} = 0$, and then all the terms with ς_{t-1} drop out from the policymaker's problem (in constraints it appears only in the definition of Π_t). Therefore there is no state variable in this problem, and thus $\bar{S}_\varsigma^e = \bar{f}_\varsigma^e = 0$.

7.5.2 Steady-state

After simplifying the system above and substituting some of the efficient steady state relationships we get

$$1 + i = \frac{1}{\beta}$$

$$\left[\frac{\varepsilon - 1}{\varepsilon}(1+s)u_C - \tilde{v}_y\right] = 0$$

$$Y = (1-\alpha)(C+G) + \alpha(C^* + G^*)$$

$$q(\varsigma) = \frac{u_C^*}{u_C}$$

$$\begin{aligned}
0 &= -\tilde{v}_Y + \phi_6 \\
0 &= u_C + \phi_5\left[u_{C^*}u_C^{-2}u_{CC}\right] + \phi_6\left[-(1-\alpha)\right] \\
0 &= g_G + \phi_6\left[-(1-\alpha)\right] \\
0 &= \phi_5\left[q'(\varsigma)\right] \\
&\quad + \phi_6\left[-(1-\alpha)\eta r'(\varsigma)(C+G) - \alpha\eta(C^* + G^*)\right]
\end{aligned}$$

This is the exact same set of equations as in the commitment steady-state. Therefore the value of production subsidy in Markov perfect equilibrium will be the same as in commitment equilibrium.

7.5.3 Linear approximation (without ZLB)

Linearizing the system of FOCs around the steady state under positive nominal interest rate gives us

$$\begin{aligned}
0 &= -\hat{Y}_t\frac{1+\phi-\kappa}{\kappa^2}\lambda + \hat{\xi_t^P}\frac{1+\phi}{\kappa^2}\lambda - \hat{\xi_t^C}\frac{\lambda}{\kappa} + \hat{\phi_{6t}} - \hat{\phi_{3t}}\varepsilon\frac{1+\phi-\kappa}{\kappa^2}\lambda \\
0 &= -\hat{\phi_{3t}}\varepsilon s\frac{\varepsilon-1}{\varepsilon}\sigma\theta_C^{-1-\sigma} + \hat{C}_t^*\sigma^2\overline{\phi_5}\theta_C^{-2-\sigma} - \hat{\varsigma}_t\frac{\lambda}{\kappa}(1-\alpha)\alpha\eta \\
&+ \hat{C}_t\sigma\left[-\theta_C^{-1-\sigma} + \left(1+\sigma-2\sigma\theta_C^{-\sigma}\right)\theta_C^{-2}\overline{\phi_{5t}}\right] \\
&+ \hat{\xi_t^C}\left[\theta_C^{-\sigma} + \theta_C^{-1}\sigma\overline{\phi_5}\right] - \sigma\theta_C^{-1}\hat{\phi_{5t}} - \hat{\phi_{6t}}(1-\alpha) \\
0 &= -\sigma'\lambda_G\theta_G^{-\sigma'-1}\hat{G}_t + \lambda_G\theta_G^{-\sigma'}\hat{\xi_t^C} - (1-\alpha)\hat{\phi_{6t}} - (1-\alpha)\alpha\eta\frac{\lambda}{\kappa}\hat{\varsigma}_t \\
0 &= \hat{\phi_{3t}} - \hat{\Pi_{H,t}}\theta_C^{\sigma}\frac{\lambda}{\kappa} \\
0 &= -\hat{C}_t\eta\alpha(1-\alpha)\frac{\lambda}{\kappa} - \hat{\phi_{3t}}\frac{\alpha\varepsilon}{\kappa}\lambda - \hat{C}_t^*\eta\alpha\frac{\lambda}{\kappa} + \hat{\phi_{5t}}(1-\alpha) - \hat{\phi_{6t}}\eta\alpha(2-\alpha) \\
&+ \hat{\varsigma}_t(\alpha-1)\alpha\left[(2-\eta)\overline{\phi_5} + \eta\frac{\lambda}{\kappa}\left(2\alpha\eta - \alpha - \eta + \frac{\eta-1}{1-\alpha}\right)\right] \\
&- \hat{G}_t\eta\alpha(1-\alpha)\frac{\lambda}{\kappa} - \hat{G}_t^*\eta\alpha\frac{\lambda}{\kappa}
\end{aligned}$$

where $\overline{\phi_5}$ is the same as before.

We can use the same constraints as in commitment problem.

