Analysis of Optimal Inflation under Discretion using a Projection Method*

Gary Anderson, Jinill Kim and Tack Yun
Federal Reserve Board

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Abstract

Since Barro and Gordon (1983), inflationary bias under optimal discretionary policy has been widely analyzed in linear quadrtic framework. In this paper, we use a prototypical New Keynesian model based on Calvo pricing in order to compare the discretion bias under its full nonlinear solution based on a projection method with that of the corresponding linear quadratic approximation. [conclusion sentence to be added]

1 Introduction

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2 Economic Structure and Optimal Policy

2.1 Economic Structure

2.1.1 Households

The preference at period 0 of the representative household is represented by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{H_t^{1+\chi}}{1+\chi} \right], \tag{1}$$

where C_t denotes consumption at period t, H_t denotes hours worked, $0 < \beta < 1$ denotes discount factor. Households purchase differentiated goods in retail market and combine them into a single composite good using a Dixit-Stiglitz (1977) aggregator, while utils of households and government activities depend only upon the amount of the composite good. The demand curve for each good z can be derived from the following cost-minimization:

$$\min \int_0^1 P_t(z) C_t(z) \, dz \;\; s.t. \;\; C_t = \left(\int_0^1 C_t(z)^{\frac{\epsilon-1}{\epsilon}} dz \right)^{\frac{\epsilon}{\epsilon-1}}, \;\; \epsilon > 1, \tag{2}$$

where $P_t(z)$ represents the nominal price at period t of good i and $C_t(z)$ is the demand of good z. The first-order condition of this cost-minimization problem then gives the demand curve of firm i:

$$C_t(z) = \left(\frac{P_t(z)}{P_t} \right)^{-\epsilon} C_t, \tag{3}$$

where the aggregate price level P_t is defined to be

$$P_t = \left(\int_0^1 P_t^{1-\epsilon}(z) \, dz \right)^{\frac{1}{1-\epsilon}}. \tag{4}$$

The household's budget constraint at period t is given by

$$C_t + E_t \left[Q_{t,t+1} \frac{B_{t+1}}{P_{t+1}} \right] = \frac{B_t}{P_t} + (1+\eta) \frac{W_t}{P_t} H_t - T_t, \tag{5}$$

where B_{t+1} is the nominal payoff at period $t+1$ of the bond-portfolio held at period t, W_t is nominal wage, and Φ_t is the real dividend income, T_t is the real lump-sum tax, and η denotes a constant rate of employment subsidy that is proportional to labor income. In addition, $Q_{t,t+1}$ is the stochastic discount factor used for computing the real value at period t of one unit of consumption goods at period $t+1$. Hence, if R_t represents the risk-free (gross) nominal rate of interest at period t, the absence of arbitrage at an equilibrium leads to

$$E_t\left[Q_{t,t+1}\frac{1}{P_{t+1}}\right]=\frac{1}{R_tP_t}. \tag{6}$$

The representative household maximizes (1) subject to flow budget constrains (5) in each period $t=0,\,1,\,\cdots,\,\infty$. The first-order conditions for the household's optimization are given by

$$C_t^{\sigma}H_t^{\chi}=(1+\eta)\,\frac{W_t}{P_t}, \tag{7}$$

$$Q_{t,t+1}=\beta\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}. \tag{8}$$

The substitution of (8) into (6) then gives the following Euler equation:

$$\beta R_tE_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\frac{P_t}{P_{t+1}}\right]=1. \tag{9}$$

2.1.2 Firms

Each firm z produces a differentiated good z using a constant returns to scale production function of the form:

$$Y_t\,(z)=A_tH_t\,(z)\,, \tag{10}$$

where $Y_t(z)$ denotes the output at period t of firm z, $H_t(z)$ denotes the hours hired by the firm.

