Linear-Quadratic Approximation to Unconditionally Optimal Policy: The Distorted Steady-State*

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Abstract

This paper establishes that one can generally obtain a purely quadratic approximation to the unconditional expectation of social welfare when the steady-state is distorted. A specific example is provided employing a canonical New Keynesian model. Unlike in the non-distorted steady state case, the approximate loss function is not defined simply over terms in inflation and output. Furthermore, optimal steady state inflation and the nominal interest rate are positive.

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1. Introduction

Taylor (1979) suggests that, in quantitative theoretical investigations under rational expectations, macroeconomic stabilization policies ought to optimize the unconditional expectation of the policymaker's objective function. That perspective on policy assessment has proven popular; some prominent recent examples include Rotemberg and Woodford (1998), Woodford (1999), Clarida, Gali and Gertler (1999), Erceg, Henderson and Levin (2000), Kollman (2002) and Schmitt-Grohe and Uribe (2007). Blake (2001) and Jensen and McCallum (2002, 2006) also provide an example of time-invariant monetary policy based on optimization of the unconditional value of the criterion function.

However, even in the simplest possible models, analytical derivations of unconditionally optimal (UO) policy have proven elusive. Damjanovic, Damjanovic and Nolan (DDN, forthcoming) develop a straightforward, intuitive and easy-to-implement approach for analytically deriving policies that are unconditionally optimal in both non-linear and linear-quadratic (LQ) settings. They also demonstrate that one can linearize a canonical New Keynesian model around the non-distorted steady-state to obtain a 'familiar' LQ approximation to the UO policy problem; that is, familiar in the sense that the loss function is defined solely over terms in inflation and output.

This paper takes up an important issue not addressed in that earlier paper: Can one devise a tractable LQ formulation to the UO policy problem when the steady-state of the model is distorted? The advantages from being able to do so include, as Benigno and Woodford (2007) note, the possibility, under certain conditions, to rank alternative policies[1]. However, there is also scepticism, (see, Benigno and Woodford, 2007), as to whether it is possible to obtain a purely

[1] See also Kim and Kim 2007.

quadratic approximation to the unconditional loss function.

In this paper the approach of DDN is extended to demonstrate that it is possible to obtain a purely quadratic approximation to social welfare around the unconditionally optimal steady state, where lump-sum subsidies are not allowed. This is a primary contribution of the paper. A specific application of the approach is provided employing the canonical New Keynesian model.

Two main results then emerge. First, unconditionally optimal monetary policy is characterized by a trend in inflation and a positive nominal interest rate. That trend in inflation complicates the linear-quadratification[2]. This explains our second result: the second-order accurate approximate loss function is no longer defined solely over terms in output and inflation, as found in DDN for the non-distorted steady-state case. However, the loss function that one obtains is easily interpreted in terms of the underlying distortions in the economy.

In section 2 the basic problem is set out in a general form. The problem is analyzed and it is shown that one can derive a purely quadratic second-order approximation to the unconditional expectation of the objective function. Section 3 begins the application; first a canonical New Keynesian, Calvo-price-setting model is set up. Section 4 formalizes the policy problem and demonstrates the application of the various steps in the approach of section 2. There is then a brief discussion of the implications for optimal monetary policy when the steady state is distorted and the authorities are optimizing over the unconditional loss function. Section 5 offers some conclusions.

[2] As shown in Damjanovic and Nolan (2006)

2. The general problem

Consider a discounted loss function of the form

$$L_t = (1-\beta)\, E_t \sum_{j=0}^{\infty} \beta^j l(x_{t+j}, \mu_{t+j}), \tag{2.1}$$

where E_t is the expectations operator *conditional on information up through date* t, β is the time discount factor, $l(x_{t+j}, \mu_{t+j})$ is the period loss function and x_t is a vector of target variables. Specifically, $x_t = [Z_t, z_t, i_t]$, where Z_t is a vector of predetermined endogenous variables (lags of variables that are included in z_t and i_t), z_t is a vector of non-predetermined endogenous variables (including 'jump' variables), the value of which will generally depend upon both policy actions and exogenous disturbances at date t, and i_t is a vector of policy instruments, the value of which is chosen in period t. μ_t denotes a vector of exogenous disturbances. For simplicity, assume that μ_t is a function of primary i.i.d. shocks, $(e_i)_{-\infty}^{t}$.

Further, let the evolution of the endogenous variables z_t and Z_t be determined by a system of simultaneous equations,

$$F\left(E_t x_{t+1,} x_t, \mu_t\right) = 0. \tag{2.2}$$

Let us further assume, following Taylor (1979), that the policy maker seeks to minimize the unconditional expectation of the loss function (2.1), subject to constraints, (2.2)[3]. That is, he or she searches for a policy rule

$$\varphi\left(E_t x_{t+1,} x_t, \mu_t\right) = 0 \tag{2.3}$$

such that

$$\varphi = \arg\min EL_t(\varphi), \tag{2.4}$$

[3]Interestingly, Taylor's approach, we think, basically boils down to a recommendation: Policymakers *ought* to seek to minimize the unconditional value of the loss function. This appears partly, perhaps largely, in response to the issue of time inconsistency. See Taylor (1979) for further discussion. McCallum (2005) is an interesting discussion of these, and related, issues.

where E is the unconditional expectations operator. We call such a policy "unconditionally optimal" and denote it 'UO-policy'.

2.1. Solution

The first step is to formulate the non-linear policy problem and identify the non-stochastic steady state around which approximation needs to take place. Next, the possibility of a second-order accurate approximation to welfare is addressed; specifically the possibility of a loss function that is solely a function of quadratic terms. However, an alternative approach to analyzing (2.2)-(2.4) is to solve a non-linear problem and to analyze the linearized optimality conditions. So, finally in this section we establish the equivalence of the LQ approach (which is the central topic of this paper) with that alternative approach of "optimize then linearize".

2.1.1. Necessary conditions for an optimum

Consider the following Lagrangian function which derives from the above optimal policy problem:

$$\mathcal{L}\left(y_{t,}x_t,\mu_t\right)=E\left(l\left(x_t,\mu_t\right)+\xi_tF\left(y_{t,}x_t,\mu_t\right)+\rho_t\left(E_tx_{t+1}-y_t\right)\right). \tag{2.5}$$

DDN (forthcoming) show that the necessary conditions for the optimality of policy, φ, is that it implies a path for the endogenous variables, x_t and y_t, and that there exists Lagrange multipliers, (ξ_t,ρ_t) , that together satisfy the first order conditions (2.6), (2.7) and constraints (2.2)[4],

$$\frac{\partial H}{\partial x_t} = \frac{\partial l\left(x_t,\mu_t\right)}{\partial x}+\xi_t\frac{\partial F\left(y_{t,}x_t,\mu_t\right)}{\partial x}+\rho_{t-1}=0; \tag{2.6}$$

$$\frac{\partial H}{\partial y_t} = \xi_t\frac{\partial F\left(y_{t,}x_t,\mu_t\right)}{\partial y_t}-\rho_t=0, \tag{2.7}$$

[4]The notation ξF is a shorthand for the tensor product, $\sum_{i=1}^{n}\xi_iF_i$.

where $H(y_{t,}x_t,\mu_t)$ is the Hamiltonian for (2.5), such that $\mathcal{L}(y_{t,}x_t,\mu_t) = E(H(y_{t,}x_t,\mu_t))$. This basic optimization problem is discussed in a little more detail in DDN (forthcoming).

Judd (1999), Woodford (2002) and Benigno and Woodford (2005) demonstrate very clearly that the choice of the steady-state is crucial (along with the solution concept for forward-looking policy problems) in being able to obtain LQ approximations to general non-linear, forward-looking policy problems. To choose the deterministic steady state, around which log-linearization takes place, one needs to solve the system of first order conditions (2.6), (2.7) and constraints (2.2). The steady state (X,ξ) is defined by the system (2.8):

$$\begin{aligned} F(X,X,\mu) &= 0; \\ \frac{\partial l(X,\mu)}{\partial x_t} + \xi\frac{\partial F(X,X,\mu)}{\partial x} + \xi\frac{\partial F(X,X,\mu)}{\partial y} &= 0, \end{aligned} \tag{2.8}$$

where X, ξ and μ indicate the vectors of steady state values of endogenous variables, Lagrange multipliers and the average value of shocks, respectively. We refer to (X,ξ) as the "unconditionally optimal steady state".

We emphasize, that the "timeless perspective" approach discussed in Woodford (2002) implies different first order conditions, and therefore, a different center of approximation. That difference will be shown to lead to very different optimal monetary policy.