7.5.4 Linear approximation (at ZLB)

Linearizing the system of FOCs around the steady state under zero nominal interest rate gives us

$$
\begin{aligned}
0 &= -\hat{Y}_t \frac{1+\phi-\kappa}{\kappa^2}\lambda + \hat{\xi_t^P}\frac{1+\phi}{\kappa^2}\lambda - \hat{\xi_t^C}\frac{\lambda}{\kappa} + \hat{\phi_{6t}} - \hat{\phi_{3t}}\varepsilon\frac{1+\phi-\kappa}{\kappa^2}\lambda \\
0 &= -\hat{\phi_{3t}}\varepsilon s\frac{\varepsilon-1}{\varepsilon}\sigma\theta_C^{-1-\sigma} - (1-\alpha)\hat{\phi}_{6t} - (1-\alpha)\eta\alpha\frac{\lambda}{\kappa}\hat{\varsigma}_t - \sigma\theta_C^{-1}\hat{\gamma}_{1t} + \hat{\xi_t^C}\left[\theta_C^{-\sigma} + \theta_C^{-1}\sigma\overline{\phi_{5t}}\right] \\
&\quad - \hat{\phi}_{5t}\theta_C^{-1}\sigma + \overline{\phi_5}\theta_C^{-2-\sigma}\sigma^2\hat{C}_t^* + \sigma\left[-\theta_C^{-1-\sigma} + \left(1+\sigma-2\sigma\theta_C^{-\sigma}\right)\theta_C^{-2}\overline{\phi_{5t}}\right]\hat{C}_t \\
0 &= -\sigma'\lambda_G\theta_G^{-\sigma'-1}\hat{G}_t + \lambda_G\theta_G^{-\sigma'}\hat{\xi_t^C} - (1-\alpha)\hat{\phi_{6t}} - (1-\alpha)\alpha\eta\frac{\lambda}{\kappa}\hat{\varsigma}_t \\
0 &= \hat{\phi_{3t}} - \hat{\Pi_{H,t}}\theta_C^{\sigma}\frac{\lambda}{\kappa} \\
0 &= -\hat{C}_t\frac{\lambda}{\kappa}\eta\alpha(1-\alpha) - \hat{\phi}_{3t}\frac{\varepsilon\alpha\lambda}{\kappa} \\
&\quad - \frac{\lambda}{\kappa}\eta\alpha\hat{C}_t^* - \hat{\phi}_{6t}\eta\alpha(2-\alpha) + \hat{\phi}_5(1-\alpha) - \beta\theta_C^{\sigma}\bar{f}_\varsigma^e\hat{\gamma}_{1t} \\
&\quad + \hat{\varsigma}_t\alpha(\alpha-1)\left[\overline{\phi_5}(2-\eta) + \frac{\lambda}{\kappa}\eta\left(2\eta\alpha - \eta - \alpha + \frac{\eta-1}{1-\alpha}\right)\right] \\
&\quad - \hat{G}_t\eta\alpha(1-\alpha)\frac{\lambda}{\kappa} - \hat{G}_t^*\eta\alpha\frac{\lambda}{\kappa}
\end{aligned}
$$

Linearized constraints are the same as in the commitment problem.

7.6 Computation algorithm

In order to solve for optimal path of linearized system first we guess that ZLB binds up to some period, and not thereafter. Then we just have to combine two linearized systems by endogenously determining time period when ZLB stops to bind. This is done essentially by using guess and verify approach. For more details on this piece-wise linear algorithm in a perfect foresight environment, please see Jung, Teranishi, and Watanabe (2005).

7.7 Proofs of propositions

7.7.1 Proposition 1

The derivation of subsidy for commitment equilibrium that makes the steady-state coincide with the efficient one is given in section 7.4.2. Section 7.5.2 shows that the same steady state subsidy applies to Markov perfect equilibrium.

7.7.2 Proposition 2

We will prove it only for productivity shock $\hat{\xi_t^P}$.

The linearized system of FOCs and constraints for the commitment equilibrium with no government spending is

$$
\begin{aligned}
0 &= -\hat{Y}_t \frac{1+\phi-\kappa}{\kappa^2}\lambda + \hat{\xi_t^P}\frac{1+\phi}{\kappa^2}\lambda - \hat{\xi_t^C}\frac{\lambda}{\kappa} + \hat{\phi_{6t}} - \hat{\phi_{3t}}\varepsilon\frac{1+\phi-\kappa}{\kappa^2}\lambda \\
0 &= -\hat{\phi_{3t}}\varepsilon s\frac{\varepsilon-1}{\varepsilon}\sigma + \hat{C}_t^*\sigma^2\overline{\phi_{5t}} + \hat{C}_t\sigma\left[(1-\sigma)\overline{\phi_{5t}} - 1\right] \\
&\quad + \hat{\xi_t^C}\left[1 + \sigma\overline{\phi_{5t}}\right] - \sigma\hat{\phi_{5t}} - \hat{\phi_{6t}}(1-\alpha) - \hat{\varsigma}_t\frac{\lambda}{\kappa}(1-\alpha)\alpha\eta \\
0 &= \hat{\phi_{3t}} - \hat{\Pi_{H,t}}\frac{\lambda}{\kappa} - \hat{\phi_{3t-1}} \\
0 &= -\hat{C}_t\eta\alpha(1-\alpha)\frac{\lambda}{\kappa} - \hat{\phi_{3t}}\frac{\alpha\varepsilon}{\kappa}\lambda - \hat{C}_t^*\eta\alpha\frac{\lambda}{\kappa} + \hat{\phi_{5t}}(1-\alpha) - \hat{\phi_{6t}}\eta\alpha(2-\alpha) \\
&\quad + \hat{\varsigma}_t(\alpha-1)\alpha\left[(2-\eta)\overline{\phi_{5t}} + \eta\frac{\lambda}{\kappa}\left(2\alpha\eta - \alpha - \eta + \frac{\eta-1}{1-\alpha}\right)\right]
\end{aligned}
$$

