Choosing Stress Scenarios for Systemic Risk Through Dimension Reduction

Matthew Pritsker*

August 26, 2016

Abstract

Regulatory stress-testing is an important tool for ensuring banking system health in many countries around the world. Current methodologies ensure banks are well capitalized against the scenarios in the test, but it is unclear how resilient banks will be to other plausible scenarios. This paper proposes a new methodology for choosing scenarios that uses a measure of systemic risk with Correlation Pursuit variable selection, and Sliced Inverse Regression factor analysis, to select variables and create factors based on their ability to explain variation in the systemic risk measure. The main result is under appropriate regularity conditions, when the banking system is well capitalized against stress-scenarios based on movements in the factors, then an approximation of systemic risk is low, i.e. the banking system will be well capitalized against the other plausible scenarios that could affect it with high probability. The paper also shows there are circumstances when several scenarios may be required to achieve systemic risk objectives. The methodology should be useful for regulatory stress-testing of banks. Although not done in this paper, the methodology can potentially be adapted for stress-testing of other financial firms including insurance companies and central counterparties.

Keywords: Stress Testing, Financial Stability, Banking
JEL Classification: G21, G28

*The authors is a member of the Risk and Policy Analysis Unit at the Federal Reserve Bank of Boston. Matt Pritsker's contact information is as follows: ph: (617) 973-3191, email: matthew.pritsker at bos.frb.org. The views expressed in this paper are those of the author but not necessarily those of the Federal Reserve Bank of Boston or other parts of the Federal Reserve System. The author thanks Maggie Li for excellent research assistance, and John Ammer, Yasushi Hamao, Christian Julliard, and Chen Zhou for valuable comments. The author also thanks seminar participants at the Federal Reserve Bank of New York, the Federal Reserve Bank of Chicago, the Federal Reserve Bank of Boston, the Bank of England, and the University of Sussex, as well as conference participants at the London School of Economics, the Midwest Finance Association, the Conference on Computational and Financial Econometrics, and the Paris Financial Management Association Conference. The author also thanks Jonathan Wright and Jens Christensen for invaluable help on term-structure modeling aspects of this work. All errors are the responsibility of the author.

1 Introduction

The great recession that accompanied the Financial Crisis of 2007-2009 underlined the role of the financial sector in real economic activity, and it highlighted the importance of controlling systemic risk, the risk that many banks and financial institutions become financially distressed at the same time and thus become impaired in their ability to provide financial intermediation for the real sector. As part of the US regulatory response to the financial crisis, stress-tests were conducted that assessed the ability of individual banks to maintain sufficient capital (measured by net worth over risk-weighted assets) to perform as financial intermediaries during a small number of adverse hypothetical macroeconomic and financial scenarios. Remedial action was required for banks that had capital shortfalls. Since the first U.S. stress test in 2009, regulatory stress testing has become the primary tool to assess the capital adequacy of many banks in the US as well as systemically important financial institutions that are not banks (nonbank-SIFIs). [1] The use of regulatory stress testing has also expanded in many other countries. The set of risks covered in the tests include the risks present in banks loan books (aka their banking books). Some stress-tests also cover the risks in large banks trading books. Separate stress scenarios are often used to cover the risks in the banking and trading books.[2]

An important goal of regulatory stress-testing is to ensure that systemic risk is low. This requires the financial system to be well capitalized against the small number of scenarios in the stress-test *and* against the much broader set of likely scenarios that the financial system may face. It is uncertain whether the current methodology for regulatory stress-testing creation can achieve this goal. An important shortcoming of the current approach is banks exposures are not formally used when creating regulatory stress-scenarios. As a consequence, regulatory scenarios may stress variables banks are not exposed to while failing to stress important variables that banks are exposed to, the missing variables problem; or they may stress variables in directions in which banks are hedged or even make profits, while missing directions in which banks take risk, the missing directions problem.

As an example of the missing variable problem, regulatory scenarios for banks loan books are often formulated on the basis of a small number of macroeconomic and financial variables that only weakly explain banks P&L [Guerrieri and Welch (2012), Bolotnyy et. al (2015)] and whose movements often fail to forecast financial crises since such crises often occur before the macroeconomy turns down [(Borio et al (2012), Alfaro and Drehmann (2009)].[3] These finding suggest that the small number of variables used for scenario formulation in banks loan books may

[1]For simplicity all of the institutions to which regulatory stress-testing is applied will be referred to as banks.

[2]For details see Federal Reserve Board (2014), European Banking Authority (2014), and Bank of England (2014).