2.2. The possibility of pure second order approximation

The value of the loss function $El(x_t,\mu_t)$ should not change if we combine it with the unconditional expectation of the constraints $EF(y_t,x_t,\mu_t)$. Thus, the appendix demonstrates that the second order approximation to this combination has a pure second order form. That is,

$$\begin{aligned} El(x_t,\mu_t) &= E\left[l(x_t,\mu_t)+\xi F(y_t,x_t,\mu_t)\right] \\ &= EQ_l+\xi EQ_F+t.i.p+O3. \end{aligned} \tag{2.9}$$

The notation $O3$ denotes third or higher order terms. Q_l and Q_F are pure second order terms of the log-approximation, around the unconditionally optimal steady state, to the loss function $l(x_t,\mu_t)$ and dynamic constraints $F(Ex_{t+1},x_t,\mu_t)$:

$$\begin{aligned} Q_l &= \frac{1}{2}\left(X^2\frac{\partial^2 l}{\partial x^2}\widehat{x}_t\widehat{x}_t+2X\mu\frac{\partial^2 l}{\partial x\partial\mu}\widehat{x}_t\widehat{\mu}_t\right); \\ Q_F &= \frac{1}{2}X^2\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\right)\widehat{x}_t\widehat{x}_t \\ &\quad +XX\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_t\widehat{x}_{t+1}+X\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{x}_t\widehat{\mu}_t+X\mu\frac{\partial^2 F}{\partial y\partial\mu}\widehat{x}_{t+1}\widehat{\mu}_t. \end{aligned}$$

It is straightforward to show that the maximization of the unconditional objective (2.9) subject to the linearized analogues of equations (2.2) yields the same solution as log-linearization of the first order conditions (2.6). This latter approach is proposed by Khan, King and Wolman (2004) in the context of conditional optimization, and is extended in DDN (2007) to unconditional optimization. See Appendix 6.2 for a confirmation of this assertion.

3. Example: Calvo model with distorted steady state

A more or less canonical dynamic New Keynesian model is developed and two issues in particular are pursued. First, what variables appear in the approximate loss function? Second, some insight is sought into the nature of optimal monetary policy although we leave for future research a full characterization of (dynamic) UO monetary policy.

3.1. The Households

As noted, the model is almost standard. However, it turns out that optimal, steady-state inflation is positive. An important implication of that trend in inflation is that it renders price dispersion, defined below, a variable of first-order importance.[5] As a result, the linear-quadratification of the model becomes a little more algebraically intensive. These issues are analyzed more fully in Damjanovic and Nolan (2006).

There are a large number of identical agents in this (closed) economy where the only input to production is labour. Each agent evaluates utility using the following criterion:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(Y_t, N_t(i)) = E_0 \sum_{t=0}^{\infty} \beta^t \left(\log(Y_t) - \frac{\lambda}{1+\upsilon} \left(\int_i N_t(i) di \right)^{1+\upsilon} \right). \quad (3.1)$$

E_t denotes the conditional expectations operator at time $t \geq 0$, β is the discount factor, Y_t is consumption and $N_t(i)$ is the quantity of labour supplied to industry i; labour is industry specific. $\upsilon \geq 0$ measures the labour supply elasticity while λ is a 'preference' parameter.

Consumption is defined over a Dixit-Stiglitz basket of goods

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}. \quad (3.2)$$

The average price-level, P_t, is known to be

$$P_t = \left[\int_0^1 p_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}}. \quad (3.3)$$

[5] Of course, it is possible to avoid this complication by appropriate indexing. Although a popular assumption in quantitative investigations, it is far from uncontroversial. On the other hand, we are interested in applying the approach of Section 2 and so the complications consequent on optimal trend inflation are of some interest to us.

The demand for each good is given by

$$Y_t(i) = \left(\frac{p_t(i)}{P_t}\right)^{-\theta} Y_t^d, \tag{3.4}$$

where $p_t(i)$ is the nominal price of the final good produced in industry i and Y_t^d denotes aggregate demand.

Agents face a flow constraint of the following sort

$$P_t Y_t + B_t = [1 + i_{t-1}] \, B_{t-1} + W_t N_t (1 - \tau) + \Pi_t. \tag{3.5}$$

As all agents are identical, the only financial assets traded in equilibrium will be those issued by the fiscal authority. Here B_t denotes the nominal value of government bond holdings, at the end of date t, $1 + i_t$ is the nominal interest rate on this 'riskless' one-period nominal asset, W_t is the nominal wage in period t (our assumptions mean that we do not need to index wages on i), and Π_t indicates any profits remitted to the individual. We assume the labour income is taxed at rate τ. The usual conditions are assumed to apply to the consumers limiting net savings behavior. Hence, necessary conditions for an optimum include:

$$-\frac{U_N'(Y_t, N_t)}{U_Y'(Y_t, N_t)} = \lambda N_t^v Y_t = w_t \,(1 - \tau)\,; \tag{3.6}$$

$$w_t = \frac{\lambda}{1 - \tau} N_t^v Y_t; \tag{3.7}$$

and

$$E_t \left\{ \frac{\beta U_Y'(Y_{t+1}, N_{t+1})}{U_Y'(Y_t, N_t)} \frac{P_t}{P_{t+1}} \right\} = \frac{1}{1 + i_t}. \tag{3.8}$$

Here w_t denotes the real wage. The complete markets assumption implies the existence of a unique stochastic discount factor,

$$Q_{t,t+k} = \beta \frac{Y_t P_t}{Y_{t+k} P_{t+k}}, \tag{3.9}$$

where

$$E_t \left\{ Q_{t,t+k} \right\} = E_t \prod_{j=0}^{k} \frac{1}{1 + i_{t+j}}.$$

3.2. Representative firm: factor demand

As noted, labour is the only factor of production. Firms are monopolistic competitors who produce their distinctive goods according to the following technology

$$Y_t(i) = A_t \left[N_t(i)\right]^{1/\phi}, \tag{3.10}$$

where $N_t(i)$ denotes the amount of labour hired by firm i in period t, A_t is a stochastic productivity shock and $1 < \phi$.

The demand for output determines the demand for labour. Hence we find that

$$N_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\theta\phi} \left(\frac{Y_t}{A_t}\right)^{\phi}. \tag{3.11}$$

We assume that there is an economy wide labour market so that all the firms pay the same wage for the same labour. As a result, as asserted above, we may write $w_t(i) = w_t, \ \ \forall i$. We assume that all households provide the same share of labour to all firms. The total amount of labour will be

$$N_t = \int N_t(i)di = \left(\frac{Y_t}{A_t}\right)^{\phi} \int \left(\frac{P_t(i)}{P_t}\right)^{-\theta\phi} di = \left(A_t^{-1}Y_t\right)^{\phi} \Delta_t, \tag{3.12}$$

where we define Δ_t as our measure of price dispersion:

$$\Delta_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\theta\phi} di. \tag{3.13}$$

3.3. Representative firm: price setting

As in Calvo (1983), each period a fixed proportion of firms are allowed to adjust prices. Those firms choose the nominal price which maximizes their expected profit given that they may have to charge the same price in k periods time, with probability α^k. As usual, we assume that firms are cost-takers. Let $p_t'(i)$ denote the choice of nominal price by a firm that is permitted to re-price in period t. As all firms who are permitted to reprice will choose the same price, optimal repricing implies

$$\left(\frac{p_t'}{P_t}\right)^{1+\theta(\phi-1)} = \frac{\left(\frac{\theta}{\theta-1}\right)\sum_{k=0}^{\infty}(\alpha\beta)^k Y_{t+k}^{-1}\left[\phi\mu_{t+k}w_{t+k}A_{t+k}^{-\phi}Y_{t+k}^{\phi}(P_t/P_{t+k})^{-\theta\phi}\right]}{\sum_{k=0}^{\infty}(\alpha\beta)^k(P_t/P_{t+k})^{1-\theta}}. \tag{3.14}$$

where μ_t is a cost-push shock. The price index then evolves according to the law of motion,

$$P_t = \left[(1-\alpha)\,p_t'^{1-\theta} + \alpha P_{t-1}^{1-\theta}\right]^{1/(1-\theta)}. \tag{3.15}$$

Because the relative prices of the firms that do not change their prices in period t, fall by the rate of inflation, we may derive a law of motion for the measure of price dispersion

$$\Delta_t = \alpha\Delta_{t-1}\pi_t^{\theta\phi} + (1-\alpha)\left(p_t'/P_t\right)^{-\theta\phi}. \tag{3.16}$$

4. UO (Monetary) Policy

Proposition 4.1 sets out the relevant Ramsey problem.