$$\beta\hat{i}_t - \hat{\xi_t^C} + \sigma\hat{C}_t = E_t\left[-\hat{\xi_{t+1}^C} + \sigma\hat{C_{t+1}} + \hat{\Pi_{H,t+1}} + \alpha\hat{\varsigma_{t+1}}\right] - \alpha\hat{\varsigma}_t$$

$$\begin{aligned} 0 = & -\varepsilon s\frac{\varepsilon-1}{\varepsilon}\sigma\hat{C}_t - \varepsilon\frac{1}{\kappa^2}\left((1+\phi-\kappa)\lambda\hat{Y}_t - (1+\phi)\lambda\hat{\xi_t^P}\right) \\ & -\varepsilon\frac{\lambda}{\kappa}\alpha\hat{\varsigma}_t + 2d_1\hat{\Pi_{H,t}} - \beta E_t\left[2d_1\hat{\Pi_{H,t+1}}\right] \end{aligned}$$

$$\hat{Y}_t = (2-\alpha)\eta\alpha\hat{\varsigma}_t + (1-\alpha)\hat{C}_t + \alpha\hat{C}_t^*$$

$$(1-\alpha)\hat{\varsigma}_t + \left(\hat{\xi_t^C} - \sigma\hat{C}_t\right) = -\sigma\hat{C}_t^*$$

Now let's plug in $\sigma = \eta = 1$, $\hat{\Pi_{H,t}} = 0$, $s = (1-\alpha)\frac{\varepsilon}{\varepsilon-1}$, $\frac{\lambda}{\kappa} = 1-\alpha$, $\overline{\phi_{5t}} = \alpha(2-\alpha)$ and let's ignore other shocks and the Euler equation (since it just defines the nominal interest rate).

$$\begin{aligned} 0 &= -\hat{Y}_t\frac{1+\phi-\kappa}{\kappa}(1-\alpha) + \hat{\xi_t^P}\frac{1+\phi}{\kappa}(1-\alpha) + \hat{\phi_{6t}} - \hat{\phi_{3t}}\varepsilon\frac{1+\phi-\kappa}{\kappa}(1-\alpha) \\ 0 &= -\hat{\phi_{3t}}\varepsilon(1-\alpha) - \hat{C}_t \\ &\quad -\hat{\phi_{5t}} - \hat{\phi_{6t}}(1-\alpha) - \hat{\varsigma}_t(1-\alpha)^2\alpha \\ 0 &= \hat{\phi_{3t}} - \hat{\phi_{3t-1}} \\ 0 &= -\hat{C}_t\alpha(1-\alpha)^2 - \hat{\phi_{3t}}\alpha\varepsilon(1-\alpha) + \hat{\phi_{5t}}(1-\alpha) - \hat{\phi_{6t}}\alpha(2-\alpha) \\ &\quad + \hat{\varsigma}_t(\alpha-1)\alpha\left[\alpha(2-\alpha) - (1-\alpha)^2\right] \end{aligned}$$

$$\begin{aligned} 0 = & -\varepsilon(1-\alpha)\hat{C}_t - \varepsilon\frac{(1-\alpha)}{\kappa}\left((1+\phi-\kappa)\hat{Y}_t - (1+\phi)\hat{\xi_t^P}\right) \\ & -\varepsilon(1-\alpha)\alpha\hat{\varsigma}_t \end{aligned}$$

$$\hat{Y}_t = (2-\alpha)\alpha\hat{\varsigma}_t + (1-\alpha)\hat{C}_t$$

$$(1-\alpha)\hat{\varsigma}_t = \hat{C}_t$$

After further simplification, this system yields

$$\begin{aligned} \hat{\phi_{6t}} &= -\hat{\xi_t^P}(1-\alpha) \\ \hat{\phi_{5t}} &= -\hat{\xi_t^P}\alpha(2-\alpha)(1-\alpha) \\ \hat{C}_t &= \hat{\xi_t^P}(1-\alpha) \\ \hat{\phi_{3t}} &= 0 \end{aligned}$$

$$\hat{Y}_t = \hat{\varsigma}_t = \hat{\xi_t^P}$$

What is left to show is that this solution also achieves the efficient allocation, which is