[3]The Federal Reserve's 2015 CCAR stress scenario for the 6 largest banks trading book positions stresses tens of thousand of variables. But, only 16 macro-financial variables were utilized for U.S. banking book exposures, and only 12 macro-financial variables were used for banking book exposures outside of the United States [Federal Reserve Board (2014)].

be inadequate, and additional variables may be needed. As an example of the missing directions problem, capital adequacy for large banks trading book positions is often assessed on the basis of a few regulatory scenarios that specify the movements of many variables for EBA/UK regulatory stress tests, and for a very large number of variables (10,000+) for US regulatory stress tests. In such a high dimensional setting, unless the scenario is chosen very carefully, important directions of risk-taking may be missed.

To overcome the missing variable and missing direction problems, U.S. regulations also require each bank to construct a scenario that stresses its most important vulnerabilities. However, because this approach is based on bank-specfic vulnerabilities, it does not ensure banks are well capitalized against common vulnerabilities, and hence cannot ensure that systemic risk is low.

This paper introduces a new methodology for creating regulatory stress scenarios; it chooses the variables to use in a stress-scenario, and the directions and amounts the variables need to be moved so that if banks are well capitalized against the scenario, then under some conditions (discussed below) systemic risk will be low as measured by an approximate systemic risk objective. The new methodology relies on dimension reduction techniques. It is premised on the idea although the value of banks positions are driven by many variables, these variables are driven by a smaller number of latent economic factors. The factors are assumed to be the most important determinant of banks risk at a portfolio level. At an individual bank level, its risk is not determined by all of the factors, but only those it has not hedged against. On an economy wide level, banks' common exposures to unhedged factors can cause them to experiene joint distress and are hence a source of systemic risk. This reasoning suggests that for stress tests to keep systemic risk low, stress testing policy needs to ensure banks remain well capitalized against movements in the most important factors that explain their joint distress. The methodology in this paper pursues this idea by illustrating how to use supervisory information and statistical techniques to identify banks exposures to the unhedged factors, and to then designs stress scenarios that keep systemic risk low.

To identify the factors, the variables that affect banks portfolios are simulated, then using supervisory information on risk exposures they are mapped into changes in the value of banks' portfolios, and into a measure of banks' joint financial distress. Then a principal components factor analysis using Sliced Inverse Regression (SIR) [Li, (1991)], is conducted that identifies an orthogonalized set of risk factors based on their ability to explain banks' joint distress.[4] Because banks' joint distress in the simulations can only depend on economic factors that banks have not hedged against, the factors identified by SIR will only depend on those factors.[5] As explained below, SIR is more accurate if it does not rely on too many variables to create the factors. To choose the variables that should be used for SIR, it is assumed that the information on banks joint

[4]Technically, the factors are chosen based on their correlation with an optimally chosen tranformation of banks joint distress.

[5]Formally, the factors identified by SIR will be spanned by banks unhedged factors, and in some conditions both sets of factors will span the same space.

distress that is contained in all of the variables, is also contained in a smaller subset of variables that are best for creating factors that explain banks' joint distress. The best variables for creating the factors are chosen using Correlation Pursuit (COP) [Zhong et al. (2012)] variable selection, which is a method for choosing the best variables to use with Sliced Inverse Regression factor analysis.

To construct a stress scenario using the factors, the factors are shocked by chosen amounts, and then all variables are set to their conditional expected values given the factor shocks. By relying on systemic risk factors to determine how variables are shocked, the scenario by construction moves variables in stressful directions from a systemic risk perspective. The shock sizes and directions in the stress-scenario are chosen so that if banks are well capitalized against the scenario, then (if feasible) regulators systemic risk objective will be achieved. Because the approach in this paper chooses one or a small number of scenarios in order to satisfy a systemic risk objective, I refer to the approach in the paper as the Systemically Chosen Scenario Approach, or SCSA.

The SIR and closely related COP method are both based on factor analysis. The purpose of classical factor analysis is to summarize the information about the joint behavior of a large number of variables by a much smaller number of factors that are linear combinations of the variables. A disadvantage of classical factor analysis is that the identified factors are chosen to explain variation in right hand side variables, but not in the dependent variable of interest. Supervised factor analysis based on inverse regression methods such as COP, SIR, and Partial Least Squares (PLS) differ from classical factor analysis in that they create factors based on their ability to explain a dependent variable. SIR and COP differ from PLS because the latter typically requires the left hand side variable to be a linear combination of the factors, while COP and SIR allow the left hand side to be a nonlinear function of the factors. Because joint distress caused by asset bubbles bursting, or asset fire sales may display nonlinear dynamics, an advantage of using SIR and COP to identify the factors is that these methodologies may still be able to uncover the factor structure even when there are nonlinearities.[6]