Proposition 4.1. *The Ramsey plan is a choice of state contingent paths for the endogenous variables* $\{\pi_{t+k,}\Delta_{t+k}, p_{t+k}, u_{t+k}, X_{t+k}, Z_{t+k}\}_{k=0}^{\infty}$ *from date* t *onwards given* $\left\{E_t A_{t+k}, E_t\mu_{t+k}\right\}_{k=0}^{\infty}$ *so as to maximize social welfare function (4.1) subject to constraints (4.2)-(4.4):*

$$\max EE_t\sum_{k=0}^{\infty}\beta^k\left(\frac{\log u_{t+k}}{(1+v)\,\phi} - \frac{1}{\phi}\log\Delta_{t+k} - u_{t+k}\right); \tag{4.1}$$

subject to:

- *The Phillips block*

$$p_t^{\theta\phi-\theta+1}X_t = Z_t; \tag{4.2}$$

$$\begin{aligned} X_t &= 1+\alpha\beta E_t X_{t+1}\pi_{t+1}^{\theta-1}; \\ Z_t &= \frac{(1+v)\phi}{\Phi}\left(\frac{\mu_t}{\mu}\frac{u_t}{\Delta_t}\right) + \alpha\beta E_t Z_{t+1}\pi_{t+1}^{\theta\phi}. \end{aligned}$$

- *The law of motion of prices*

$$\Delta_t = \alpha \Delta_{t-1} \pi_t^{\theta\phi} + (1-\alpha)\, p_t^{-\theta\phi}. \tag{4.3}$$

- *Prices*: p_t is the relative price set by firms updating at time t,

$$p_t = \left(\frac{1 - \alpha \pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{1}{1-\theta}}. \tag{4.4}$$

It is useful in formalizing this policy problem to define some variables as follows: Discounted marginal revenue is $X_t := E_t \sum_{k=0}^{\infty} (\beta\alpha)^k \left(\frac{P_t}{P_{t+k}}\right)^{1-\theta}$; discounted marginal cost is $Z_t := E_t \sum_{k=0}^{\infty} (\beta\alpha)^k \frac{\mu_{t+k}}{\mu} \frac{(1+v)\phi}{\Phi} \frac{u_{t+k}}{\Delta_t} \left(\frac{P_t}{P_{t+k}}\right)^{-\theta\phi}$; period marginal cost is $u_{t+k} := \frac{\lambda}{1+v} \Delta_{t+k}^{v+1} \left(A_{t+k}^{-1} Y_{t+k}\right)^{(v+1)\phi}$; and $\Phi := \frac{\theta-1}{\theta} \frac{1-\tau}{\mu} < 1$, which indexes the steady state distortions in this economy.

One can set up the Hamiltonian for this problem, as proposed in section 2, as follows:

$$\begin{aligned} H \;=\;& \left(\frac{1}{(v+1)\,\phi} \log u_t - \frac{1}{\phi} \log \Delta_t - u_t \right) \\ & + \rho_t \left(X_t - 1 - \beta\alpha \pi_{t+1}^{\theta-1} X_{t+1} \right) \\ & + \varphi_t \left(Z_t - \frac{\mu_t}{\mu} \frac{(1+v)\phi}{\Phi} \frac{u_t}{\Delta_t} - \beta\alpha \pi_{t+1}^{\theta\phi} Z_{t+1} \right) \\ & + \xi_t \left(Z_t - p_t^{\theta\phi-\theta+1} X_t \right) \\ & + \eta_t \left(\Delta_t - \alpha \Delta_{t-1} \pi_t^{\theta\phi} - (1-\alpha)\, p_t^{-\theta\phi} \right) \\ & + \delta_t \left(p_t - \left(\frac{1 - \alpha \pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{1}{1-\theta}} \right). \end{aligned}$$

The necessary conditions for an optimum include:

$$\begin{aligned}
u_t \frac{\partial}{\partial u_t} H &= \frac{1}{(v+1)\,\phi} - u_t - \varphi_t \frac{\mu_t}{\mu} \frac{(1+v)\phi}{\Phi} \frac{u_t}{\Delta_t}; \\
\frac{\partial}{\partial \Delta_t} H &= \left(-\frac{1}{\phi \Delta_t} \right) + \varphi_t \left(\frac{\mu_t}{\mu} \frac{(1+v)\phi}{\Phi} \frac{u_t}{\Delta_t^2} \right) + \eta_t - E_t \alpha \eta_{t+1} \pi_{t+1}^{\theta\phi}; \\
\frac{\partial}{\partial X_t} H &= \rho_t - \rho_{t-1} \beta \alpha \pi_t^{\theta - 1} - \xi_t p_t^{\theta\phi - \theta + 1}; \\
\frac{\partial}{\partial Z_t} H &= \varphi_t - \varphi_{t-1} \beta \alpha \pi_t^{\theta\phi} + \xi_t; \\
\pi_t \frac{\partial}{\partial \pi_t} H &= -\left(\theta - 1\right) \rho_{t-1} \beta \alpha \pi_t^{\theta-1} X_t - \varphi_{t-1} \beta \alpha \theta \phi \pi_t^{\theta\phi} Z_t \\
& \quad - \eta_t \alpha \theta \phi \Delta_{t-1} \pi_t^{\theta\phi} - \delta_t p_t^{\theta} \frac{\alpha \pi_t^{\theta-1}}{1-\alpha}; \\
p_t \frac{\partial}{\partial p_t} H &= -\xi_t \left(\theta\phi - \theta + 1\right) X_t p_t^{1-\theta+\theta\phi} + \theta \phi \eta_t \left(1-\alpha\right) p_t^{-\theta\phi} + p_t \delta_t.
\end{aligned} \tag{4.5}$$

To reduce a little on notation, denote

$$c_t := \left(\frac{\mu_t}{\mu} \frac{(1+v)\phi}{\Phi} \frac{u_t}{\Delta_t} \right), \tag{4.6}$$

which represents marginal production costs.

4.1. The steady state

The value of the endogenous variables in steady state should solve the system of constraints (4.2), (4.3), (4.4), (4.6) and the first order conditions (4.5). As a result one obtains the following steady state equations:

$$\begin{array}{rl}
p = \left(\frac{1-\alpha\pi^{\theta-1}}{1-\alpha}\right)^{\frac{1}{1-\theta}}; & \varphi\left(v+1\right)\phi c = 1 - \Phi \Delta c; \\
\Delta = \left(\frac{1-\alpha}{1-\alpha\pi^{\theta\phi}}\right) p^{-\theta\phi}; & \eta \Delta \left(1 - \alpha \pi^{\theta\phi}\right) = \left(\frac{1}{\phi} - \varphi c\right); \\
X = \frac{1}{1-\beta\alpha\pi^{\theta-1}}; & \xi = -\varphi\left(1 - \alpha\beta\pi^{\theta\phi}\right); \\
Z = X p^{\theta\phi-\theta+1}; & \rho = \xi X p^{\theta\phi-\theta+1} = -\varphi\left(1-\alpha\beta\pi^{\theta\phi}\right) X p^{\theta\phi-\theta+1}; \\
c = \left(1 - \alpha\beta\pi^{\theta\phi}\right) Z; & \delta p = \left(\theta\phi - \theta + 1\right)\rho - \theta\phi\eta\left(1-\alpha\right) p^{-\theta\phi}. \\
u = \Phi \frac{c\Delta}{(1+v)\phi}; &
\end{array} \tag{4.7}$$

Using these equations, one can derive the following expression

$$(\theta-1)\,\rho\beta\alpha\pi^{\theta-1}X+\varphi\beta\alpha\theta\phi\pi^{\theta\phi}Z+\eta\alpha\theta\phi\Delta\pi^{\theta\phi}+\delta p^{\theta}\frac{\alpha\pi^{\theta-1}}{1-\alpha}=0, \tag{4.8}$$

which can be used to infer a value for the optimal steady-state inflation rate. Using parameter values typically found in the literature, that expression implies that optimal steady state inflation is of the order of 0.2% a year. This small positive trend in inflation reflects a number of conflicting effects. On the one hand, a small amount of inflation can boost demand, as it partially offsets other distortions in the economy. On the other hand price dispersion, which is rising in inflation, acts rather like a cost shock on firms, for reasons analyzed in Damjanovic and Nolan (2006). Hence, one finds that optimal trend inflation has a U-shaped relation to price stickiness α; it is increasing in α when initial price dispersion is relatively small, and declines once initial price dispersion is sufficiently large. Optimal inflation declines in the discount factor, β. As discussed in more detail in DDN (forthcoming), UO policy, in contrast, say, to timeless perspective policy, gives some weight to the distribution of initial conditions. In particular, it reacts to the value of the initial output gap. That is partly why some stimulation of output via inflation is desirable. So, the smaller the discount factor the higher is the relative weight on initial conditions and the higher the optimal inflation rate. Finally, we note that the nominal interest rate is positive in the UO steady state. That conclusion follows from the Euler equation (3.8) which yields $1/(1+i)=\beta/\pi<1$. We leave a fuller characterization of UO monetary policy to further research.