$$\begin{aligned} 0 = & \hat{C}_t(1-\alpha)\left[\frac{\kappa}{\lambda}\frac{\sigma-1}{\sigma} + (1-\alpha)\left(\frac{1}{\sigma} - \eta\right)\right] - \hat{\xi_t^P}\frac{(1-\alpha)}{\sigma\lambda}(1+\phi) \\ & + \hat{\varsigma}_t(1-\alpha)\left[\eta\alpha(2\eta\alpha - \alpha - \eta) - \eta(\eta\alpha - 1) - \frac{\eta}{(1-\alpha)}\right. \\ & \left. + \frac{(1-\alpha)}{\sigma}(2\eta\alpha - \alpha - 1) + \frac{\kappa}{\sigma\lambda}(\alpha + 1 - \eta\alpha)\right] \\ & + \hat{Y}_t\left[\eta + \frac{(1-\alpha)}{\sigma\lambda}(1+\phi-\kappa)\right] \end{aligned}$$

$$\hat{Y}_t = \hat{\varsigma}_t\left[(1-\alpha)\eta\alpha + \alpha\eta\right] + \hat{C}_t(1-\alpha) + \hat{C}_t^*\alpha$$

$$\hat{\varsigma}_t (1-\alpha) + \left(-\sigma \hat{C}_t + \hat{\xi}_t^C\right) = -\sigma \hat{C}_t^*$$

After plugging in values for the special case considered it becomes

$$0 = -\hat{\xi}_t^P \frac{(1+\phi)}{\kappa} + \hat{Y}_t \left[1 + \frac{1+\phi-\kappa}{\kappa}\right]$$

$$\hat{Y}_t = (2-\alpha)\,\alpha\hat{\varsigma}_t + (1-\alpha)\,\hat{C}_t$$

$$(1-\alpha)\,\hat{\varsigma}_t = \hat{C}_t$$

which shows clearly that this is the same solution as the commitment one.

7.7.3 Proposition 3

Again, we will prove it only for productivity shock $\hat{\xi_t^P}$.

The linearized system of FOCs and constraints for the Markov perfect equilibrium with no government spending is

$$\begin{aligned}
0 &= -\frac{1+\phi-\kappa}{\kappa^2}\lambda\hat{Y}_t + \frac{1+\phi}{\kappa^2}\lambda\hat{\xi}_t^P - \frac{\lambda}{\kappa}\hat{\xi}_t^C + \hat{\phi}_{6t} + \hat{\phi}_{3t}\left[-\varepsilon\frac{1+\phi-\kappa}{\kappa^2}\lambda\right] \\
0 &= -\hat{\phi}_{3t}\varepsilon\frac{\lambda}{\kappa}\sigma - (1-\alpha)\,\hat{\phi}_{6t} - (1-\alpha)\,\eta\alpha\frac{\lambda}{\kappa}\hat{\varsigma}_t \\
&\quad - \hat{\phi}_{5t}\sigma + \overline{\phi_{5t}}\sigma^2\hat{C}_t^* + \sigma\left[\overline{\phi_{5t}}\,(1-\sigma) - 1\right]\hat{C}_t + \hat{\xi}_t^C\left[\sigma\overline{\phi_{5t}} + 1\right] \\
0 &= \hat{\phi}_{3t} - \frac{\lambda}{\kappa}\hat{\Pi}_{H,t} \\
0 &= -\hat{C}_t\frac{\lambda}{\kappa}\eta\alpha\,(1-\alpha) - \hat{\phi}_{3t}\frac{\varepsilon\alpha\lambda}{\kappa} \\
&\quad - \frac{\lambda}{\kappa}\eta\alpha\hat{C}_t^* - \hat{\phi}_{6t}\eta\alpha\,(2-\alpha) + \hat{\phi}_{5t}\,(1-\alpha) \\
&\quad + \hat{\varsigma}_t\alpha\,(\alpha-1)\left[\overline{\phi_{5t}}\,(2-\eta) + \frac{\lambda}{\kappa}\eta\left(2\eta\alpha - \eta - \alpha + \frac{\eta-1}{1-\alpha}\right)\right]
\end{aligned}$$

$$\beta\hat{i}_t - \hat{\xi_t^C} + \sigma\hat{C}_t = E_t\left[-\hat{\xi_{t+1}^C} + \sigma\hat{C_{t+1}} + \hat{\Pi_{H,t+1}} + \alpha\hat{\varsigma_{t+1}}\right] - \alpha\hat{\varsigma}_t$$

$$\begin{aligned}
0 &= -\varepsilon s\frac{\varepsilon-1}{\varepsilon}\sigma\hat{C}_t - \varepsilon\frac{1}{\kappa^2}\left((1+\phi-\kappa)\,\lambda\hat{Y}_t - (1+\phi)\,\lambda\hat{\xi_t^P}\right) \\
&\quad - \varepsilon\frac{\lambda}{\kappa}\alpha\hat{\varsigma}_t + 2d_1\hat{\Pi_{H,t}} - \beta E_t\left[2d_1\hat{\Pi_{H,t+1}}\right]
\end{aligned}$$

$$\begin{aligned}
\hat{Y}_t &= (2-\alpha)\,\eta\alpha\hat{\varsigma}_t + (1-\alpha)\,\hat{C}_t + \alpha\hat{C}_t^* \\
(1-\alpha)\,\hat{\varsigma}_t &+ \left(\hat{\xi_t^C} - \sigma\hat{C}_t\right) = -\sigma\hat{C}_t^*
\end{aligned}$$

The only difference from the commitment problem in 7.7.2 is the absence of the term with $\hat{\phi_{3t-1}}$ in the third FOC. Since in the solution to the commitment problem $\hat{\phi_{3t}} = 0$, it is clear that the same solution as before will satisfy this system.