Further, firms set prices as in the sticky price model of Calvo (1983). Specifically, each period a fraction of firms $(1-\alpha)$ are allowed to change prices, whereas the other fraction, α, do not change. Let P_t^* be the new price set by firms resetting prices in period t. Then, firms resetting prices in period t chooses a new optimal price in order to maximize the following expected discounted sum of profits obtained in current and future periods:

$$\sum_{k=0}^{\infty}\alpha^kE_t\left[Q_{t,t+k}\left((1+\psi)\,\frac{P_t^*}{P_{t+k}}-\frac{W_{t+k}}{A_{t+k}P_{t+k}}\right)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon}Y_{t+k}\right]. \tag{11}$$

Differentiating (11) with respect to P_t^* gives the first-order condition for P_t^*:

$$\sum_{k=0}^{\infty} \alpha^k E_t \left[Q_{t,t+k} \left((1+\psi) \frac{P_t^*}{P_{t+k}} - \frac{\epsilon}{\epsilon - 1} \frac{W_{t+k}}{A_{t+k} P_{t+k}} \right) P_{t+k}^{\epsilon} Y_{t+k} \right] = 0. \tag{12}$$

Besides, the Calvo type staggering leads equation (4) to become

$$P_t = \left[(1-\alpha) \left(P_t^* \right)^{1-\epsilon} + \alpha P_{t-1}^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}}. \tag{13}$$

Next, we will show that the profit maximization condition (12) can be rewritten in a recursive way. In order to do this, note that substituting (8) into (12) and then rearranging, we have

$$(1+\psi) \sum_{k=0}^{\infty} (\alpha\beta)^k E_t \left[\left(\frac{Y_{t+k}}{C_{t+k}^{\sigma}} \right) \left(\frac{P_{t+k}}{P_t} \right)^{\epsilon - 1} \right] \frac{P_t^*}{P_t} \tag{14}$$

$$= \frac{\epsilon}{\epsilon - 1} \sum_{k=0}^{\infty} (\alpha\beta)^k E_t \left[\left(\frac{W_{t+k} Y_{t+k}}{A_{t+k} P_{t+k} C_{t+k}^{\sigma}} \right) \left(\frac{P_{t+k}}{P_t} \right)^{\epsilon} \right]. \tag{15}$$

It is now useful to define two variables, F_t and V_t, as follows.

$$F_t = (1+\psi) \sum_{k=0}^{\infty} (\alpha\beta)^k E_t \left[\left(\frac{Y_{t+k}}{C_{t+k}^{\sigma}} \right) \left(\frac{P_{t+k}}{P_t} \right)^{\epsilon - 1} \right], \tag{16}$$

$$S_t = \frac{\epsilon}{\epsilon - 1} \sum_{k=0}^{\infty} (\alpha\beta)^k E_t \left[\left(\frac{W_{t+k} Y_{t+k}}{A_{t+k} P_{t+k} C_{t+k}^{\sigma}} \right) \left(\frac{P_{t+k}}{P_t} \right)^{\epsilon} \right]. \tag{17}$$

We then have the following recursive representations of the two variables F_t and S_t, defined in (16), respectively.

$$F_t = (1+\psi) \frac{Y_t}{C_t^{\sigma}} + \alpha\beta E_t \left[\Pi_{t+1}^{\epsilon-1} F_{t+1} \right], \tag{18}$$

$$S_t = \frac{\epsilon}{\epsilon - 1} \left(\frac{W_t}{P_t A_t} \right) \left(\frac{Y_t}{C_t^{\sigma}} \right) + \alpha\beta E_t \left[\Pi_{t+1}^{\epsilon} S_{t+1} \right], \tag{19}$$

with two terminal conditions of the forms,

$$\lim_{T\to\infty} (\alpha\beta)^T E_t \left[\left(\prod_{k=1}^{T} {\Pi_{t+k}}^{\epsilon-1} \right) F_{t+T} \right] = 0, \quad \lim_{T\to\infty} (\alpha\beta)^T E_t \left[\left(\prod_{k=1}^{T} \Pi_{t+k}^{\epsilon} \right) S_{t+T} \right] = 0.$$

Here, Π_t $(= P_t/P_{t-1})$ is the ratio of the price level at period t to the price level at period $t-1$. We now substitute the definitions of F_t and S_t specified in (16) into the profit maximization condition (14) to yield

$$\frac{P_t^*}{P_t} = \frac{S_t}{F_t}. \tag{20}$$

In addition, substituting equation (20) into (13) leads to

$$1 = (1-\alpha)\left(\frac{S_t}{F_t}\right)^{1-\epsilon} + \alpha\Pi_t^{\epsilon-1}. \tag{21}$$

As a result, we have expressed the profit maximization condition (14) and the price level definition (13) in terms of F_t and S_t with their intertemporal evolution equations (18) and (19).