4.2. The quadratic form

Having recovered the optimal steady state, one can obtain a quadratic loss function, an equation of the form (2.9). The details of the derivation are set out in the appendix:

$$EU=EQ_l+\rho EQ_X+\varphi EQ_Z+\xi EQ_{ZX}+\eta EQ_{\Delta}+\delta EQ_p,$$

where

$$
\begin{aligned}
Q_l &= -\frac{1}{2}u\widehat{u}_t^2; \\
Q_X &= \frac{1}{2}X\left(\widehat{X}_t^2\right) - \frac{1}{\beta\alpha\pi^{\theta-1}}X\frac{1}{2}\widehat{X}_t^2 = -\frac{1}{2}\frac{1}{\beta\alpha\pi^{\theta-1}}\widehat{X}_t^2; \\
Qz &= \frac{1}{2}Z\widehat{Z}_t^2 - \frac{1}{2}c\widehat{c}_t^2 - \frac{1}{2}\beta\alpha\pi^{\theta\phi}Z\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right)^2; \\
Q_{xz} &= \frac{1}{2}Z\widehat{Z}_t^2 - Z\frac{1}{2}\left(\left(\theta\phi - \theta + 1\right)\widehat{p}_t + \widehat{X}_t\right)^2 = 0; \\
Q_\Delta &= \frac{1}{2}\Delta\widehat{\Delta}_t^2 - (1-\alpha)\,p^{\theta\phi}\frac{1}{2}\left(\theta\phi\widehat{p}_t\right)^2 - \frac{1}{2}a\Delta\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right)^2; \\
Q_p &= \frac{1}{2}p\widehat{p}_t^2 - \frac{1}{2}p\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\left(\theta\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} - 1\right)\widehat{\pi}_t{}^2.
\end{aligned}
$$

One can simplify this expression in a number of ways; consider the following expression:[6]

$$EU = -\frac{1}{2}E\left[u\widehat{u}_t^2 + \varphi c\widehat{c}_t^2 + \Lambda_x\widehat{X}_t^2 + \Lambda_\Delta\widehat{\Delta}_t^2 + \Lambda_\pi\widehat{\pi}_t{}^2\right]. \tag{4.9}$$

It is possible to write equation (4.9) in a way that relates it more clearly to the 'standard' loss function often employed which is defined over simply output and inflation. First, recall the definitions of $\widehat{u}_t$:

$$\widehat{u}_t = (v+1)\,\widehat{\Delta}_t + (v+1)\,\phi\left(\widehat{Y}_t - \widehat{A}_t\right).$$

Now note that $\widehat{c}_t$ can be represented as

$$\widehat{c}_t = \widehat{u}_t - g_t.$$

$\widehat{g}_t$ can be thought of as the 'labour wedge' of inefficiency (note the role price dispersion):

$$g_t := \frac{\partial U}{\partial N}/\frac{\partial F}{\partial N} = \frac{u_t}{c_t} = \left(\frac{\mu}{\mu_t}\frac{\Phi}{(1+v)\phi}\Delta_t\right),$$

which in log-linearized form is simply:

$$\widehat{g}_t = \widehat{\Delta}_t - \widehat{\mu}_t.$$

So we can further simplify (4.9) as

$$EU = -\frac{1}{2}E\left[\phi\,(1+v)\left(\widehat{Y}_t - \widehat{Y}_t^*\right)^2 + G\widehat{g}_t^2 + \Lambda_x\widehat{X}_t^2 + \Lambda_\Delta\widehat{\Delta}_t^2 + \Lambda_\pi\widehat{\pi}_t{}^2\right]. \tag{4.10}$$

[6] The coefficients of equation (4.9) are positive for reasonable calibration.

The term $\widehat{Y}_t^*$ represents the 'target' level of output $Y_t^* = \widehat{A}_t - \widehat{\mu}_t - \upsilon_\Delta \widehat{\Delta}_t$ (and where details concerning coefficients are again given in the Appendix). The 'target' rate is increasing in productivity and declining in the cost-push shock; it is also declining in price dispersion. The variable $\widehat{X}_t$ represents, in effect, the losses to the firm forced to charge suboptimal prices due to price stickiness and expected inflation, to which they may not be able to react.

This form of the loss function can easily be nested to familiar cases, either the non-distorted steady state where $\Phi = 1$, or where the steady state of the model economy remains distorted but where the social discount rate is equal to the private rate of discount, $\beta = 1$ (in which case the UO policy and the timeless perspective policies coincide). In both special cases optimal monetary policy corresponds to price stability and the loss function (4.10) reduces to a familiar form defined simply over inflation and output. Specifically, if the optimal steady state is characterized by price stability, then $\Lambda_x = 0$. Moreover one can easily show that price dispersion, $\widehat{\Delta}_t$, is a second order term in that case. Lastly, the labour wedge $\widehat{g}_t$ is then simply a cost -push shock, $\widehat{\mu}_t$, and can be considered as a term independent of policy.

Finally, for completeness, the full set of linearized equations of the model economy are set out at the end of the Appendix.

5. Conclusion

This paper developed a straightforward approach for analytically deriving UO (monetary) policy. It demonstrated that, in general, one is able to obtain a purely quadratic approximate unconditional loss function in the case of a model economy with a distorted steady state. In an application, it was shown that the loss function may be somewhat more complex than in a model with no steady-state distortions; inflation and output are no longer the sole arguments in the loss function. However, the loss function so obtained is easily interpreted in terms of the underlying distortions in the economy. Furthermore, optimal inflation and nominal interest rates are positive in the steady state. From the perspective of UO policy, therefore, one may have the beginnings of a theory as to why central banks appear to target a positive rate of inflation.

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6. Appendixes

6.1. The possibility of the second order approximation

The first part of the appendix demonstrates the key result in Section 2.2, namely the existence of the quadratic form, (2.9). The first line of the following block of equations corresponds to the top line of (2.9), the subsequent lines being its quadratic approximation:

$$\begin{aligned} El\left(x_t,\mu_t\right) &= E\left[l\left(x_t,\mu_t\right)+\xi F\left(y_t,x_t,\mu_t\right)\right]= \\ &= E\left(l+X\frac{\partial l}{\partial x}\widehat{x}_t+\frac{1}{2}\left(X^2\frac{\partial^2 l}{\partial x^2}\widehat{x}_t\widehat{x}_t+X\frac{\partial l}{\partial x}\widehat{x}_t\widehat{x}_t+2X\mu\frac{\partial l}{\partial x\partial\mu}\widehat{x}_t\widehat{\mu}_t\right)\right) \\ &\quad +E\xi\left(F+X\frac{\partial F}{\partial x}\widehat{x}_t+X\frac{\partial F}{\partial y}\widehat{y}_t\right) \\ &\quad +\frac{1}{2}\xi\left(X\frac{\partial F}{\partial x}+X^2\frac{\partial^2 F}{\partial x^2}\right)E\widehat{x}_t\widehat{x}_t+\frac{1}{2}\xi\left(X\frac{\partial F}{\partial y}+XX\frac{\partial F}{\partial y^2}\right)E\widehat{y}_t\widehat{y}_t \\ &\quad +\xi E\left(XX\frac{\partial F}{\partial x\partial y}\widehat{x}_t\widehat{y}_t+X\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{x}_t\widehat{\mu}_t+X\mu\frac{\partial F}{\partial y\partial\mu}\widehat{y}_t\widehat{\mu}_t\right)+O3. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} El\left(x_t,\mu_t\right) &= l+\xi F+XE\widehat{x}_t\left(\frac{\partial l}{\partial x}+\xi\frac{\partial F}{\partial x}+\xi\frac{\partial F}{\partial y}\right) \\ &\quad +\frac{1}{2}XE\widehat{x}_t\widehat{x}_t\left(\frac{\partial l}{\partial x}+\xi\frac{\partial F}{\partial x}+\xi\frac{\partial F}{\partial y}\right)+EQ_l+\xi EQ_F+t.i.p+O3. \end{aligned} \tag{6.1}$$