7.7.4 Proposition 4

We will prove it only for productivity shock $\hat{\xi_t^P}$.

Let's substitute terms of trade instead of consumption using risk sharing condition and by doing this get rid of consumption in the system of FOCs from 7.4.4 and get

$$
\begin{aligned}
0 &= -\hat{Y}_t \frac{1+\phi-\kappa}{\kappa^2}\lambda + \hat{\xi_t^P} \frac{1+\phi}{\kappa^2}\lambda + \hat{\phi_{6t}} - \hat{\phi_{3t}}\varepsilon \frac{1+\phi-\kappa}{\kappa^2}\lambda \\
0 &= -\hat{\phi_{3t}}\varepsilon s \frac{\varepsilon-1}{\varepsilon}\sigma + \hat{\varsigma}_t \left[(1-\sigma)\frac{\lambda}{\kappa}\eta\alpha\frac{(2-\alpha)}{[1-\alpha]} - 1 - \frac{\lambda}{\kappa}\alpha\eta\right](1-\alpha) \\
&\quad - \sigma\hat{\phi_{5t}} - \hat{\phi_{6t}}(1-\alpha) + \sigma\hat{\gamma}_{1t-1} - \sigma\hat{\gamma}_{1t} \\
0 &= \hat{\phi_{3t}} + \frac{1}{2d_1}\hat{\gamma}_{1t-1} - \hat{\Pi_{H,t}}\frac{\lambda}{\kappa} - \hat{\phi_{3t-1}} \\
0 &= -\hat{\phi_{3t}}\frac{\alpha\varepsilon}{\kappa}\lambda + \hat{\phi_{5t}}(1-\alpha) - \hat{\phi_{6t}}\eta\alpha(2-\alpha) \\
&\quad + \hat{\varsigma}_t(\alpha-1)\alpha\left[(2-\eta)\frac{\lambda}{\kappa}\eta\alpha\frac{(2-\alpha)}{[1-\alpha]} + \eta\frac{\lambda}{\kappa}\left(2\alpha\eta-\alpha-\eta+\frac{\eta-1}{1-\alpha}\right) + \eta\frac{\lambda}{\kappa}\frac{(1-\alpha)}{\sigma}\right] \\
&\quad + \hat{\gamma}_{1t-1}\alpha - \hat{\gamma}_{1t}\beta\alpha
\end{aligned}
$$

Next let's combine some of this conditions to get rid of $\hat{\phi_{5t}}$ and $\hat{\phi_{6t}}$, and let's denote

$$
\begin{aligned}
\tilde{x}_t &\equiv \hat{Y}_t\frac{1+\phi-\kappa}{\kappa^2}\lambda\left[\eta\alpha(2-\alpha)+\frac{(1-\alpha)^2}{\sigma}\right] - \hat{\xi_t^P}\frac{1+\phi}{\kappa^2}\lambda\left[\eta\alpha(2-\alpha)+\frac{(1-\alpha)^2}{\sigma}\right] \\
&\quad - \hat{\varsigma}_t(\alpha-1)\left[(2-\eta)\frac{\lambda}{\kappa}\eta\alpha^2\frac{(2-\alpha)}{[1-\alpha]} + \eta\alpha\frac{\lambda}{\kappa}\left(2+2\alpha\eta-2\alpha-\eta+\frac{\eta-1}{1-\alpha}\right) - \frac{\eta\alpha^2}{\sigma}\frac{\lambda}{\kappa} - \frac{(\alpha-1)}{\sigma}\right] \\
&= \hat{x}_t\frac{1+\phi-\kappa}{\kappa^2}\lambda\left[\eta\alpha(2-\alpha)+\frac{(1-\alpha)^2}{\sigma}\right] \\
&\quad - \hat{\varsigma_t^{gap}}(\alpha-1)\left[(2-\eta)\frac{\lambda}{\kappa}\eta\alpha^2\frac{(2-\alpha)}{[1-\alpha]} + \eta\alpha\frac{\lambda}{\kappa}\left(2+2\alpha\eta-2\alpha-\eta+\frac{\eta-1}{1-\alpha}\right) - \frac{\eta\alpha^2}{\sigma}\frac{\lambda}{\kappa} - \frac{(\alpha-1)}{\sigma}\right] \\
&\quad + \hat{\xi_t^P}\left[\frac{1+\phi-\kappa}{\kappa^2}\lambda\left[\eta\alpha(2-\alpha)+\frac{(1-\alpha)^2}{\sigma}\right]\Psi_7 - \frac{1+\phi}{\kappa^2}\lambda\left(\eta\alpha(2-\alpha)+\frac{(1-\alpha)^2}{\sigma}\right)\right. \\
&\quad - (\alpha-1)\left[(2-\eta)\frac{\lambda}{\kappa}\eta\alpha^2\frac{(2-\alpha)}{[1-\alpha]} + \eta\alpha\frac{\lambda}{\kappa}\left(2+2\alpha\eta-2\alpha-\eta+\frac{\eta-1}{1-\alpha}\right) - \frac{\eta\alpha^2}{\sigma}\frac{\lambda}{\kappa} - \frac{(\alpha-1)}{\sigma}\right]\times \\
&\quad \left.\times\left[(2-\alpha)\eta\alpha+\frac{(1-\alpha)^2}{\sigma}\right]^{-1}\Psi_7\right]
\end{aligned}
$$