2.1.3 Social Resource Constraint

In any sticky price model with staggered price setting, relative prices can differ across firms. Besides, if firms have different relative prices, there are distortions that create an wedge between the aggregate output measured in terms of production factor inputs and aggregate demand measured in terms of the composite goods. In order to see such relative price distortions, individual outputs are linearly aggregated:

$$A_t H_t = Y_t \int_0^1 \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon} dz,$$

where $H_t = \int_0^1 H_t(z)dz$. Hence, one can define a measure of relative price distortions as

$$\Delta_t = \int_0^1 \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon} dz. \tag{22}$$

As a result, the aggregate production function can be written as follows:

$$Y_t = \frac{A_t}{\Delta_t} H_t. \tag{23}$$

In order to obtain a law of motion for the measure of relative price distortion described above, note that the Calvo type staggering allows one to rewrite the measure of relative price distortions specified in equation (22) as

$$\Delta_t = (1-\alpha)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon} + \alpha\Pi_t^{\epsilon}\Delta_{t-1}. \tag{24}$$

Hence, substituting (13) into (24), one can derive an expression of how the measure of relative price distortions evolves over time:

$$\Delta_t = (1-\alpha)\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{\epsilon}{\epsilon-1}} + \alpha\Pi_t^{\epsilon}\Delta_{t-1}. \tag{25}$$

Finally, the aggregate market clearing condition is given by

$$Y_t = C_t + G_t, \tag{26}$$

where G_t denotes government expenditures at period t. The social resource constraint in period t is therefore given by

$$\frac{A_t}{\Delta_t}H_t = C_t + G_t. \tag{27}$$

2.2 Optimal Policy under Commitment

2.3 Optimal Policy under Discretion

Having described equilibrium conditions, we consider optimal policy problem under discretion for the economy discussed in the previous section.

Before proceeding, it is worthwhile to discuss implementation conditions of of the optimal policy problem, which constraint the feasible allocations of the social planner. First, the household budget constraint is not included as a constraint for the optimal policy problem because of the lump-sum tax. Second, the size of employment subsidy rate determines whether the profit maximization condition is included as an implementation condition in the optimal policy problem.

In order to get some insights about the role of employment subsidy rate, we describe equilibrium condition of the flexible price model and then compare them with those of the first-best equilibrium. Since $\alpha = 0$ corresponds to the flexible price model, it follows from (12) that the profit maximization condition for the flexible price model turns out to be

$$\frac{\bar{W}_t}{\bar{P}_t} = (1+\psi)\left(1-\epsilon^{-1}\right)A_t. \tag{28}$$

where $\bar{W}_t$ and $\bar{P}_t$ are the nominal wage rate and the price level in the flexible price model. Combining (7) with (28), we can see that the relationship between MRS and MPL in the flexible price model is given by

$$\bar{C}_t^{\sigma}\bar{H}_t^{\chi} = \tau(1+\psi)A_t, \tag{29}$$

where $\bar{C}_t$ and $\bar{H}_t$ denote consumption and labor in the flexible price model. Here, the parameter τ is defined as

$$\tau = \left(1 - \epsilon^{-1}\right)(1 + \eta). \tag{30}$$

It is clear from (30) that employment subsidy rate determines the value of τ, given a value of demand elasticity. In particular, we can see from (29) that when we set $\tau(1 + \psi) = 1$, the flexible price model can achieve the level of output, which would be attained at the perfectly competitive equilibrium.