Here Q_l and Q_F are pure second order terms:

$$\begin{aligned} Q_l &= \frac{1}{2}\left(X^2\frac{\partial^2 l}{\partial x^2}\widehat{x}_t\widehat{x}_t+2X\mu\frac{\partial l}{\partial x\partial\mu}\widehat{x}_t\widehat{\mu}_t\right); \\ Q_F &= \frac{1}{2}X^2\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\right)\widehat{x}_t\widehat{x}_t+XX\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_t\widehat{x}_{t+1}+X\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{x}_t\widehat{\mu}_t+X\mu\frac{\partial^2 F}{\partial y\partial\mu}\widehat{x}_{t+1}\widehat{\mu}_t. \end{aligned}$$

Using the constraints $E_t x_{t+1} = y_t$, the steady state conditions (2.8) and the property of unconditional expectations that $Ez_{t+1} = Ez_t$, we can show that the firts line of expression (6.1) equals to $l + \xi F = l$ is a steady state value of loss function, which is a term independent of policy ($t.i.p.$). Thus we have proved that the loss function can be represented in a pure quadratic form.

$$El\left(x_t,\mu_t\right)=EQ_l+\xi EQ_F+t.i.p+O3.$$

6.2. Alternative approaches to recovering UO policy

The approach of some researchers is to solve non-linear problems and then linearize the resulting optimality conditions. For example, in the context of *conditionally* optimal monetary policy, that is the approach taken by Khan, King and Wolman (2003). This section demonstrates that this alternative approach also works in the case of *unconditionally* optimal policy. Specifically, the maximization of the unconditional objective (2.9) subject to the linearized analogues of equations (2.2) yields the same solution as log-linearization of the first order conditions (2.6). The first order conditions to the non-linear problem are written as

$$\begin{aligned}
\frac{\partial H}{\partial x_t} &= \frac{\partial l\left(x_t,\mu_t\right)}{\partial x}+\xi_t\frac{\partial F\left(y_t,x_t,\mu_t\right)}{\partial x}+\rho_{t-1}=0;\\
\frac{\partial H}{\partial y_t} &= \xi_t\frac{\partial F\left(y_t,x_t,\mu_t\right)}{\partial y_t}-\rho_t.
\end{aligned}$$

The log-linearized versions of these equations are:

$$\begin{aligned}
\frac{\partial H}{\partial x_t} &= \frac{\partial l}{\partial x}+X\frac{\partial^2 l}{\partial x^2}\widehat{x}_t+\mu\frac{\partial^2 l}{\partial x\partial\mu}\widehat{\mu}_t\\
&\quad+\xi\frac{\partial F}{\partial x}+\xi\frac{\partial F}{\partial x}\widehat{\xi}_t+\xi X\frac{\partial^2 F}{\partial x^2}\widehat{x}_t+\xi X\frac{\partial^2 F}{\partial x\partial y}\widehat{y}_t+\xi\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{\mu}_t\\
&\quad+\rho+\rho\widehat{\rho}_{t-1}+O2; \qquad (6.2)\\
\frac{\partial H}{\partial y_t} &= \xi\frac{\partial F}{\partial y}+\xi\frac{\partial F}{\partial y}\widehat{\xi}_t+\xi X\frac{\partial^2 F}{\partial y^2}\widehat{y}_t+\xi X\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_t+\xi X\frac{\partial^2 F}{\partial\mu\partial y}\widehat{\mu}_t\\
&\quad-\rho-\rho\widehat{\rho}_t+O2; \qquad (6.3)
\end{aligned}$$

which we can simplify by plugging (6.3) into (6.2) and using the steady state conditions (2.8) as

$$\begin{aligned}
\frac{\partial H}{\partial x_t} &= X\frac{\partial^2 l}{\partial x^2}\widehat{x}_t+\mu\frac{\partial^2 l}{\partial x\partial\mu}\widehat{\mu}_t\\
&\quad\xi\frac{\partial F}{\partial x}\widehat{\xi}_t+\xi X\frac{\partial^2 F}{\partial x^2}\widehat{x}_t+\xi X\frac{\partial^2 F}{\partial x\partial y}\widehat{y}_t+\xi X\frac{\partial^2 F}{\partial x\partial y}\widehat{y}_t+\xi\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{\mu}_t\\
&\quad\xi\frac{\partial F}{\partial y}\widehat{\xi}_{t-1}+\xi X\frac{\partial^2 F}{\partial y^2}\widehat{y}_{t-1}+\xi X\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_{t-1}+\xi\mu\frac{\partial^2 F}{\partial\mu\partial y}\widehat{\mu}_{t-1}=0. \qquad (6.4)
\end{aligned}$$

We turn now to the LQ approach. Utility can be represented as (2.9). Hence, the relevant optimization problem is

$$\max El\left(\widehat{x}_t,\widehat{y}_t,\widehat{\mu}_t\right)=\max\frac{1}{2}E\left(X^2\frac{\partial^2 l}{\partial x^2}\widehat{x}_t\widehat{x}_t+2X\mu\frac{\partial l}{\partial x\partial\mu}\widehat{x}_t\widehat{\mu}_t\right)$$
$$+\frac{1}{2}X^2\xi E\left(\frac{\partial^2 F}{\partial x^2}\widehat{x}_t\widehat{x}_t+\frac{\partial^2 F}{\partial y^2}\widehat{y}_t\widehat{y}_t\right)$$
$$+\xi XX\frac{\partial^2 F}{\partial x\partial y}E\widehat{x}_t\widehat{y}_t+\xi X\mu\frac{\partial^2 F}{\partial x\partial\mu}E\widehat{x}_t\widehat{\mu}_t+\xi X\mu\frac{\partial^2 F}{\partial y\partial\mu}E\widehat{y}_t\widehat{\mu}_t,$$

subject to log-linearized constraints

$$F\left(E_t x_{t+1},x_t,\mu_t\right)=X\frac{\partial F}{\partial x_t}\widehat{x}_t+X\frac{\partial F}{\partial y}\widehat{y}_t+\mu\frac{\partial F}{\partial\mu_t}\widehat{\mu}_t=0; \quad (6.5)$$
$$\widehat{y}_t=E_t\widehat{x_{t+1}}. \quad (6.6)$$

The new Hamiltonian can be written as

$$\begin{aligned}\widetilde{H} &= \frac{1}{2}\left(X^2\frac{\partial^2 l}{\partial x^2}\widehat{x}_t\widehat{x}_t+2X\mu\frac{\partial l}{\partial x\partial\mu}\widehat{x}_t\widehat{\mu}_t\right)+\frac{1}{2}X^2\xi\left(\frac{\partial^2 F}{\partial x^2}\widehat{x}_t\widehat{x}_t+\frac{\partial^2 F}{\partial y^2}\widehat{y}_t\widehat{y}_t\right)\\&+\xi XX\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_t\widehat{y}_t+\xi X\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{x}_t\widehat{\mu}_t+\xi X\mu\frac{\partial^2 F}{\partial y\partial\mu}\widehat{y}_t\widehat{\mu}_t\\&+s_t\left(X\frac{\partial F}{\partial x_t}\widehat{x}_t+X\frac{\partial F}{\partial y}\widehat{y}_t+\mu\frac{\partial F}{\partial\mu_t}\widehat{\mu}_t\right)+r_t\widehat{y}_t-r_{t-1}\widehat{x}_t.\end{aligned}$$

where s_t and r_t are the corresponding Lagrange multipliers attached to linearized constraints (6.5, 6.6). The resulting first order conditions are

$$\begin{aligned}\frac{\partial\widetilde{H}}{\partial\widehat{x}_t} &= \left(X^2\frac{\partial^2 l}{\partial x^2}\widehat{x}_t+2X\mu\frac{\partial l}{\partial x\partial\mu}\widehat{\mu}_t\right)+\xi X^2\frac{\partial^2 F}{\partial x^2}\widehat{x}_t\\&+\xi XX\frac{\partial^2 F}{\partial x\partial y}\widehat{y}_t+\xi X\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{\mu}_t+s_tX\frac{\partial F}{\partial x_t}\widehat{x}_t-\widehat{r}_{t-1},\end{aligned}$$

and

$$\frac{\partial\widetilde{H}}{\partial\widehat{y}_t}=\xi X^2\frac{\partial^2 F}{\partial y^2}\widehat{y}_t+\xi XX\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_t+\xi X\mu\frac{\partial^2 F}{\partial y\partial\mu}\widehat{\mu}_t+\widehat{s}_tX\frac{\partial F}{\partial y}+\rho\widehat{r}_t.$$

