The zero lower bound on the interest rate and a Neoclassical Phillips curve

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Abstract
I derive the exact analytical solution for optimal monetary policy given a Neoclassical Phillips curve and a zero lower bound on the nominal interest rate. There is a particular range of interest rate rule parameters that may close the output gap. One way of closing the output gap involves stable but high inflation (the divine coincidence). In the general case inflation is variable and potentially lower. Thus, one can achieve stable OR low inflation, but not both. When the productivity shock has an unbounded support, only the variable inflation version of optimal policy is implementable. Optimal policy then involves a lagged interest rate response to shocks and a random walk price level.

*The views in this paper are those of the author. They do not necessarily represent those of Norges Bank.
1 Introduction

This paper discusses monetary policy in a model with a Neoclassical Phillips curve and a productivity shock. In one version of the model, the productivity shock has a bounded support (and a uniform distribution). In another version, the shock has an unbounded support (and a lognormal distribution). One result is that the zero lower bound does not prevent the central bank from achieving the first best allocation. I describe a simple interest rate rule that implements this allocation. The intuition is that with a Neoclassical Phillips curve, expected inflation is not costly. In equilibrium, the nominal rate may be stable while a variable inflation rate delivers a real interest rate that tracks the natural real rate.

A second result is that the log of the price level follows a random walk (with drift) under optimal monetary policy. A lagged interest rate response to shocks is required, except in a limit case where the inflation rate is high and perfectly stabilized. Since the first best allocation is attainable in the present model, there is no commitment problem associated with delivering a lagged interest rate response. Welfare is the same whether inflation is stabilized or not, as long as any movements in inflation are known one period in advance - which is the case under the set of optimal policies considered here.

A third result is that more volatile structural shocks require either a higher inflation rate or a more volatile inflation rate, if output is to be stabilized. The risky steady states for output and inflation\(^1\) depend on the distribution of disturbances and monetary policy. With a bounded productivity shock, there is a range of interest rate rule parameters that deliver output stabilization. One way of closing the output gap in this case involves stable but high inflation (the divine coincidence). That same policy choice - with a high enough inflation target - would deliver output stabilization also given a New Keynesian Phillips Curve, when productivity shocks have a bounded support. The straightforward reason for this is that a high enough inflation target gives room enough for maneuver, so that real shocks always can be absorbed by a variable nominal interest rate. But with a Neoclassical Phillips Curve there are more options for output stabilization, and inflation may be variable and potentially lower under optimal policy. The latter point is a result highlighted in this paper.

The menu of possible policies that will deliver an optimal outcome narrows to only one possible choice in the context of an unbounded productivity shock; Output is fully stabilized if and only if inflation varies around its trend, and tracks the natural real interest rate. This policy choice would not deliver output stabilization under a New Keynesian Phillips Curve, where fully expected but variable inflation is costly.

\(^{1}\)I use the definition of the risky steady state established by Coeurdacier, Rey, and Winant (2011).
If authorities deviate from optimal policy, and instead choose to stabilize inflation closer to its trend, there will be episodes of negative output gaps. The distribution of inflation and output gaps will then be skewed to the left. This policy is not time consistent. The frequency of hitting the zero lower bound will be increasing in the variance of productivity shocks, decreasing in the level of the inflation target, and increasing in the degree to which inflation is stabilized.

This paper is not about price level determinacy or the potential multiplicity of equilibria associated with the zero lower bound on interest rates\(^2\). Rather, it is concerned with situations where monetary policy may be prevented from being expansionary enough to stabilize output, as discussed in seminal work by Krugman (1998) and Eggertsson and Woodford (2003).

Much of the literature on the zero lower bound for the nominal interest rate relates to the case of a New Keynesian Phillips curve\(^3\). As discussed in among others Wolman (1998), Adam and Billi (2007) and Nakov (2008), the degree of intrinsic (endogenous) inflation stickiness determines the costs of the zero lower bound constraint. The value of being able to implement policy under commitment is higher when inflation is more sticky.

In this paper, some agents set prices flexibly, while some agents set prices one period in advance. Synchronized price setting creates a Neoclassical Phillips curve and enables me to derive an exact analytical solution, following Henderson and Kim (2001). The case of a Neoclassical Phillips curve may be of interest because it represents a limit case; Systematic monetary policy is useful\(^4\), but anticipated policy beyond the next period does not have real effects even though agents are fully forward looking. The Neoclassical Phillips curve establishes an example where price level targeting is unhelpful in a low inflation environment. It may on the other hand be argued that the New Keynesian Phillips curve describes a case where anticipated policy too far into the future is powerful, and where the value of commitment might be overestimated.

As shown in Alstadheim (2013), the Neoclassical Phillips curve may be derived as the limit of the New Keynesian Phillips Curve, when firms fully index their prices to expected future inflation. However, a model with the generalized Phillips curve cannot be solved without approximation. In this paper, I study

\(^2\)See e.g. Benhabib, Schmitt-Grohe, and Uribe (2001) and also Chapter 2 in Woodford (2003). See also Alstadheim and Henderson (2006). For a more recent discussion, see Aruoba and Schorfheide (2013).

\(^3\)But see Fuhrer and Madigan (1997) and Wolman (2005). Adam and Billi (2006) and Adam and Billi (2007) are important contributions to the literature on the zero lower bound in New Keynesian models. See also Braun and Korber (2011).

\(^4\)In Sargent and Wallace (1975), a Neoclassical Phillips curve setup is applied, and there monetary policy is useless for stabilization purposes. In their setup, the expected real interest rate calculated with inflation expectation as of yesterday appears in the IS curve, \(i_t - E_{t-1} \pi_{t+1}\). In this paper, the Euler equation includes the real interest rate calculated as of today, \(i_t - E_t \pi_{t+1}\), and that makes monetary policy have real effects.
an exact solution in order to integrate the treatment of level effects and stabilization effects of policy. I therefore focus on the limit case of a Neoclassical version only.

I abstract from distortions other than the one-sector price stickiness. This means that if policy removes the price stickiness distortion, any potential commitment problem also disappears in this paper. The results in Adam and Billi (2007) and Ngo (2014) highlight that level effects of policy under discretionary policy may have particular importance, given a potentially binding zero lower bound constraint, when optimal policy is not time-consistent. Those issues are not covered here.

The simple rule considered in this paper is optimal in the sense described in Woodford (2001). I use a public finance approach, where I solve for the optimal allocation in the economy, and then back out the set of parameters of the simple rule that deliver the first best outcome. The optimal rule responds directly to productivity shocks, and not only to endogenous variables.

The next section describes the model. In section 3, I solve the flexible-price version of the model. In section 4, I derive the sticky-price solution and present optimal monetary policy in the case of a uniform distribution of the productivity shock. I calibrate constant terms describing the risky steady states such that nominal levels increase when the volatility of shocks, and policy parameters, imply more interest rate volatility. In this way, the nominal interest rate never violates the zero lower bound constraint. It is thereby shown that a more stable inflation rate necessarily goes with a higher steady state inflation rate under optimal policy. I present a menu of different optimal monetary policies. In section 5, I derive the solution for the model given a lognormal distribution of the shock, and I present optimal policy in that case. There, I also simulate the model, in order to illustrate the link between the degree of inflation stabilization and the frequency with which the zero lower bound is encountered. Section 6 provides concluding remarks.