to get

$$
\begin{aligned}
0 &= \hat{\Pi_{H,t}} - \frac{\kappa}{\lambda}\hat{\phi_{3t}} + \frac{\kappa}{\lambda}\hat{\phi_{3t-1}} - \frac{1}{2d_1}\frac{\kappa}{\lambda}\hat{\gamma}_{1t-1} \\
0 &= \frac{1}{(1+\beta\alpha-\alpha)}\tilde{x}_t + \hat{\gamma}_{1t} - \frac{1}{(1+\beta\alpha-\alpha)}\hat{\gamma}_{1t-1} \\
&\quad + \hat{\phi_{3t}}\varepsilon\frac{\lambda}{\kappa}\frac{1}{(1+\beta\alpha-\alpha)}\left[\frac{1+\phi-\kappa}{\kappa}\left(\eta\alpha(2-\alpha)+\frac{(1-\alpha)^2}{\sigma}\right)+1\right]
\end{aligned}
$$

After introducing the following notation

$$\tilde{\lambda} \equiv \frac{1}{(1+\beta\alpha-\alpha)}$$
$$\tilde{\phi_{1t}} \equiv \hat{\gamma}_{1t}$$
$$\tilde{\beta} \equiv (1+\beta\alpha-\alpha)$$
$$\tilde{\phi_{2t}} \equiv -\frac{\kappa}{\lambda}\hat{\phi_{3t}}$$
$$\tilde{\kappa} \equiv \varepsilon\frac{\lambda^2}{\kappa^2}\frac{1}{(1+\beta\alpha-\alpha)}\left[\frac{1+\phi-\kappa}{\kappa}\left(\eta\alpha\,(2-\alpha)+\frac{(1-\alpha)^2}{\sigma}\right)+1\right]$$
$$\tilde{\pi}_t \equiv \hat{\Pi_{H,t}}$$
$$\tilde{\sigma} \equiv \frac{1}{2d_1}\frac{\kappa}{\lambda}(1+\beta\alpha-\alpha)$$

the same system becomes

$$\tilde{\lambda}\tilde{x}_t+\tilde{\phi_{1t}}-\frac{1}{\tilde{\beta}}\tilde{\phi_{1t-1}}-\tilde{\kappa}\tilde{\phi_{2t}}=0$$
$$\tilde{\pi}_t+\tilde{\phi_{2t}}-\tilde{\phi_{2t-1}}-\frac{\tilde{\sigma}}{\tilde{\beta}}\tilde{\phi_{1t-1}}=0$$

which is isomorphic to Eggertsson and Woodford (2003). Therefore price level target rule will have the same form.

7.7.5 Reformulation of targeting rule and PSE conditions

Note that the same targeting rule as in previous section could be expressed thorugh different gaps, that is

$$\tilde{x}_t = \tilde{Y_t}^{gap}\frac{1+\phi-\kappa}{\kappa^2}\lambda\left[\eta\alpha\,(2-\alpha)+\frac{(1-\alpha)^2}{\sigma}\right]$$
$$-\tilde{\varsigma_t}^{gap}(\alpha-1)\left[(2-\eta)\frac{\lambda}{\kappa}\eta\alpha^2\frac{(2-\alpha)}{[1-\alpha]}+\eta\alpha\frac{\lambda}{\kappa}\left(2+2\alpha\eta-2\alpha-\eta+\frac{\eta-1}{1-\alpha}\right)-\frac{\eta\alpha^2}{\sigma}\frac{\lambda}{\kappa}-\frac{(\alpha-1)}{\sigma}\right]$$

where

$$\tilde{Y_t}^{gap} \equiv \hat{Y}_t-\hat{\xi_t^P}\frac{1+\phi}{1+\phi-\kappa}$$
$$\tilde{\varsigma_t}^{gap} \equiv \hat{\varsigma}_t-\hat{\xi_t^P}0.$$