So, it follows that we may write

$$\begin{aligned}\frac{1}{X}\frac{\partial\widetilde{H}}{\partial\widehat{x}_t} &= \left(X\frac{\partial^2 l}{\partial x^2}\widehat{x}_t+\mu\frac{\partial^2 l}{\partial x\partial\mu}\widehat{\mu}_t\right)+\xi X\frac{\partial^2 F}{\partial x^2}\widehat{x}_t\\&+\xi X\frac{\partial^2 F}{\partial x\partial y}\widehat{y}_t+\xi\mu\frac{\partial^2 F}{\partial x\partial\mu}\widehat{\mu}_t+s_t\frac{\partial F}{\partial x_t}\\&\xi X\frac{\partial^2 F}{\partial y^2}\widehat{y}_{t-1}+\xi X\frac{\partial^2 F}{\partial x\partial y}\widehat{x}_{t-1}+\xi\mu\frac{\partial^2 F}{\partial y\partial\mu}\widehat{\mu}_{t-1}+\widehat{s}_{t-1}\frac{\partial F}{\partial y}.\end{aligned}$$

This is identical to (6.4) with the following relations between Lagrange multipliers $s_t = \xi\widehat{\xi}_t,\ \ r_t = \rho\widehat{\rho}_t$.

6.3. A2: The second order approximation to unconditional welfare.

In Section 4.2 of the main text we asserted the existence of the following quadratic equation,

$$EU = E\left(Q_l + \rho Q_X + \varphi Q_Z + \xi Q_{ZX} + \eta Q_\Delta + \delta Q_p\right).$$

where Q_l, is the second order term of the loss function, and $Q_X,\ Q_Z, Q_{ZX}, Q_\Delta,\ Q_p$ are the second order terms of the log linear approximation to constraints (4.2)-(4.4). This section demonstrates how one derives that equation. The model can be rewritten in the following linear-quadratic representation

$$\begin{aligned}
& (\frac{1}{(v+1)\,\phi}\log u_t - \frac{1}{\phi}\log\Delta_t - u_t) - O3 \\
= \;& \frac{1}{(v+1)\,\phi}\widehat{u}_t - \frac{1}{\phi}\widehat{\Delta}_t - u\left(\widehat{u}_t + \frac{1}{2}\widehat{u}_t^2\right) + tip; \\
& \left(X_t - 1 - \beta\alpha\pi_{t+1}^{\theta-1}X_{t+1}\right) - O3 \\
= \;& X\left(\widehat{X}_t + \frac{1}{2}\widehat{X}_t^2\right) - \beta\alpha\pi^{\theta-1}X\left(\widehat{X}_{t+1} + (\theta-1)\,\widehat{\pi}_{t+1} + \frac{1}{2}\left(\widehat{X}_{t+1} + (\theta-1)\,\widehat{\pi}_{t+1}\right)^2\right); \\
& \left(Z_t - c_t - \beta\alpha\pi_{t+1}^{\theta\phi}Z_{t+1}\right) - O3 \\
= \;& Z\left(\widehat{Z}_t + \frac{1}{2}\widehat{Z}_t^2\right) - c\left(\widehat{c}_t + \frac{1}{2}c_t^2\right) - \beta\alpha\pi^{\theta\phi}Z\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1} + \frac{1}{2}\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right)^2\right); \\
& \left(Z_t - p_t^{\theta\phi-\theta+1}X_t\right) - O3 \\
= \;& Z\left(\widehat{Z}_t + \frac{1}{2}\widehat{Z}_t^2\right) - p^{\theta\phi-\theta+1}X\left((\theta\phi-\theta+1)\,\widehat{p}_t + \widehat{X}_t + \frac{1}{2}\left((\theta\phi-\theta+1)\,\widehat{p}_t + \widehat{X}_t\right)^2\right); \\
& \Delta_t - \alpha\Delta_{t-1}\pi_t^{\theta\phi} - (1-\alpha)\,p_t^{\theta\phi} - O3 \\
= \;& \Delta\left(\widehat{\Delta}_t + \frac{1}{2}\widehat{\Delta}_t^2\right) - a\Delta\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t + \frac{1}{2}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right)^2\right) \\
& - (1-\alpha)\,p^{\theta\phi}\left(\theta\phi\widehat{p}_t + \frac{1}{2}\left(\theta\phi\widehat{p}_t\right)^2\right); \\
& p_t - \left(\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha}\right)^{\frac{1}{1-\theta}} - O3; \\
= \;& p\left(\widehat{p}_t + \frac{1}{2}\widehat{p}_t^2\right) - p\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\widehat{\pi}_t - \frac{1}{2}p\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\left(\theta\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} - 1\right)\widehat{\pi}_t{}^2.
\end{aligned}$$

The linear relations are therefore,

$$\widehat{X}_t - \beta\alpha\pi^{\theta-1}\left(\widehat{X}_{t+1} + (\theta-1)\,\widehat{\pi}_{t+1}\right) = O2; \tag{6.7}$$

$$\widehat{Z}_t - \frac{c}{Z}\widehat{c}_t - \beta\alpha\pi^{\theta\phi}\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right) = O2; \tag{6.8}$$

$$\widehat{Z}_t - \left((\theta\phi - \theta + 1)\,\widehat{p}_t + \widehat{X}_t\right) = O2; \tag{6.9}$$

$$\widehat{\Delta}_t - a\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right) - (1-\alpha)\,\frac{p^{\theta\phi}}{\Delta}\theta\phi\widehat{p}_t = O2; \tag{6.10}$$

$$\widehat{p}_t - \frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\widehat{\pi}_t = O2, \tag{6.11}$$

and the following are the quadratic relations:

$$\begin{aligned}
Q_l &= -\frac{1}{2}u\widehat{u}_t^2; \\
Q_X &= \frac{1}{2}X\left(\widehat{X}_t^2\right) - \frac{1}{\beta\alpha\pi^{\theta-1}}X\frac{1}{2}\widehat{X}_t^2 = -\frac{1}{2}\frac{1}{\beta\alpha\pi^{\theta-1}}\widehat{X}_t^2; \\
Qz &= \frac{1}{2}Z\widehat{Z}_t^2 - \frac{1}{2}c\widehat{c}_t^2 - \frac{1}{2}\beta\alpha\pi^{\theta\phi}Z\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right)^2; \qquad (6.12) \\
Q_{xz} &= \frac{1}{2}Z\widehat{Z}_t^2 - Z\frac{1}{2}\left((\theta\phi - \theta + 1)\,\widehat{p}_t + \widehat{X}_t\right)^2 = 0; \\
Q_\Delta &= \frac{1}{2}\Delta\widehat{\Delta}_t^2 - (1-\alpha)\,p^{\theta\phi}\frac{1}{2}\left(\theta\phi\widehat{p}_t\right)^2 - \frac{1}{2}a\Delta\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right)^{2;} \qquad (6.13) \\
Q_p &= \frac{1}{2}p\widehat{p}_t^2 - \frac{1}{2}p\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\left(\theta\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} - 1\right)\widehat{\pi}_t{}^2. \qquad (6.14)
\end{aligned}$$

One can simplify these expressions as follows.

Simplification of Qp : Use (6.11) in (6.14) to find that

$$Q_p = -\frac{1}{2}\Pi\widehat{\pi}_t{}^2,$$

where we define

$$\Pi := \left(\frac{1-\alpha\pi^{\theta-1}}{1-\alpha}\right)^{\frac{1}{1-\theta}}\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\left(\frac{(\theta-1)\,\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} - 1\right).$$