2 The model

The model includes a continuum of agents who are yeoman farmers. The agents belong either to a flexible price goods production sector or a one-period-in-advance price setting sector, and there is monopolistic competition. Goods from the two sectors are combined into one composite consumption good. Agents learn which sector they will belong to in the next period at the point in time when the sticky price agents need to set their price. They buy state contingent claims before they learn which sector they will belong to, which ensures perfect risk sharing.
Equation (1) below describes that the representative agent maximizes utility with respect to the composite consumption good $c$, her output price $p$ and money $m$ and bonds $b$, subject to a period budget constraint, where $\lambda_t$ is the Lagrange multiplier. $\beta$ is the agent’s discount factor on period utility. $\rho$ describes the inverse of the intertemporal elasticity of substitution in consumption. The period utility term $-\frac{1}{2} \kappa_t y_t^2$ represents disutility from producing output $y_t$ in period $t$. $\kappa$ is an i.i.d. supply shock that will be common to the two production sectors. It will determine the natural real interest rate.

The last term in the period utility function, $f(\frac{m_t}{P_t})$, represents utility from holding real money balances. I let $\mu$ in equation (4) be a number close to zero and disregard any welfare effects from money holdings in the following. However, modelling money demand is necessary for completeness and in order to capture the zero lower bound on interest rates. $P_t$ is the price of $c_t$ in terms of $m_t$. $\delta$ represents a satiation level of real money balances.

Each agent maximizes utility subject to the constraint that income from production after taxes or subsidies, $(1+\omega)p_t y_t$, plus financial assets and their return brought over from last period (money $m_{t-1}$, bonds $b_{t-1}$ and $(1+i_{t-1})b_{t-1}^g$) must equal taxes $t_t$, consumption expenditure $P_t c_t$ and new holdings of financial assets. $b_t^g$ is the nominal value of risk free government bonds, while $b_t$ is a vector of quantities of state contingent claims, and $\delta_{t,t+1}$ is the vector of the prices of those claims. Each state contingent claim pays one unit of currency in the subsequent period given a particular realization of the state in that period. The gross risk free nominal interest rate, $1+i_t$ (I will also use $I_t$ for this variable) is therefore equal to $[\delta_{t,t+1}\cdot1]^{-1}$, where 1 is a vector of ones. The maximization problem is

$$Max_E E_n \left\{ \sum_{t=n}^{\infty} \beta^{t-n} \left( \frac{c_t^{1-\rho}}{1-\rho} - \frac{1}{2} \kappa_t y_t^2 + f(\frac{m_t}{P_t}) + \lambda_t[(1+\omega)p_t y_t + m_{t-1} + (1+i_{t-1})b_{t-1}^g + b_{t-1} - t_t - P_t c_t - m_t - b_t^g - \delta_{t,t+1} b_t] \right) \right\},$$

(1)

where

$$c_t = \frac{c_{s,t}^{1-\gamma} c_{f,t}^{1-\gamma}}{\gamma^{(1-\gamma)}},$$

(2)

$$c_{s,t} = \int_0^1 (c_{j,t})^{\frac{\gamma-1}{\gamma}} dj \frac{\gamma}{\gamma-1}, \quad c_{f,t} = \int_0^1 (c_{i,t})^{\frac{\gamma-1}{\gamma}} di \frac{\gamma}{\gamma-1}.$$  (3)

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5 In a yeoman farmer model, the labor market is internalized. $\kappa$ may be interpreted as a labor supply shock or a productivity shock. In particular, following Obstfeld and Rogoff (1996), the productivity variable $\kappa$ may be understood as follows: Let disutility from work effort $l$ be given by $-\phi l$ and the production function be $Al^\alpha$, $\alpha < 1$. Inverting the production function gives $l = (\frac{A}{\phi})^{1/\alpha}$. Given $\alpha = \frac{1}{2}$ and $\kappa = \frac{2\phi}{\sqrt{\alpha}}$, we get $-\phi (\frac{A}{\phi})^{1/\alpha} = -\frac{1}{2} \kappa y^2$.

6 Without a satiation level, the nominal interest rate can reach the zero lower bound only in the limit when real balances go to infinity.
and
\[ f\left(\frac{m_t}{P_t}\right) = \begin{cases} -\frac{1}{2}\mu(\delta - \frac{m_t}{P_t})^2, & \frac{m_t}{P_t} \leq \delta \\ 0, & \frac{m_t}{P_t} > \delta \end{cases}. \quad (4) \]

\(c_{s,t}\) in (2) is defined in (3) as the composite sticky-price good, where \(j\) indexes producers of different period \(t\) ‘sticky-price’ \((s)\) goods \(c_{j,s,t}\). \(i\) indexes the ‘flexible-price’ \((f)\) goods \((c_{i,f,t})\) that go into the \(c_{f,t}\) composite flexible-price good. \(\gamma\) indicates the strength of preference for the good produced in the sticky price sector.

Equilibrium conditions are derived in the online appendix A, where the effect of monopolistic competition on output is eliminated by the subsidy \(1 + \frac{\theta}{\sigma - 1}\). Symmetry across producers and risk sharing among consumers simplifies aggregation. Since every period in the model may be described by the following equations (and in this sense the model is not dynamic), I replace time subscript \(t\) by \(+1\) and \(-1\) time subscripts. Capital letters denote aggregate quantities, and inflation \(\Pi \equiv \frac{P}{P_{-1}}\):

\[ \Pi = \left[ \frac{\kappa Y_f}{(\frac{1}{2}Y)^{-\rho}} \right]^{1-\gamma} \left[ \frac{E_{-1}(\kappa Y_f^2)}{E_{-1}(Y_s(\frac{1}{2}Y)^{-\rho} \Pi^{-1})} \right]^\gamma, \quad \text{(price equation)} \quad (5) \]

\[(\frac{1}{2}Y)^{-\rho} = \beta(1 + i)E\Pi^{-1}(\frac{1}{2}Y_{-1})^{-\rho}, \quad \text{(demand)} \quad (6)\]

\[ Y_f = \left( \frac{\kappa Y_f}{(\frac{1}{2}Y)^{-\rho}} \right)^{-1}(1 - \gamma)Y, \quad \text{(flex-price output)} \quad (7) \]

and

\[ Y_s = \left[ \left( \frac{E_{-1}(\kappa Y_s^2)}{E_{-1}(Y_s(\frac{1}{2}Y)^{-\rho} \Pi^{-1})} \right)^{-1}\Pi \gamma Y \right]. \quad \text{(sticky-price output)} \quad (8) \]

Equations (5)-(8) may be used together with some specification for monetary and fiscal policy to solve for inflation \(\Pi\), aggregate sticky price sector output \(Y_s\), aggregate flexible price sector output \(Y_f\), aggregate total output \(Y\) and \(1 + i\). I will use both \(I\) and \(1 + i\) to denote the gross nominal interest rate in the following.

### 2.1 Monetary and fiscal policy

Similar to Kim and Henderson (2005), I assume that authorities use an interest rate rule in the class:

\[ 1 + i = I^* \beta^{-1} \kappa^{\lambda_s} \kappa_{-1}^{\lambda_s-1} \Pi^{\lambda_s}. \quad \text{(interest-rate rule)} \quad (9) \]

Woodford (2001) studies conditions under which a rule like the Taylor rule (see Taylor (1993), Taylor (1999)) delivers optimal outcomes in a simple New
Keynesian model. (9) is in the spirit of the results in Woodford (2001), except that it is in levels rather than in logs. The size of the response coefficient $\lambda_\kappa$ may potentially be constrained by the zero lower bound and thus be state dependent in this paper.