We now characterize the optimal policy problem under discretion. The government at period 0 chooses a set of decision rules for $\{ C_t, H_t, F_t, S_t, \Pi_t, \Delta_t \}_{t=0}^{\infty}$ in order to maximize

$$V(\Delta_{t-1}) = \max \left\{ \frac{C_t^{1-\sigma} - 1}{1 - \sigma} - \frac{H_t^{1+\chi}}{1 + \chi} + \beta E_t \left[V(\Delta_t) \right] \right\}, \tag{31}$$

subject to the following equilibrium conditions in each period $t = 0, 1, \cdots, \infty$:

$$C_t + G_t \leq \frac{A_t}{\Delta_t} H_t, \tag{32}$$

$$F_t = (1 + \psi) \frac{A_t H_t}{\Delta_t C_t^{\sigma}} + \alpha\beta E_t \left[\Pi_{t+1}^{\epsilon-1} F_{t+1} \right], \tag{33}$$

$$S_t = \frac{H_t^{1+\chi}}{\tau \Delta_t} + \alpha\beta E_t \left[\Pi_{t+1}^{\epsilon} S_{t+1} \right], \tag{34}$$

$$\Delta_t = (1 - \alpha) \left(\frac{1 - \alpha \Pi_t^{\epsilon-1}}{1 - \alpha} \right)^{\frac{\epsilon}{\epsilon-1}} + \alpha \Pi_t^{\epsilon} \Delta_{t-1}, \tag{35}$$

$$S_t = F_t^{\frac{1}{1-\epsilon}} \left(\frac{1 - \alpha \Pi_t^{\epsilon-1}}{1 - \alpha} \right). \tag{36}$$

Having described the optimal policy problem under discretion, the first-order conditions can be summarized as follows:

$$1 + \sigma(1 + \psi) \frac{A_t H_t}{\Delta_t C_t} \phi_{2t} = \phi_{1t} C_t^{\sigma}, \tag{37}$$

$$\Delta_t C_t^{\sigma} H_t^{\chi} + (1 + \psi) A_t \phi_{2t} + \frac{1 + \chi}{\tau} \phi_{3t} C_t^{\sigma} H_t^{\chi} = \phi_{1t} A_t C_t^{\sigma}, \tag{38}$$

$$\phi_{2t} = \phi_{5t}^{\frac{1}{1-\epsilon}} \left(\frac{1 - \alpha \Pi_t^{\epsilon-1}}{1 - \alpha} \right), \tag{39}$$

$$\phi_{3t} = -\phi_{5t}, \tag{40}$$

$$\epsilon \left(\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha} \right)^{\frac{1}{\epsilon-1}} - \Pi_t \Delta_{t-1} \right) \phi_{4t} = \frac{1}{1-\alpha} \left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha} \right)^{\frac{-\epsilon}{\epsilon-1}} F_t \phi_{5t}, \tag{41}$$

$$(1+\psi) \frac{A_t H_t}{\Delta_t^2 C_t^\sigma} \phi_{2t} + \phi_{3t} \frac{H_t^{1+\chi}}{\tau \Delta_t^2} - \phi_{4t} + \beta E_t \left[V'(\Delta_t) \right] = \phi_{1t} \frac{A_t H_t}{\Delta_t^2} + \alpha\beta E_t \left[\phi_{2t} L'(\Delta_t) + \phi_{3t} M'(\Delta_t) \right], \tag{42}$$

where ϕ_{1t}, ϕ_{2t}, ϕ_{3t}, ϕ_{4t}, and ϕ_{5t} are Lagrange multipliers for (32), (33), (34), (35), and (36) respectively. In addition, auxiliary functions $L(\Delta_t)$ and $M(\Delta_t)$ are defined as

$$L(\Delta_t) = \Pi_{t+1}^{\epsilon-1} F_{t+1}, \tag{43}$$
$$M(\Delta_t) = \Pi_{t+1}^{\epsilon} S_{t+1}. \tag{44}$$

3 Conclusion

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