Simplification of Qz : Use (6.8) in (6.12),

$$
\begin{aligned}
2\frac{EQz}{Z} &= \widehat{Z}_t^2 - \left(1-\alpha\beta\pi^{\theta\phi}\right)\widehat{c}_t^2 - \beta\alpha\pi^{\theta\phi}\left(\widehat{Z}_t + \theta\phi\widehat{\pi}_t\right)^2 \qquad (6.15)\\
&= \left(1-\beta\alpha\pi^{\theta\phi}\right)\widehat{Z}_t^2 - \left(1-\alpha\beta\pi^{\theta\phi}\right)\widehat{c}_t^2 - \beta\alpha\pi^{\theta\phi}\left(\theta\phi\widehat{\pi}_t\right)\left(2\widehat{Z}_t + \theta\phi\widehat{\pi}_t\right)\\
&= \left((\theta\phi-\theta+1)\widehat{p}_t + \widehat{X}_t\right)\left(\left(1-\beta\alpha\pi^{\theta\phi}\right)\left((\theta\phi-\theta+1)\widehat{p}_t + \widehat{X}_t\right) - 2\beta\alpha\pi^{\theta\phi}(\theta\phi)\widehat{\pi}_t\right)\\
&\quad - \left(1-\alpha\beta\pi^{\theta\phi}\right)\widehat{c}_t^2 - \beta\alpha\pi^{\theta\phi}(\theta\phi)^2\widehat{\pi}_t^2\\
&= \widehat{X}_t^2\left(1-\alpha\beta\pi^{\theta\phi}\right) - \left(1-\alpha\beta\pi^{\theta\phi}\right)\widehat{c}_t^2\\
&\quad -\beta\alpha\pi^{\theta\phi}(\theta\phi)^2\widehat{\pi}_t^2 + \left(1-\beta\alpha\pi^{\theta\phi}\right)(\theta\phi-\theta+1)^2\widehat{p}_t^2 - (\theta\phi-\theta+1)2\beta\alpha\pi^{\theta\phi}(\theta\phi)\widehat{\pi}_t\widehat{p}_t\\
&\quad 2\left(\frac{\theta\phi\left(\alpha\pi^{\theta-1} - \beta\alpha\pi^{\theta\phi}\right) - (\theta-1)\alpha\pi^{\theta-1}\left(1-\alpha\beta\pi^{\theta\phi}\right)}{1-\alpha\pi^{\theta-1}}\right)\widehat{\pi}_t\widehat{X}_t.
\end{aligned}
$$

Furthermore from (6.7) one can find an expression for $E2\widehat{\pi}_t\widehat{X}_t$

$$
E\widehat{X}_t^2 = E\left(\beta\alpha\pi^{\theta-1}\right)^2\left(\widehat{X}_{t+1}^2 + 2(\theta-1)\widehat{\pi}_{t+1}\widehat{X}_{t+1} + (\theta-1)^2\widehat{\pi}_{t+1}^2\right),
$$

which implies that

$$
2E\widehat{\pi}_t\widehat{X}_t = \frac{1}{(\theta-1)}\left(\frac{1-\left(\beta\alpha\pi^{\theta-1}\right)^2}{\left(\beta\alpha\pi^{\theta-1}\right)^2}\right)\widehat{X}_t^2 - (\theta-1)\widehat{\pi}_t^2. \qquad (6.16)
$$

Now, combining (6.16) with (6.15)

$$
EQz = -Z\frac{1}{2}E\left[\left(1-\alpha\beta\pi^{\theta\phi}\right)\widehat{c}_t^2 + Z_\pi\widehat{\pi}_t{}^2 + Z_x\widehat{X}_t^2\right],
$$

where

$$
\begin{aligned}
Z_\pi &= \frac{(\theta\phi-\theta+1)\left[\theta\phi\left(\alpha\beta\pi^{\theta\phi} - \alpha\pi^{\theta-1}\alpha\pi^{\theta-1}\right) + (\theta-1)\alpha\pi^{\theta-1}\left(1-\beta\alpha\pi^{\theta\phi}\right)\right]}{\left(1-\alpha\pi^{\theta-1}\right)^2};\\
Z_x &= \frac{1-\beta\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta-1}}\frac{1-\beta^2\alpha\pi^{\theta-1}}{\beta^2\alpha\pi^{\theta-1}} + \frac{\theta\phi}{\theta-1}\frac{1-\left(\beta\alpha\pi^{\theta-1}\right)^2}{\left(\beta\alpha\pi^{\theta-1}\right)^2}\frac{\beta\alpha\pi^{\theta\phi} - \alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}.
\end{aligned}
$$

Simplification of Q_Δ :

$$
\frac{2}{\Delta}Q_\Delta = \widehat{\Delta}_t^2 - (1-\alpha)\frac{p^{-\theta\phi}}{\Delta}(\theta\phi\widehat{p}_t)^2 - a\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right)^2
$$

One can simplify (6.13) using (6.10)

$$\begin{aligned}
a\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1}+\theta\phi\widehat{\pi}_t\right)^2 &= \frac{1}{a\pi^{\theta\phi}}\left(\widehat{\Delta}_t+(1-\alpha)\frac{p^{-\theta\phi}}{\Delta}\frac{\theta\phi\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\widehat{\pi}_t\right)^2\\
&= \frac{1}{a\pi^{\theta\phi}}\widehat{\Delta}_t^2+\frac{1}{a\pi^{\theta\phi}}\left(\frac{\theta\phi\left(1-\alpha\pi^{\theta\phi}\right)\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\right)^2\widehat{\pi}_t{}^2\\
&\quad +2\theta\phi\frac{1-\alpha\pi^{\theta\phi}}{a\pi^{\theta\phi}}\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\widehat{\Delta}_t\widehat{\pi}_t.
\end{aligned}$$

Next, using constraint (6.10), one finds $E2\widehat{\Delta}_t\widehat{\pi}_t$

$$a\pi^{\theta\phi}\widehat{\Delta}_{t-1}=\widehat{\Delta}_t+\theta\phi\left((1-\alpha)\frac{p^{-\theta\phi}}{\Delta}\frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}-a\pi^{\theta\phi}\right)\widehat{\pi}_t.$$

Recall that $\Delta=\left(\frac{1-\alpha}{1-\alpha\pi^{\theta\phi}}\right)p^{-\theta\phi}$, so that

$$a\pi^{\theta\phi}\widehat{\Delta}_{t-1}=\widehat{\Delta}_t+\theta\phi\left(\frac{\alpha\pi^{\theta-1}-\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta-1}}\right)\widehat{\pi}_t+O2.$$

This implies

$$E\left(a\pi^{\theta\phi}\widehat{\Delta}_{t-1}\right)^2=E\widehat{\Delta}_t^2+E\left(\theta\phi\frac{\alpha\pi^{\theta-1}-\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta-1}}\right)^2\widehat{\pi}_t{}^2+2\theta\phi\left(\frac{\alpha\pi^{\theta-1}-\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta-1}}\right)E\widehat{\pi}_t\widehat{\Delta}_t.$$

One can simplify the final terms in the expression as follows

$$\begin{aligned}
2\frac{\theta\phi}{1-\alpha\pi^{\theta-1}}E\widehat{\pi}_t\widehat{\Delta}_t &= \\
&-\frac{1-\left(a\pi^{\theta\phi}\right)^2}{\alpha\pi^{\theta-1}-\alpha\pi^{\theta\phi}}E\widehat{\Delta}_t^2-\frac{\theta\phi^2}{1-\alpha\pi^{\theta-1}}\frac{\alpha\pi^{\theta-1}-\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta-1}}E\widehat{\pi}_t{}^2;
\end{aligned}\tag{6.17}$$

$$\frac{2}{\Delta\left(1-\alpha\pi^{\theta\phi}\right)}EQ_\Delta=-E\frac{1-a\pi^{\theta\phi}\alpha\pi^{\theta-1}}{\alpha\pi^{\theta\phi}-\alpha\pi^{\theta-1}}\widehat{\Delta}_t^2-E\alpha\pi\left[\frac{\theta\phi}{1-\alpha\pi^{\theta-1}}\right]^2\widehat{\pi}_t{}^2.$$

Hence, using these simplifications, we return to the quadratic expression.

$$\begin{aligned}
EU &= E\left(Q_l+\rho Q_X+\varphi Q_Z+\xi Q_{ZX}+\eta Q_\Delta+\delta Q_p\right);\\
&= -\frac{1}{2}Eu\widehat{u}_t^2-\frac{1}{2}\rho\frac{1}{\beta\alpha\pi^{\theta-1}}E\widehat{X}_t^2-\frac{1}{2}\varphi ZE\left[\frac{c}{Z}\widehat{c}_t^2+Z_\pi\widehat{\pi}_t{}^2+Z_x\widehat{X}_t^2\right]\\
&\quad -\frac{1}{2}\Delta\eta\left(1-\alpha\pi^{\theta\phi}\right)E(\alpha\pi^{\theta-1}\left[\frac{\theta\phi}{1-\alpha\pi^{\theta-1}}\right]^2\widehat{\pi}_t{}^2+\frac{1-a\pi^{\theta\phi}\alpha\pi^{\theta-1}}{\alpha\pi^{\theta\phi}-\alpha\pi^{\theta-1}}\widehat{\Delta}_t^2)\\
&\quad -\frac{1}{2}\Pi\delta E\widehat{\pi}_t{}^2;\\
&= -\frac{1}{2}E\left(u\widehat{u}_t^2+\varphi c\widehat{c}_t^2+\Lambda_x\widehat{X}_t^2+\Lambda_\pi\widehat{\pi}_t{}^2+\Lambda_\Delta\widehat{\Delta}_t^2\right),
\end{aligned}\tag{6.18}$$

where

$$\Lambda_x = \varphi Z \left(Z_x - \frac{1 - \beta\alpha\pi^{\theta\phi}}{\beta\alpha\pi^{\theta-1}} \right);$$

$$\Lambda_\pi = \varphi Z Z_\pi + \Delta\eta D_\pi + \Pi\delta;$$

$$\Lambda_\Delta = \Delta\eta \left(1 - \alpha\pi^{\theta\phi}\right) \frac{\left(1 - a\pi^{\theta\phi}\alpha\pi^{\theta-1}\right)}{\alpha\pi^{\theta\phi} - \alpha\pi^{\theta-1}} > 0.$$