Woodford notes that a standard Taylor rule will deliver the optimal allocation of output in a simple New Keynesian model under certain conditions: First, the rule needs to respond directly to the natural real interest rate in order to deliver the optimal outcome. As Woodford, p. 235, points out: "such a variable intercept is actually in the spirit of Taylor’s prescription, which describes the intercept as incorporating "the central bank’s estimate of the equilibrium real rate of interest" (Taylor, 1999, p.325)." In this paper, the term $\beta^{-1} k^{\lambda_\kappa-\kappa-1}$ reduces to the natural real interest rate given certain choices of the policy parameters $\lambda_k$ and $\lambda_{\kappa-1}$.

A second condition noted in Woodford (2001), is that the rule needs to prescribe a response to an appropriately defined output gap. In this paper, the divine coincidence applies, and a direct response to output is therefore not needed in order to close the output gap. Also, with a Neoclassical rather than a New Keynesian Phillips curve, inflation stabilization will only be one among several alternative ways of stabilizing output. A direct response to the output gap will not be needed under those alternative optimal policies either.

Woodford also notes that the interest rate rule should adhere to the Taylor principle by responding sufficiently strongly to the inflation rate. I will fix the calibration of $\lambda_\pi$ at $\lambda_\pi = 1.5$ throughout this paper. My approach is to derive the optimal (flexible price) output level and then back out the required policy parameters $\lambda_k$ and $\lambda_{\kappa-1}$ that support this allocation, while $\lambda_\pi$ may be left at for example 1.5. Among optimal policy parameters, I will then select those that produce equilibria that are consistent with the zero lower bound on the interest rate. This approach means that I will not consider implications of possible off equilibrium path expectations.

The policy parameter $I^*$ governs the gross level of nominal variables. Since I do not impose certainty equivalence, the steady state levels will depend on policy and the variance of shocks. But in order to aid intuition, we may for a moment consider the nonstochastic case; A given gross inflation target $\Phi$ would be supported by the policy parameter choice $I^* = \Phi^{1-\lambda_\pi}$, where the exponent $1 - \lambda_\pi$ is needed in order to account for the exponent $\lambda_\pi$ on the level of inflation in the rule. The associated steady state gross nominal interest rate would then be $I^{SS} = \Phi \beta^{-1}$.

I will apply two alternative ways of anchoring the policy choice of $I^*$ and hence $\Phi$. In sections 3 and 4, I will calibrate $I^*$ to make the levels of nominal rates and inflation rates as low as possible, while still staying clear off the zero lower bound in equilibrium under different levels of variance for the structural
shock \( \kappa \). The risky steady state nominal interest rates and inflation rate will therefore be higher when the variance of shocks is higher and when policy requires the nominal rate to be more variable. In section 5, I will calibrate \( I^* \) to make the risky steady state nominal interest rate \( I^{ss} \) stable as the variance of disturbances changes. In both cases, \( I^* \) will depend on the variance of \( \kappa \) in order to produce the desired nominal paths.

Utility maximization and the No-Ponzi-game condition (see online appendix A) together establish the transversality condition;

\[
E_t \left\{ \lim_{s \to \infty} \frac{m_{t+s} + b_{t+s}}{P_{t+s}} \Pi_{j=t}^{j=s} (1 + \epsilon_j)^{-1} = 0 \right\}.
\]

(10)

I assume throughout that fiscal policy makes sure that the transversality condition holds, so that fiscal policy is passive, as in Leeper (1991). When initial net public debt \( m_n + b_n \) is positive it will be satisfied with e.g. a balanced budget rule for fiscal policy, if the nominal interest rate is at least marginally positive with some positive probability.

3 The flexible-price model

Flexible prices in both sectors mean that the relative price is determined by the fixed parameter \( \gamma \);

\[
\frac{P_s}{P_f} = \left\{ \frac{\gamma}{1 - \gamma} \right\}^\frac{1}{\gamma},
\]

(11)

and output in the two sectors are given by

\[
Y_f = \kappa^{-1} \left\{ \frac{\gamma}{1 - \gamma} \right\}^{-\frac{1}{\gamma}} \left\{ \frac{1}{2} Y \right\}^{-\rho}, \quad Y_s = \kappa^{-1} \left\{ \frac{\gamma}{1 - \gamma} \right\}^{\frac{1}{\gamma}(1 - \gamma)} \left\{ \frac{1}{2} Y \right\}^{-\rho}.
\]

(12)

Substituting the above into the expression for aggregate output

\[
Y \equiv \frac{(Y_s)(Y_f)^{1 - \gamma}}{\gamma^{\gamma}(1 - \gamma)^{1 - \gamma}},
\]

implies that

\[
Y = K \cdot \kappa^{-\frac{1}{\gamma}} \cdot \frac{1}{\gamma}, \quad K = \left[(1 - \gamma)^{(1 - \gamma)^{\frac{1}{\gamma}}} \cdot 2 \frac{\epsilon}{\gamma} \right],
\]

(13)

and

\[
C = \frac{1}{2} Y = \frac{1}{2} K \cdot \kappa^{-\frac{1}{\gamma}}.
\]

(14)

In the symmetric case, where \( \gamma = \frac{1}{2} \), we have \( \frac{P_s}{P_f} = 1 \), \( C = \kappa^{-\frac{1}{\gamma}} \) and \( Y = 2\kappa^{-\frac{1}{\gamma}} \).
With output given by equation (13), the unknown variables are the inflation rate and the nominal interest rate. The demand equation and the interest-rate rule now give me a system of two equations in two unknowns, \( \Pi \) and \( I \),

\[
\kappa_\Sigma = \beta (1 + i) E(\Pi_{t+1}^{-1} \kappa_\Sigma) \quad \text{(demand)} \tag{15}
\]

and

\[
1 + i = I^* \beta^{-1} \kappa^\lambda \kappa_{-1}^{\lambda - 1} \Pi^\lambda. \quad \text{(interest-rate rule)} \tag{16}
\]

### 3.1 The flexible-price model solution.

In order to solve (15) and (16), I follow Henderson and Kim (2001) and use the method of undetermined coefficients. The guess for the solution for price inflation is

\[
\Pi = \Phi \kappa^{\phi - \phi_{\kappa-1}}. \quad \text{(guess for } \Pi). \tag{17}
\]

An assumption about the distribution of \( \kappa \) has to be made, and here I assume that \( \kappa \) has a uniform distribution between \( \kappa_L \) and \( \kappa_H \), \( \kappa_L < \kappa_H \), and I will assume \( E(\kappa) = 1 \). In the online appendix B, it is shown that (17) is a solution of (15) and (16) with parameters as given in table 1.