Then linearized PSE conditions in terms of these gaps could be represented as

$$\begin{aligned}\tilde{Y_t}^{gap} &= E_t\tilde{Y_{t+1}}^{gap}-\frac{(1-\alpha)}{\sigma}\tilde{r_t}^{gap}\\ &\quad+(2-\alpha)\,\eta\alpha\,(\tilde{\varsigma_t}^{gap}-E_t\tilde{\varsigma_{t+1}}^{gap})\\ &\quad-\frac{1+\phi}{1+\phi-\kappa}\left(\hat{\xi_t^P}-E_t\hat{\xi_{t+1}^P}\right)\end{aligned}$$

$$\begin{aligned}\hat{\Pi_{H,t}} &= \beta E_t\hat{\Pi_{H,t+1}}\\ &\quad+\frac{\varepsilon}{2d_1}\frac{\lambda}{\kappa}\left[\frac{(1+\phi-\kappa)}{\kappa}+\frac{\sigma}{1-\alpha}\right]\tilde{Y_t}^{gap}\\ &\quad+\frac{\varepsilon}{2d_1}\frac{\lambda}{\kappa}\left[\alpha-\frac{(2-\alpha)}{1-\alpha}\eta\sigma\alpha\right]\tilde{\varsigma_t}^{gap}\\ &\quad+\left[\frac{\varepsilon}{2d_1}\frac{\lambda}{\kappa}\left[\frac{(1+\phi-\kappa)}{\kappa}+\frac{\sigma}{1-\alpha}\right]\left(\frac{1+\phi}{1+\phi-\kappa}-\Psi_7\right)\right.\\ &\quad\left.-\frac{\varepsilon}{2d_1}\frac{\lambda}{\kappa}\left[\alpha-\frac{(2-\alpha)}{1-\alpha}\eta\sigma\alpha\right]\Psi_7+\Psi_{22}\right]\hat{\xi_t^P}\end{aligned}$$

$$\tilde{Y}_t^{gap} = \left[(2-\alpha)\eta\alpha + \frac{(1-\alpha)^2}{\sigma}\right]\tilde{\varsigma}_t^{gap} - \left(\left[(2-\alpha)\eta\alpha + \frac{(1-\alpha)^2}{\sigma} - 1\right]\Psi_7 + \frac{1+\phi}{1+\phi-\kappa}\right)\hat{\xi_t^P}$$

where

$$\tilde{r_t}^{gap} = \hat{r_t}.$$

7.7.6 Proposition 5

We will prove it only for productivity shock $\hat{\xi_t^P}$.

Repeating the steps from the previous subsection, but for the system from 7.5.4 yields

$$\tilde{x}_t = -\hat{\Pi_{H,t}}\varepsilon\frac{\lambda^2}{\kappa^2}\left[\frac{1+\phi-\kappa}{\kappa}\left(\eta\alpha(2-\alpha) + \frac{(1-\alpha)^2}{\sigma}\right) + 1\right]$$

where $\tilde{x}_t$ is the same as in previous subsection. Moreover, in notation from 7.7.4 this targeting rule becomes

$$\tilde{x}_t = -\frac{\tilde{\kappa}}{\tilde{\lambda}}\hat{\Pi_{H,t}}$$

7.7.7 Proposition 6

We will prove it only for productivity shock $\hat{\xi_t^P}$.

Let's plug in $\sigma = \sigma' = \eta = 1$, $\hat{\Pi_{H,t}} = 0$ into the system of FOCs and constraints from 7.4.3. Also let's ignore other shocks and the Euler equation (since it just defines the nominal interest rate).

$$\begin{aligned}
0 &= -\hat{Y}_t\frac{1+\phi-\kappa}{\kappa^2}\lambda + \hat{\xi_t^P}\frac{1+\phi}{\kappa^2}\lambda + \hat{\phi_{6t}} - \hat{\phi_{3t}}\varepsilon\frac{1+\phi-\kappa}{\kappa^2}\lambda \\
0 &= -\hat{\phi_{3t}}\varepsilon s\frac{\varepsilon-1}{\varepsilon}\theta_C^{-2} - \hat{\varsigma}_t\frac{\lambda}{\kappa}(1-\alpha)\alpha \\
&\quad + \hat{C}_t\left[-\theta_C^{-2} + \left(2 - 2\theta_C^{-1}\right)\theta_C^{-2}\overline{\phi_{5t}}\right] - \theta_C^{-1}\hat{\phi_{5t}} - \hat{\phi_{6t}}(1-\alpha) \\
0 &= -\lambda_G\theta_G^{-2}\hat{G}_t - (1-\alpha)\hat{\phi_{6t}} - (1-\alpha)\alpha\frac{\lambda}{\kappa}\hat{\varsigma}_t \\
0 &= \hat{\phi_{3t}} - \hat{\phi_{3t-1}} \\
0 &= -\hat{C}_t\alpha(1-\alpha)\frac{\lambda}{\kappa} - \hat{\phi_{3t}}\frac{\alpha\varepsilon}{\kappa}\lambda + \hat{\phi_{5t}}(1-\alpha) - \hat{\phi_{6t}}\alpha(2-\alpha) \\
&\quad + \hat{\varsigma}_t(\alpha-1)\alpha\left[\overline{\phi_5} + \frac{\lambda}{\kappa}(\alpha-1)\right] \\
&\quad - \hat{G}_t\alpha(1-\alpha)\frac{\lambda}{\kappa}
\end{aligned}$$