6.3.1. Further simplification

We log-linearize the expression of the marginal disutility from labour u_t as

$$\widehat{u}_t = (v+1)\,\widehat{\Delta}_t + (v+1)\,\phi\left(\widehat{Y}_t - \widehat{A}_t\right).$$

We employ the following representation of marginal production costs

$$\widehat{c}_t = \widehat{u}_t - \widehat{g}_t,$$

where $\widehat{g}_t$ is the labour wedge defined as

$$g_t := \frac{\partial U}{\partial N} / \frac{\partial F}{\partial N} = \frac{u_t}{c_t} = \frac{\mu}{\mu_t} \frac{\Phi}{(1+v)\phi} \Delta_t. \tag{6.19}$$

The first two terms in the quadratic loss function (6.18) can be simplified as follows:

$$
\begin{aligned}
u\widehat{u}_t^2+\varphi c\widehat{c}_t^2 &= \widehat{u}_t^2+\varphi c\left(\widehat{u}_t-\widehat{g}_t\right)^2 \\
&= \left(u+\varphi c\right)\widehat{u}_t^2+\varphi c\left(\widehat{g}_t\right)^2-2\varphi c\widehat{u}_t\widehat{g}_t \\
&= \left(u+\varphi c\right)\left(\widehat{u}_t-\frac{\varphi c}{\left(u+\varphi c\right)}\widehat{g}_t\right)^2+\varphi c\left(1-\left(u+\varphi c\right)\varphi c\right)\widehat{g}_t^2 \\
&= \frac{1}{\left(v+1\right)\phi}\left(\left(v+1\right)\widehat{\Delta}_t+\left(v+1\right)\phi\left(\widehat{Y}_t-\widehat{A}_t\right)-\frac{\varphi c}{u+\varphi c}\left(\widehat{\Delta}_t-\widehat{\mu}_t\right)\right)^2 \\
&\quad +\varphi c\left(1-\frac{\varphi c}{\left(v+1\right)\phi}\right)\widehat{g}_t^2 \\
&= \left(v+1\right)\phi\left(\frac{1}{\phi}\widehat{\Delta}_t+\widehat{Y}_t-\widehat{A}_t-\varphi c\left(\widehat{\Delta}_t-\widehat{\mu}_t\right)\right)^2 \\
&\quad +\frac{\varphi c}{\left(v+1\right)\phi}\left(\left(v+1\right)\phi-\varphi c\right)\widehat{g}_t^2 \\
&= \left(v+1\right)\phi\left(\widehat{Y}_t-\left(\widehat{A}_t-\widehat{\mu}_t-\left(\frac{v}{\left(v+1\right)\phi}+u\right)\widehat{\Delta}_t\right)\right)^2 \\
&\quad +\frac{\varphi c}{\left(v+1\right)\phi}\left(\left(v+1\right)\phi-\varphi c\right)\widehat{g}_t^2 \\
&= \left(v+1\right)\phi\left(\widehat{Y}_t-Y_t^*\right)^2+G\widehat{g}_t^2,
\end{aligned}
$$

where we define Y_t^* and G as

$$
\begin{aligned}
Y_t^* &:= \widehat{A}_t-\widehat{\mu}_t-\left(\frac{v}{\left(v+1\right)\phi}+u\right)\widehat{\Delta}_t \\
G &:= \frac{\varphi c}{\left(v+1\right)\phi}\left(\left(v+1\right)\phi-\varphi c\right)
\end{aligned}
$$

To obtain this result we recall that the steady state value of the Lagrange multiplier φ satisfies the following equation:

$$
\varphi c+u=\frac{1}{\left(v+1\right)\phi}\left(1-\Phi\Delta c\right)+u=\frac{1}{\left(v+1\right)\phi}
$$

6.4. Linearized equations of the model

For completeness, we provide details of the linear approximate model, consisted of the first order conditions (4.5) a system of constraints (4.2), (4.3), (4.4), (4.6).

The linearized block of equations is thus:

$$
\begin{aligned}
u_t \frac{\partial}{\partial u_t} H &= -u\widehat{u}_t - \varphi c\left(\widehat{\varphi}_t + \widehat{c}_t\right) \\
\Delta \frac{\partial}{\partial \Delta_t} H &= -\frac{1}{\phi}\widehat{\Delta}_t + \varphi c\left(\widehat{\varphi}_t + \widehat{c}_t - \widehat{\Delta}_t\right) + \Delta\eta\widehat{\eta}_t - \alpha\Delta\eta\pi^{\theta\phi} E_t\left(\widehat{\eta}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right) \\
\frac{\partial}{\partial X_t} H &= \rho\widehat{\rho}_t - \rho\beta\alpha\pi^{\theta-1}\left(\widehat{\rho}_{t-1} + (\theta-1)\widehat{\pi}_t\right) - \xi p^{\theta\phi-\theta+1}\left(\widehat{\xi}_t + (\theta\phi-\theta+1)\widehat{p}_t\right) \\
\frac{\partial}{\partial Z_t} H &= \varphi\widehat{\varphi}_t - \varphi\beta\alpha\pi^{\theta\phi}\left(\widehat{\varphi}_{t-1} + \theta\phi\widehat{\pi}_t\right) + \xi\widehat{\xi}_t \\
\pi_t \frac{\partial}{\partial \pi_t} H &= -(\theta-1)\rho\beta\alpha\pi^{\theta-1} X\left(\widehat{\rho}_{t-1} + (\theta-1)\widehat{\pi}_t + \widehat{X}_t\right) - \varphi\beta\alpha\theta\phi\pi^{\theta\phi} Z\left(\widehat{\varphi}_{t-1} + \theta\phi\widehat{\pi}_t + \widehat{Z}_t\right) \\
&\quad -\eta\alpha\theta\phi\Delta\pi^{\theta\phi}\left(\widehat{\eta}_t + \widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right) - \delta p^{\theta}\frac{\alpha\pi^{\theta-1}}{1-\alpha}\left(\widehat{\delta}_t + \theta\widehat{p}_t + (\theta-1)\widehat{\pi}_t\right) \\
p_t \frac{\partial}{\partial p_t} H &= -\xi(\theta\phi-\theta+1) X p^{1-\theta+\theta\phi}\left(\widehat{\xi}_t + \widehat{X}_t + (1-\theta+\theta\phi)\widehat{p}_t\right) \\
&\quad +\theta\phi\eta p^{-\theta\phi}(1-\alpha)\left(\widehat{\eta}_t - \theta\phi\widehat{p}_t\right) + \delta p\left(\widehat{\delta}_t + \widehat{p}_t\right)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{X}_t - \beta\alpha\pi^{\theta-1} E_t\left(\widehat{X}_{t+1} + (\theta-1)\widehat{\pi}_{t+1}\right) &= 0 \\
Z\widehat{Z}_t - c\widehat{c}_t - \beta\alpha\pi^{\theta\phi} Z E_t\left(\widehat{Z}_{t+1} + \theta\phi\widehat{\pi}_{t+1}\right) &= 0 \\
\widehat{Z}_t - \left((\theta\phi-\theta+1)\widehat{p}_t + \widehat{X}_t\right) &= 0 \\
\Delta\widehat{\Delta}_t - a\Delta\pi^{\theta\phi}\left(\widehat{\Delta}_{t-1} + \theta\phi\widehat{\pi}_t\right) - (1-\alpha) p^{-\theta\phi}\theta\phi\widehat{p}_t &= 0 \\
\widehat{p}_t - \frac{\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\widehat{\pi}_t &= 0 \\
-\widehat{c}_t + \widehat{u}_t + \widehat{\mu}_t - \widehat{\Delta}_t &= 0
\end{aligned}
$$