<table>
<thead>
<tr>
<th>Table 1: The flexible-price model solution</th>
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<tbody>
<tr>
<td>( \phi_\kappa = \left[ \frac{\phi}{1+\phi} - \lambda_\kappa + \phi_{\kappa-1} \right] \frac{1}{\lambda_\kappa} )</td>
</tr>
<tr>
<td>( \phi_{\kappa-1} = -\frac{\lambda_\kappa - 1}{\lambda_\kappa} )</td>
</tr>
<tr>
<td>( \Phi = \beta \kappa_L^{-\lambda_\kappa - \phi - \phi_{\lambda_\kappa}} \frac{1}{1-\phi} \left( \kappa_H^{-\phi + 1} - \kappa_L^{-\phi + 1} \right) \frac{1}{\kappa_H - \kappa_L} )</td>
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The risky steady state level of the inflation rate, equal to \( \Pi = \Phi (1)^{\phi_\kappa} (1)^{\phi_{\kappa-1}} = \Phi \), is determined by the \( I^* \) level in the interest rate rule, as well as by the volatility of shocks and monetary policy. The expression for \( \Phi \) in table 1 reflects an \( I^* \) that is normalized to be as low as possible, given the zero lower bound on interest rates (see the online appendix) and the volatility of disturbances. In particular, it turns out that

\[
I^* \equiv \beta \kappa_L^{-\lambda_\kappa - \phi - \phi_{\lambda_\kappa}} \Phi^{-\lambda_\kappa} \tag{18}
\]

minimizes the feasible steady state nominal interest rate and inflation rate. With this \( I^* \), the solution for \( \Phi \) is given in (1.3), and the equilibrium nominal interest rate is given by:

\[
E(\kappa^a) = \int_{\kappa_L}^{\kappa_H} \kappa^a (\frac{1}{\kappa_H - \kappa_L}) \lesssim \int \frac{1}{1+a} \kappa_1^{1+a} \frac{1}{\kappa_1 - \kappa_1} \lesssim \frac{1}{1+a} (\kappa_H^{1+a} - \kappa_L^{1+a}) \frac{1}{\kappa_H - \kappa_L}.
\]
\[ 1 + i = (\frac{\kappa}{\kappa_L})^{(\lambda_\kappa + \phi_\kappa \lambda_\pi)} \geq 1. \]  

(19)

The equilibrium nominal interest rate reaches its minimum when \( \kappa = \kappa_L \), and its maximum when \( \kappa = \kappa_H \), and the risky steady state is \( I^{SS} = (\frac{1}{\tau_L})^{(\lambda_\kappa + \phi_\kappa \lambda_\pi)} \). This is true as long as \( \lambda_\kappa + \phi_\kappa \lambda_\pi \geq 0 \), which will be the case in this paper. Intuitively, the nominal interest rate might need to be relatively high when the shock takes on a high value, because then productivity is expected to increase (\( \kappa \) is expected to fall), consumption and potential output are expected to increase, and this situation is characterized by a relatively high natural real interest rate.

3.2 Strict inflation targeting in the flexible-price model

From the general solution for inflation in equation (17) together with table 1, we know that in order to stabilize the inflation rate perfectly we need \( \phi_\kappa = \phi_{\kappa-1} = 0 \). If authorities respond to shocks directly, they can let

\[ \lambda_\kappa = \frac{\rho}{1 + \rho} \quad \text{and} \quad \lambda_{\kappa-1} = 0, \]

(20)

in which case inflation is constant and given by

\[ \Pi = \Phi = \beta E[(\frac{\kappa}{\kappa_L})^{\frac{\phi_\kappa}{\tau_\pi}}]. \]

(21)

The risky steady state inflation rate, \( \Phi \), is increasing in the variance of \( \kappa \) (mean preserving spreads around \( \kappa = 1 \)). The equilibrium nominal interest rate is now variable and given by

\[ 1 + i = (\frac{\kappa}{\kappa_L})^{\frac{\phi_\kappa}{\tau_\pi}}. \]

(22)

If authorities choose to let \( \lambda_{\pi} \rightarrow \infty \), they can stabilize the inflation rate completely regardless of \( \lambda_\kappa \) and \( \lambda_{\kappa-1} \) since \( \phi_\kappa = \phi_{\kappa-1} = 0 \) also in that case. The constant inflation rate and the equilibrium nominal interest rate are still given by (21) and (22).

3.3 A low but variable inflation level in the flexible-price model

For future reference, it is useful to derive the flexible price solution with different types of monetary policy. The lowest possible inflation level is attainable with a nominal rate as low as possible in steady state. From equation (19), I know that a constant nominal interest rate at zero requires:

\[ \lambda_\kappa = -\phi_\kappa \lambda_\pi. \]

(23)

Given equation (1.1) and (1.2) in table 1, this requires

\[ \lambda_{\kappa-1} = \frac{\rho}{1 + \rho} \lambda_\pi, \]

(24)
while $\lambda_\kappa$ and $\lambda_\pi$ can be chosen according to (23). With this policy, and given (18), the equilibrium inflation rate is

$$
\Pi = \beta \kappa_1^0 \frac{1}{1 - \phi_\kappa + \frac{\rho}{\phi_\kappa}} (\kappa_H^{1 + \frac{\lambda_\kappa}{\phi_\kappa}} + \frac{\rho}{\phi_\kappa} - \kappa_L^{1 + \frac{\lambda_\kappa}{\phi_\kappa}} - \frac{\rho}{\phi_\kappa}) \frac{1}{\kappa_H - \kappa_L} \cdot \kappa^{\frac{\lambda_\kappa}{\phi_\kappa} \kappa_{-1}^{\frac{\rho}{\phi_\kappa}}} \quad (25)
$$

while the equilibrium solution for the nominal interest rate is

$$
1 + i = 1. \quad (26)
$$

With this policy, authorities stabilize the nominal interest rate by letting the expected inflation rate instead of the nominal interest rate move along with the natural real rate in equilibrium. In order to achieve this, we have seen in equation (24) that the policymaker has to respond to the lagged productivity shock. The interest rate rule also responds to inflation, and the net effect is a stable nominal interest rate in equilibrium.

The inflation rate may also vary with the contemporaneous shock if there is an interest rate response $\lambda_\kappa$, but the net effect is again that the equilibrium nominal interest rate is stable at zero regardless of this.

4 The sticky-price model with a uniform distribution of the productivity shock

I use the same approach as in the flexible-price case, but with price stickiness in the model I need to solve for inflation and output simultaneously. I first simplify the model in order to get a pair of equations in $\Pi$ and $Y$ only. Next, I use the interest-rate rule and guesses for output and inflation solutions to solve using the method of undetermined coefficients.

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$I^\star$ is in the case of $\lambda_\kappa > 0$ given by:

$$
I^\star = \beta \left[ \beta \frac{1}{1 + \frac{\rho}{\phi_\kappa}} (\kappa_H^{1 + \frac{\lambda_\kappa}{\phi_\kappa}} + \frac{\rho}{\phi_\kappa} - \kappa_L^{1 + \frac{\lambda_\kappa}{\phi_\kappa}} - \frac{\rho}{\phi_\kappa}) \frac{1}{\kappa_H - \kappa_L} \right]^{-\lambda_\pi} = \beta^{1 - \lambda_\pi} [E(\kappa^{1 + \frac{\lambda_\kappa}{\phi_\kappa}} + \frac{\rho}{\phi_\kappa})]^{-\lambda_\pi}
$$

Inserting the above expression and the solution for $1 + i$ in (16) and using $\lambda_{\kappa-1} = \frac{\rho}{\phi_\kappa} \lambda_\pi$ we get

$$
1 + i = \beta^{1 - \lambda_\pi} \left[ E(\kappa^{1 + \frac{\lambda_\kappa}{\phi_\kappa}} + \frac{\rho}{\phi_\kappa}) \right]^{-\lambda_\pi} \cdot \beta^{-1} \kappa^{\lambda_\kappa} \kappa_{-1}^{\frac{\rho}{\phi_\kappa} \lambda_\pi} \left( \beta E_{-1}(\kappa^{1 + \frac{\lambda_\kappa}{\phi_\kappa}} + \frac{\rho}{\phi_\kappa}) \kappa_{-1}^{\frac{\rho}{\phi_\kappa} \lambda_\pi} \right)^{\lambda_\pi} = 1
$$
4.1 Simplifying the price equation