$$0 = -\varepsilon s\frac{\varepsilon-1}{\varepsilon}\theta_C^{-2}\hat{C}_t - \varepsilon\frac{1}{\kappa^2}\left((1+\phi-\kappa)\lambda\hat{Y}_t - (1+\phi)\lambda\hat{\xi_t^P}\right) - \varepsilon\frac{\lambda}{\kappa}\alpha\hat{\varsigma}_t$$

$$\hat{Y}_t = (2-\alpha)\alpha\hat{\varsigma}_t + (1-\alpha)\left(\hat{C}_t + \hat{G}_t\right)$$

$$(1-\alpha)\hat{\varsigma}_t = \theta_C^{-1}\hat{C}_t$$

After we further simplify it and plug some steady state relationships like

$$\overline{\phi_5} = \alpha\frac{(2-\alpha)}{\theta_C} = \alpha\lambda_G\theta_G^{-1}\frac{(2-\alpha)}{(1-\alpha)^2} = \alpha\frac{\lambda}{\kappa}\frac{(2-\alpha)}{(1-\alpha)}$$

we get

$$\hat{Y}_t = \hat{\varsigma}_t = \hat{\xi_t^P}$$

$$\hat{\phi_{6t}} = -\hat{\xi_t^P}\frac{\lambda}{\kappa}$$
$$\hat{G_t} = \hat{\xi_t^P}\left(1-\alpha\right)\theta_G$$
$$\hat{\phi_{3t}} = 0$$
$$\hat{C_t} = \left(1-\alpha\right)\theta_C\hat{\xi_t^P}$$
$$\hat{\phi_{5t}} = \left[2\left(1-\theta_C^{-1}\right)\theta_C^{-1}-1\right]\alpha\left(2-\alpha\right)\left(1-\alpha\right)\hat{\xi_t^P}$$

What is left to show is that this solution also achieves the efficient allocation, which is (after we plug in values for our special case in the system from 7.2.2)

$$\begin{aligned}
0 = &-\hat{\xi}_t^P\frac{(1-\alpha)}{\lambda}\left(1+\phi\right)+\hat{Y}_t\left[1+\frac{(1-\alpha)}{\lambda}\left(1+\phi-\kappa\right)\right]\\
&-\hat{G}_t\left(1-\alpha\right)^2+\hat{\varsigma}_t\left(1-\alpha\right)\left[\left(\alpha-1\right)^2-\frac{1}{(1-\alpha)}\right.\\
&\left.-\left(\alpha-1\right)^2\theta_C+\frac{\kappa}{\lambda}\right]\\
0 = &-\lambda_G\theta_G^{-2}\hat{G}_t-\left(1-\alpha\right)\frac{1+\phi-\kappa}{\kappa^2}\lambda\hat{Y}_t\\
&+\left(1-\alpha\right)\frac{1+\phi}{\kappa^2}\lambda\hat{\xi}_t^P-\frac{\lambda}{\kappa}\left(1-\alpha\right)\alpha\hat{\varsigma}_t
\end{aligned}$$
$$\hat{Y}_t = \left(2-\alpha\right)\alpha\hat{\varsigma}_t+\left(1-\alpha\right)\left(\hat{C}_t+\hat{G}_t\right)$$
$$\left(1-\alpha\right)\hat{\varsigma}_t-\theta_C^{-1}\hat{C}_t = 0$$

Further simplification yields

$$\begin{aligned}
0 = &-\hat{\xi}_t^P\frac{(1+\phi)}{\kappa}\theta_C+\hat{Y}_t\left[1+\theta_C\frac{(1+\phi-\kappa)}{\kappa}\right]\\
&-\hat{G}_t\left(1-\alpha\right)^2+\hat{\varsigma}_t\left(\left(1-\alpha\right)^3-1\right)\left(1-\theta_C\right)\\
0 = &-\theta_G^{-1}\hat{G}_t-\frac{1+\phi-\kappa}{\kappa}\hat{Y}_t+\frac{1+\phi}{\kappa}\hat{\xi}_t^P-\alpha\hat{\varsigma}_t
\end{aligned}$$

$$\hat{Y}_t = \left(2-\alpha\right)\alpha\hat{\varsigma}_t+\left(1-\alpha\right)\left(\hat{C}_t+\hat{G}_t\right)$$

$$\left(1-\alpha\right)\theta_C\hat{\varsigma}_t = \hat{C}_t$$

Straightforward substitution of the solution to the commitment problem shows that this system is also satisfied at this solution.

7.7.8 Proposition 7

Direct comparison of the systems in 7.4.3 and 7.5.3 shows that the only difference between them is the absence of the term with $\hat{\phi_{3t-1}}$ in the forth FOC in the discretion problem. But since optimal solution under commitment implied $\hat{\phi_{3t}} = 0$, then the previous commitment solution should also satisfy the discretion optimality conditions.