In order to write the price equation in terms of aggregate output and inflation only, I derive the sticky-price sector output and the flexible-price sector output as functions of total output. Use (5) to substitute out \( E_{-1}(\frac{\kappa Y^f}{2Y}) \) in (8) to get

\[ Y_s = \left[ \Pi^\frac{1}{\gamma} \frac{\kappa Y^f}{2Y} - \Pi \right]^{1-\gamma} \Pi \gamma Y. \]  

(27)

Rearrange (7) to get an expression for \( Y_f \) in terms of \( Y \),

\[ Y_f = \kappa \frac{-\frac{1}{2}}{\frac{1}{2}} \left( \frac{1}{2} \right)^{\frac{1}{2}} Y^f \left( 1 - \gamma \right)^{\frac{1}{2}}. \]  

(28)

Substituting out for \( Y_f \) in (27) and simplifying gives \( Y_s \) as a function of \( Y \) only,

\[ Y_s = \kappa \frac{1-\gamma}{\gamma} \left( \frac{1}{2} \right)^{\frac{1}{2}} Y^f \left( 1 - \gamma \right)^{\frac{1}{2}} \gamma. \]  

(29)

Substituting for (28) and (29) in equation (5) gives the Neoclassical Phillips curve:

\[ \Pi = \left[ \kappa \frac{1}{2} Y^f \gamma \left( 1 - \gamma \right) \gamma \left( \frac{1}{2} \right)^{\gamma} + \rho \right] \frac{E_{-1}(\frac{\kappa Y^f}{2Y}) \left( 1 - \gamma \right)^{\frac{1}{2}} \gamma}{E_{-1}(\kappa \frac{1}{2} Y^f \gamma \left( 1 - \gamma \right)^{\frac{1}{2}} \gamma).} \]  

(30)

This equation says that the inflation rate is determined by the expected inflation rate as of the last period, actual output and expected output.

4.2 Solving the sticky-price model with a uniform distribution of the productivity shock

I now have the price equation and the demand equation, and I add an interest-rate rule:

\[ \Pi = (1 - \gamma)^{\frac{1}{\gamma}} \left( \frac{1}{2} \right)^{\frac{1}{2}} \left[ \kappa \frac{1}{2} Y^f \gamma \left( 1 - \gamma \right) \gamma \left( \frac{1}{2} \right)^{\gamma} + \rho \right] \frac{E_{-1}(\kappa \frac{1}{2} Y^f \gamma \left( 1 - \gamma \right)^{\frac{1}{2}} \gamma)}{E_{-1}(\kappa \frac{1}{2} Y^f \gamma \left( 1 - \gamma \right)^{\frac{1}{2}} \gamma)}. \]  

(31)

\[ Y^{-\rho} = \beta(1 + i)E(\Pi_{-1}^{-1}Y^{-\rho}). \]  

(32)

and

\[ I = \beta^{-1} \kappa \lambda \Pi^{-1} \Pi. \]  

(33)

Equations (31)-(33) can be solved for output, inflation and the nominal interest rate, as shown in the online appendix C. The solutions for output and inflation are given by

\[ Y = \Psi \kappa^{\psi} \quad \text{and} \quad \Pi = \Phi \kappa^{\phi} \kappa^{-1}. \]
where the distribution of $\kappa$ again is uniform between $\kappa_L$ and $\kappa_H$, and the the coefficients are as given in table 2. Now, $\Psi$ is the risky steady state of output, while $\Phi$ still is the risky steady state of inflation.

<table>
<thead>
<tr>
<th>Table 2: The sticky-price model solution, uniform distribution of $\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi = -\frac{\lambda_\kappa + \lambda_{\psi} \left( \frac{1 - \gamma}{2\gamma} \right)^{\gamma}}{[\rho + \frac{1 - \gamma + \gamma \lambda_\kappa}{2\gamma}]}$ (2.1)</td>
</tr>
<tr>
<td>$\phi_\kappa = \frac{1 - \gamma}{2\gamma} + \frac{(1 + \rho)(1 - \gamma)}{2\gamma} \psi$ (2.2)</td>
</tr>
<tr>
<td>$\phi_{\kappa - 1} = -\frac{\lambda_{\kappa - 1}}{\lambda_\kappa}$ (2.3)</td>
</tr>
<tr>
<td>$\Phi = { \beta \kappa_L \left( -\phi_\kappa \lambda_\kappa \right) } \left( \kappa_H^{-1} - \phi_\kappa - \rho \psi - \kappa_L^{-1} - \phi_\kappa - \rho \psi \right)$</td>
</tr>
<tr>
<td>$(1 - \phi_\kappa - \rho \psi)^{-1} \left( \kappa_H - \kappa_L \right)^{-1}$ (2.4)</td>
</tr>
<tr>
<td>$\Psi = K \cdot \left{ \kappa_H \phi^{\left( \frac{1 + \rho - (1 - \gamma) \gamma + 2\gamma}{\gamma} \right)} + \frac{1 + \gamma}{\gamma} - \frac{1}{\gamma} - \frac{1}{\gamma} \right}$</td>
</tr>
<tr>
<td>$(1 + \frac{1 + \gamma}{2\gamma} + \frac{1 + \gamma}{2\gamma} \psi^{\left( 1 + \rho - (1 - \gamma) \gamma + 2\gamma \right)} - \psi - \phi_\kappa)$</td>
</tr>
<tr>
<td>$(\kappa_H^{-1} - \phi_\kappa - \rho \psi - \kappa_L^{-1} - \phi_\kappa - \rho \psi)^{\frac{1}{\gamma} \theta}$</td>
</tr>
<tr>
<td>$(1 + \frac{1 + \gamma}{2\gamma} + \psi^{\left( 1 + \rho - (1 - \gamma) \gamma + 2\gamma \right)} - \psi - \phi_\kappa)^{\frac{1}{\gamma} \theta}$ (2.5)</td>
</tr>
</tbody>
</table>

Memo: Equil. nom. interest rate: $1 + \frac{1}{\phi_{\kappa - 1}} \geq 1$, $K = (1 - \gamma)^{\frac{1 - \gamma}{1 + \gamma}} \Phi^{-\lambda_\kappa}$

Memo: $I^* = \beta \kappa_L^{-\lambda_\kappa - \phi_\kappa \lambda_\kappa}$

Equation (2.1) shows that output now depends on both current monetary policy, and monetary policy one period ahead, through $\lambda_\kappa$ and $\lambda_{\kappa - 1}$. In order to get some intuition for the constant terms $\Phi$ and $\Psi$, see footnote 7.

Again, $\Phi$ embeds a normalization of $I^*$ to produce the lowest feasible nominal interest rate and inflation rate, given the zero lower bound and structural shocks.

Under certain conditions, the "sticky price part" of the risky steady state of output $\Psi$ - which are the extra terms in addition to $K$ (compare equation (2.5) to equation (13)) - drops out, and the risky steady state of output is equal to its flexible price level. In particular, this happens under optimal stabilization policy, in the particular case of log utility ($\rho = 1$). See the online appendix E.

### 4.3 The divine coincidence: output stabilization with a stable inflation rate

Authorities can achieve perfect stabilization of output if they stabilize the inflation rate; $\phi_\kappa = \phi_{\kappa - 1} = 0$ is achieved by $\lambda_\kappa = \frac{\rho}{1 + \rho}$ and $\lambda_{\kappa - 1} = 0$ (or by $\lambda_\kappa \to \infty$). Technically, $\psi$ then reduces to $\frac{1}{1 + \rho}$, so that the time-varying part of
output follows the exact path of the time-varying part of flexible-price output, see equation (13). With $\rho = 1$, the constant term $\Psi$ then is also equal to its flexible-price counterpart ($\Psi = K$).

The reason why a constant inflation rate eliminates output gap distortions, is that with a fully expected constant inflation rate, sticky price sector price setters know which price to set one period in advance. There is then no distortion of relative prices. This makes sure that the allocation of production and consumption across sectors is efficient. The constant inflation rate, equal to the risky steady state inflation rate, is now given by

$$\Pi = \Phi = \{\beta \kappa \frac{1}{\kappa_L} \frac{1}{\kappa L} \} E[\kappa \frac{\tau}{\tau^*}].$$

(34)

The solution for the nominal rate is (like in the flexible price case with inflation targeting) given by:

$$1 + i = \frac{K}{\kappa_L} \frac{1}{\kappa L} \frac{1}{\kappa L}.$$

The risky steady state for the nominal interest rate, with $E(\kappa) = 1$, is given by

$$1 + i_{ss} = (\frac{1}{\kappa_L}) \frac{1}{\kappa L},$$

which also is increasing in mean preserving spreads of $\kappa$ around $\kappa = 1$.

4.4 Output stabilization with a low but variable inflation rate

There is another way of stabilizing output, besides strict inflation targeting: If authorities respond to the current shock in the next period, and agents observe the current shock before they set next period’s prices, the variation in the inflation rate comes as no surprise and is then not costly in terms of output in this model. Monetary authorities can then track the natural real interest rate to the extent that they can determine the inflation rate in the next period - which they do in this model.

Technically, for the purpose of output stabilization, we are still looking for a rule that yields

$$\psi = -\frac{1}{1 + \rho}. \quad (35)$$

(35) is satisfied if

$$\lambda_{\kappa} + \frac{\lambda_{\kappa-1}}{\lambda_{\kappa}} = \frac{\rho}{1 + \rho}. \quad (36)$$

\[\text{9Any constant inflation rate higher than } \Phi \text{ is attainable by adjusting } I^* \text{ correspondingly.}\]
\( \phi_k = 0 \) whenever (35) holds. But the case of

\[
\frac{\lambda_{k-1}}{\lambda_\pi} = \frac{\rho}{1 + \rho} \quad \text{and} \quad \lambda_\kappa = 0
\]

(37)

is of particular interest. The reason is that the equilibrium nominal interest rate is given by

\[
1 + i = \left( \frac{K}{\kappa L} \right)^{\lambda_\kappa + \phi_\kappa \lambda_\pi}.
\]

Since \( \phi_\kappa = 0 \) whenever (35) holds, the nominal interest rate is also stable and equal to zero when \( \lambda_\kappa = 0 \) and (35) holds and \( \lambda_\pi \) is bounded. This policy then establishes the minimum possible nominal equilibrium that is attainable in this model.

Note that authorities cannot postpone the effect of the \( \kappa \) shock on the inflation rate further by responding to shocks lagged more than one period, and still stabilize output. The reason is that authorities rely on the current variation in the inflation rate to create a real interest rate equal to the natural rate.\(^{10}\)

Using \( \lambda_\kappa = \phi_\kappa = 0 \), the interest rate rule that establishes the minimal inflation rate and a stable nominal interest rate is given by

\[
I = I^* \beta^{-1} K_{-1}^{\frac{1}{\kappa L}} \Pi^{\lambda_\pi} = \Phi^{-\lambda_\pi} K_{-1}^{\frac{1}{\kappa L}} \Pi^{\lambda_\pi}.
\]

In equilibrium, the solution for the interest rate will be given by

\[
1 + i = \left( \frac{K}{\kappa L} \right)^0 = 1.
\]

The risky steady state inflation rate is given by

\[
\Phi = \{ \beta K_L^0 \} E[\kappa^{\frac{\hat{\pi}}{\bar{\pi}}} \Pi^{\lambda_\pi}] = \beta E[\kappa^{\frac{\hat{\pi}}{\bar{\pi}}}],
\]

and the equilibrium inflation rate is

\[
\Pi = \Phi K_{-1}^{\frac{\hat{\pi}}{\bar{\pi}}} = \beta E[\kappa^{\frac{\hat{\pi}}{\bar{\pi}}} K_{-1}^{\frac{\hat{\pi}}{\bar{\pi}}}].
\]

\(^{10}\)It might seem that authorities could infer the lagged supply shock from the solution for the inflation rate, since the solution is \( \Pi = \Phi K_{\kappa-1}^{\frac{\hat{\pi}}{\bar{\pi}}} \). One might therefore think that responding to a function of the inflation rate instead of the lagged shock directly could yield the first-best solution. However, the inflation rate that closes the output gap depends on the lagged shock only because monetary authorities respond to the lagged shock, as can be seen from the expression for \( \phi_{\kappa-1} \) in table 2. If we eliminate \( \lambda_{\kappa-1} \) from the interest-rate rule and let the authorities respond to the appropriate function of the inflation rate instead, all response parameters in the reaction function cancel out. Authorities are left with a rule that says they should peg the nominal rate at zero.
4.5 Output stabilization in the general case: The price level is a random walk

We may make (36) hold with any set of parameters chosen from a particular set;

\[ \lambda_\kappa = \alpha \frac{\rho}{1+\rho} \text{ and } \lambda_{\kappa-1} = (1-\alpha) \frac{\rho}{1+\rho} \lambda_\pi, \text{ where } \alpha \in (0,1). \]  

(39)

With parameters chosen from this set, \( \phi_\kappa = 0 \), and \( \phi_{\kappa-1} = (1-\alpha) \frac{\rho}{1+\rho} \), and furthermore \( \Phi = \{ \beta \kappa_L^{(\alpha-\varphi \rho)} \} E(\kappa \frac{\rho}{1+\rho}), \) \( 1 + i = \left( \frac{\kappa}{\kappa_L} \right)^{\alpha-\varphi \rho}, \) and \( \Pi = \{ \beta \kappa_L^{(\alpha-\varphi \rho)} \} E(\kappa \frac{\rho}{1+\rho}) \kappa_{\kappa-1}^{-(1-\alpha) \frac{\rho}{1+\rho}}. \) Hence, with a lower \( \alpha \), more emphasis is put on interest rate stability, and less on inflation stability. The output gap is stabilized with any \( \kappa \), but the minimum feasible risky steady state inflation level (given by \( \Phi \)) is lower with a lower \( \alpha \). Unless \( \alpha = 1 \) (a perfectly stabilized inflation rate), the inflation rate inherits the stochastic process of \( \kappa \). The log of the price level is a random walk with drift under optimal monetary policy:

\[ \log(\Pi) = \log P - \log P_{-1} = \log \Phi - (1-\alpha) \frac{\rho}{1+\rho} \log \kappa_{-1}. \]  

(40)

<p>| Table 3: A menu of optimal policies. Sticky price model with ( \kappa \sim u[\kappa_L, \kappa_H] ). |</p>
<table>
<thead>
<tr>
<th>Policy:</th>
<th>Endogenous variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lowest possible inflation</td>
<td>( \Pi = \beta E[\kappa \frac{\rho}{1+\rho}] \kappa_{-1}^{\frac{\rho}{1+\rho}} )</td>
</tr>
<tr>
<td>Policy parameters in above case</td>
<td>( \lambda_\kappa = 0, \lambda_{\kappa-1} = \frac{\rho}{1+\rho} \cdot \lambda_\pi )</td>
</tr>
<tr>
<td>Intermediate case</td>
<td>( \Pi = \beta \kappa_L^{\left(\frac{\rho}{1+\rho}\right)} E(\kappa \frac{\rho}{1+\rho}) \kappa_{\kappa-1}^{-(1-\alpha) \frac{\rho}{1+\rho}} )</td>
</tr>
<tr>
<td>Policy parameters in above case</td>
<td>( \lambda_\kappa = \frac{\rho}{1+\rho} \cdot \lambda_\pi ), ( \lambda_{\kappa-1} = (1-\alpha) \cdot \frac{\rho}{1+\rho} \cdot \lambda_\pi )</td>
</tr>
<tr>
<td>Fully stable inflation</td>
<td>( \Pi = \beta E[(\frac{\rho}{1+\rho}) \frac{\rho}{1+\rho}] )</td>
</tr>
<tr>
<td>Policy parameters in above case</td>
<td>( \lambda_\kappa \to \infty ), or ( \alpha \to 1 )</td>
</tr>
</tbody>
</table>

Figure 1 illustrates how stabilization policy works in this model. A productivity shock (\( \kappa \) decreases by 0.5%) that increases potential output in period 1 by 0.25 percentage points is illustrated in the first panel. A one percentage point (annualized) lower real interest rate for one period is needed in order to keep the output gap closed. A higher \( \rho \) would require an even stronger interest rate response (intertemporal substitution would then be less elastic). An optimally designed "quite strict inflation targeting policy" (\( \alpha = 0.8 \)) requires the nominal interest rate to fall in the same period. But since the fall in the nominal interest rate is not made quite strong enough to establish the required fall in the real interest rate when \( \alpha < 1 \), authorities will need to respond somewhat to the shock with an expansionary policy in the next period as well (\( \lambda_{\kappa-1} > 0 \)).
Inflation increases somewhat in period 2 (by 0.2 percentage points annualized). The net effect on the nominal interest rate in period 2 is zero.

A much more moderate contemporaneous interest rate response ($\alpha = 0.2$) is also consistent with output stabilization. The inflation rate increases by 0.8 percentage points in period 2 under this policy, while the nominal rate goes down by 20 basis points in period 1. Again, the real interest rate falls by one percentage point for one period. The high inflation rate in the next period, combined with the lagged response to the productivity shock in that period, mean that the net equilibrium change in the nominal interest rate in the next period again will be zero. The output gap - the deviation of output from its flexible price solution given in equation 13 - will be zero under any $\alpha^{11}$.

Implications for the levels of risky steady states and the volatility of interest rates and inflation rates under different policies are presented in table 4 for a benchmark case (where $\beta = 0.995$, $\gamma = 0.5$, $\rho = 1$, $E(\kappa) = 1$, and $\kappa_L$, $\kappa_H$ are chosen to give a standard deviation of flexible price output equal to 0.5%, which is the case when the standard deviation of $\kappa$ is 1%). A strict inflation targeting policy implies increasing steady states for interest rates and inflation, as the volatility of the productivity shock increases. Recall that the intercept term of the interest rate rule adjusts in order to keep all possible interest rate outcomes at or above zero, for any given $\kappa_L$ parameter.

11The gap that optimal stabilization policy closes, is the deviation of output from its sticky-price risky steady state level, minus the deviation of flexible price output from its flexible-price risky steady state level. In this model, the risky steady state under optimal policy and sticky prices is the same as the risky steady state under flexible prices IF $\rho = 1$. See discussion in appendix E.
Figure 2: Risky steady states and standard deviations. Uniform distribution of $\kappa$. Alternative optimal policies.

<table>
<thead>
<tr>
<th>Table 4: Risky steady state$^a$ (S.D.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark calibration. $^{1^{st}}$ $^{\pi}$</td>
</tr>
<tr>
<td>$\alpha = 0.0$</td>
</tr>
<tr>
<td>$\alpha = 0.5$</td>
</tr>
<tr>
<td>$\alpha = 1.0$</td>
</tr>
</tbody>
</table>

$^a$ The risky steady state is calculated from the analytical expressions, annualized net value in percent. The standard deviations are derived from simulations, 10000 periods.

Effects of exogenous volatilities on risky steady states and standard deviations of endogenous variables under different policies are illustrated in figure 2. The "stable interest rate policy" implies an increasing standard deviation of the inflation rate, as the volatility of the productivity shock increases (see the lower left panel of figure 2). The risky steady state interest rate is unaffected by the volatility of the productivity shock in this case, see upper right panel of figure 2.

Along with the lower risky steady state nominal rate, we also observe a lower risky steady state inflation rate. With a lower $\alpha$, and thus a less stable inflation rate, a lower risky steady state inflation rate is attainable for any given volatility of exogenous shocks.
5 The sticky-price model with a lognormal distribution

In this section, a case where the support of \( \kappa \) is unbounded is considered. In particular, \( \kappa \sim LN(0, \sigma^2) \). Now, the steady state levels of the interest rate and inflation rate can, under strict inflation targeting, never be high enough to always close the output gap: there will always be cases where the natural real rate is lower than the negative of the inflation target. The menu of optimal policies consistent with the zero lower bound in table 3 is therefore narrowed to \( \alpha = 0 \) only. I now calibrate the policy parameter \( I^* \) such that the risky steady state interest rate stays constant as the parameter describing exogenous volatility increases (see online appendix D).

Section 5.1 below derives the solution of the model with a lognormal distribution of \( \kappa \). In section 5.2, I consider nonlinear policies, where authorities emphasize inflation stabilization (\( \alpha > 0 \)) during low-volatility, or normal times. The response to contemporaneous shocks is lowered to make the interest rate stay at zero each time the shock takes on a particularly large negative value and the zero lower bound is encountered. With the response-coefficients now being constrained in certain periods, a negative output gap will be unavoidable during such zero lower bound episodes.

5.1 Solving the model with a lognormal distribution

Using the method of undetermined coefficients, the solutions for output and inflation are given by

\[
Y = \Psi \kappa^\psi \quad \text{and} \quad \Pi = \Phi \kappa^{\phi_{\psi} - \phi_{\kappa} - 1},
\]

where the solution is given in table 5. The dynamics of endogenous variables, described by equations (5.1)-(5.3), is the same as before. The parameters describing the distribution of \( \kappa \) again appear in the risky steady states of output, \( \Psi \) and inflation \( \Phi \). The solution is derived in the online appendix D.
Table 5: The sticky-price model solution, $\kappa \sim LN(\mu, \sigma^2)$.

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$-\lambda_\kappa + \lambda_\kappa \left( \frac{1 - \gamma}{2} \right) + \frac{\lambda_{\kappa-1}}{2\gamma}$</th>
<th>(5.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_\kappa$</td>
<td>$\frac{1-\gamma}{\gamma^2} + \frac{(1+\rho)(1-\gamma)}{2\gamma} \psi$</td>
<td>(5.2)</td>
</tr>
<tr>
<td>$\phi_{\kappa-1}$</td>
<td>$\frac{-\lambda_{\kappa-1}}{\kappa}$</td>
<td>(5.3)</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>$\Pi^* \cdot \exp\left{ \frac{1}{2} (-\phi_\kappa - \rho \psi)^2 \sigma^2 \right}$</td>
<td>(5.4)</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>$\exp[-\frac{1}{1+\rho^2} \left( \psi \left( \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} \right) + \frac{1}{\gamma} \right)^2 - \left{ \psi \left( \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} \right) \right}^2 - \rho \phi_\kappa + \frac{1-\gamma}{2\gamma})^2 \sigma^2] = \Pi^* \Phi^{\frac{-\lambda_\kappa}{\kappa}}$</td>
<td>(5.5)</td>
</tr>
<tr>
<td>$K$</td>
<td>$(1-\gamma)^{-\frac{2+\gamma}{1+2\gamma}} \sigma^{-\frac{2+\gamma}{1+2\gamma}} \frac{2+\gamma}{1+2\gamma}$</td>
<td></td>
</tr>
</tbody>
</table>

Memo: $I^\kappa \equiv \Pi^* \cdot \exp\left\{ \frac{1}{2} (-\phi_\kappa - \rho \psi)^2 \sigma^2 \right\}^{-\lambda_\kappa} = \Pi^* \Phi^{\frac{-\lambda_\kappa}{\kappa}}$

Table 6: Benchmark calibration, lognormal distribution

| $\beta$ | 0.995 | $\gamma$ | 0.5 | $\rho$ | 1 | $\sigma_\kappa^2$ | 0.01 | $\pi^*$ | 0.005 |

$\sigma_\kappa^2 = 0.01$ gives a S.D. of potential output of 0.5%

$\pi^* = 0.005$ quarterly net inflation corresponds to 2% annual inflation

With the benchmark calibration in table 6, figure 3 illustrates how risky steady states and standard deviations of endogenous variables depend on the standard deviation of the productivity shock under different optimal policies ($\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$). I let the parameter $\Pi^*$ scale the inflation level that $I^\kappa$ produces. Under all optimal policies, the risky steady state quarterly gross nominal rate will be $I^{SS} = \Pi^* \beta^{-1}$, while $\Phi = \Pi^* \cdot \exp\left\{ \frac{1}{2} \left( \frac{\psi}{1+\rho^2} \right)^2 \sigma^2 \right\}$. Figure 3 shows annualized net levels in percent.

Only the $\alpha = 0$ (stable optimal interest rate) policy is consistent with a possible equilibrium, since the productivity shock may take on negative values of unlimited size. The two other alternatives illustrated in figure 3 abstract from the zero lower bound. In the case of a lognormal distribution, optimal stabilization policy will produce $\Psi = K$, so that also the level of output will be equal to the flexible price level under all structural parameters (see online appendix E).

5.2 Nonlinear policies and the frequency of zero lower bound episodes

In this subsection I assume that authorities choose $\alpha > 0$. When authorities observe a large negative $\kappa$-shock, the natural real interest rate will be particu-
larly low, and it might be impossible to respond fully to that shock during the present period, since (in logs) we have

\[ i = i^* - \ln \beta + \lambda_{\kappa} \ln \kappa + \lambda_{\kappa-1} \ln \kappa_{-1} + \lambda_\pi \pi. \]

One might think that it would be possible to increase the inflation target in the interest rate rule during such zero lower bound episodes, in order to create enough inflation to stay off the zero lower bound, and keep the response coefficients optimal at any given \( \alpha \). However, technically the solution derived does not apply if \( i^* \) is stochastic. And we know that inflation surprises create output gaps in this model.

One might also think that one could stay off the zero lower bound by switching to an \( \alpha = 0 \) policy during zero lower bound episodes only. However, the choice of \( \alpha \) impinges not only on the contemporaneous response coefficient \( \lambda_\kappa \), but also on the response coefficient \( \lambda_{\kappa-1} \) and hence the assumed response to the contemporaneous shock in the next period. For the change to \( \alpha = 0 \) today not to create an output distortion in the next period, authorities would have to stick to \( \alpha = 0 \) in the next period as well, and in each period after that. An \( \alpha > 0 \) policy will therefore be dominated by an \( \alpha = 0 \) policy; Sooner or later the zero lower bound will be binding, at which point authorities would like to lower \( \alpha \). Since first best optimal policy is implementable at all times with \( \alpha = 0 \), any other policy will require commitment and be time inconsistent.

In order to shed light on how authorities anyway could implement inflation stabilization (assuming a commitment technology is available) during periods...
where the zero lower bound is not binding, I simulate the model. Authorities choose a small enough $\lambda_k$, to exactly stay at the zero lower bound, in periods where the constraint otherwise would be violated. This will produce a negative output gap during zero lower bound episodes, because $\lambda_k < \alpha \frac{\rho}{1+\rho}$ while $\lambda_{k-1} = (1 - \alpha) \frac{\rho}{1+\rho} \lambda_{\pi}$. Inflation will be relatively low during such zero lower bound episodes, because a too contractionary monetary policy pulls inflation down.

The simulations are conducted by searching for the maximum possible contemporaneous response coefficient $\lambda_k$ each time the zero lower bound is encountered. A $\lambda_k$ close to the optimal $\alpha \frac{\rho}{1+\rho}$ is feasible when $\kappa$ is not too negative and $\alpha$ is low. An outcome for the equilibrium interest rate in a small range around zero is permitted in the simulations (the zero lower bound is in the range of +/- 20 basis points here), and hence the interest rate does not stop at exactly zero during each zero lower bound episode.

Figures 4-6 show simulations of the model with the calibration given in table 6. In figure 4, $\alpha = 1$ during "normal" times, but inflation falls below target if a zero lower bound event kicks in. Two such events occur in the simulation shown. The distribution of inflation is therefore skewed to the left. With lower ambitions regarding inflation stabilization around trend ($\alpha = 0.5$) in figure 5, no zero lower bound episode is triggered in the 100 period (25 year) simulation shown. This result is sensitive to the calibration of the model. With a higher exogenous volatility or a lower inflation target, more zero lower bound episodes will be triggered. In figure 6, where $\alpha = 0$, inflation is more volatile and the interest rate is constant at its risky steady state level.

6 Concluding remarks

Policy prescriptions regarding the zero lower bound constraint appear to be sensitive to the nature of the Phillips Curve. They have been shown to be sensitive to other distortions than those originating from price stickiness as well, but in this paper the focus is on the effects of price stickiness only.

In this model, welfare is maximized when the output gap is closed; I consider the cashless limit, and there are no other allocations than money and output to be concerned about. With the Neoclassical Phillips Curve, the "divine coincidence" result from the New Keynesian Phillips Curve is modified.

First, if we disregard the zero lower bound or consider a bounded support for the exogenous shock, a wide set of nominal paths is consistent with output gap stabilization and welfare maximization.
Figure 4: Simulation of endogenous variables under $\alpha = 1$ policy. Lognormal distribution of $\kappa$.

Figure 5: Simulation of endogenous variables under $\alpha = 0.5$ policy. Lognormal distribution of $\kappa$. 

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When one introduces the zero lower bound, it has been shown elsewhere that inflation stabilization is no longer time consistent in the simplest New Keynesian Phillips Curve case. However, in the Neoclassical Phillips Curve case studied here, there is still one type of policy that is time consistent and optimal. Under that policy, the price level follows a random walk with drift, while the interest rate rule prescribes a response to shocks with a lag.

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