This paper contributes to both the practice and theory of regulatory stress-testing.[7] Current regulatory practice uses different approaches to specify scenarios for banks' loan and trading books. Loan book stresses are usually based on macroeconomic models, and consistent with those models, utilize a relatively small number of macroeconomic and financial variables that differ from the variables banks use to model their loan book risks. The disconnect between the variables used by banks and regulators is a likely contributor to the missing variable problem in loan book stress tests. By contrast, in the trading book, regulators specify stress scenarios using a very large

[6] Gibson and Pritsker (2001), and Giglio et al (2012) use PLS in the context of dimension reduction for risk measurement. A novel aspect of Giglio et al is it uses quantile regression with partial least squares, and hence is an early attempt to estimate a nonlinear factor structure using PLS. In addition, Kelly and Pruitt (2013) use PLS for stock market prediction, and Groen and Kapetanios discuss its use for macroeconomic forecasting.

[7] Seminal contributions to systemic risk stress-testing include the macro-financial models of the Bank of Austria [Boss et. al. (2006)] and the Bank of England [Alessandri et. al. (2009)]. Bookstaber et. al. (2013) and Schuermann (2013) provide critical reviews of the literature and regulatory practice.

number of variables based on the variables banks use to model their risks. This approach to trading book scenarios mitigates the missing variables problem, but exacerbates difficulties in choosing the direction and magnitude by which trading book variables should be shocked in a stress scenario. The methods advocated in this paper have potential to improve stress testing for both sets of books. Loan book variable selection is improved by choosing from among the variables that banks use to model loan book risks, and then selecting the loan book variables that are most useful for modeling systemic risk. To address the missing directions problem in both the loan and trading books, the paper uses dimension reduction to identify the direction and magnitude of banks vulnerabilities to a smaller number of identified systemic risk factors, and chooses stress scenarios to achieve regulatory objectives based on this information.

This paper is related to a growing literature on systemic risk measurement [Bisias et. al.(2012)]. The methodology in this paper does not address all types of systemic risk, but it is related to banks becoming financially distressed by becoming undercapitalized together. The approach to derive stress scenarios in this paper requires a measure of systemic risk based on banks' joint distress. Measures that could be used in this paper include aggregated across banks versions of Systemic Expected Shortfall (SES) [Acharya et al, 2010], the Distressed Insurance Premium (DIP) [Huang et al (2009)], or System Assets in Distress (SAD) [Pritsker, (2014)].[8] Although not pursued in this paper, it should be possible to alternatively measure systemic risk based on commonality in financial institutions' liquidity mismatches, and then identify factors that explain vulnerability to this commonality.[9]

This paper is closely related to a few papers in the stress-testing literature. Pritsker (2014) proposes a methodology to achieve systemic risk objectives at lowest capital cost in a framework that uses a very large number of stress scenarios. This paper proposes a complementary approach that attempts to accomplish a similar objective by utilizing a smaller number of stress scenarios that are very carefully chosen. Reliance on a few scenarios is more consistent with regulatory practice, and may be more practical to implement if computing many scenarios is too costly. Another complementary paper is Kapinos and Mitnik (2014). They use LASSO regression and factor analysis to improve on variable selection and P & L modeling as part of stress-testing. Their application differs from this paper in several important ways, most importantly they do not choose variables or factors based on their ability to explain systemic risk, and their analysis does not have a systemic risk objective.[10]

[8] System Assets in Distress is aggregated across banks.

[9] See BIS 2013 for a discussion of liquidity stress testing.

[10] Kapinos and Mitnik (KM) (2014) choose which macro-variables and transforms of macro-variables explain components of banking-book P&L using LASSO regression; they then apply principal components to the chosen variables and use the components to model how banks respond to changes in macro variables as part of stress-testing. Although this paper and KM are similar in choosing variables and factors, their focuses are different. KM chooses variables to estimate models that relate the variables to banks P&L. By contrast, this paper takes the relationships between the variables and banks P&L as given; it then uses these relationships to chooses variables and factors to create stress-scenarios to attain systemic risk objectives.

One criticism of regulatory stress tests is they do not have a clearly defined objective. That is, it is unclear what objective function justifies their choice of the scenarios they focus on given the many scenarios they could choose. This paper addresses the criticism by choosing regulatory stress scenarios in order to keep systemic risk below a target level. Some papers on stress scenario selection pursue a related approach for firm stress-testing by choosing a stress scenario that generates the largest losses for the firm from among a set of possible scenarios. If the firm is well capitalized against the worst-case scenario, it is well capitalized against all scenarios in the set with a probability exceeding the probability of the set [Breuer et al (2009), Flood and Korenko (2013)].[11] An advantage of this approach is if the objective for a firm is to achieve capital adequacy with a given probability, then if the set has that probability, assuring capital adequacy against the set meets and exceeds the objective. A disadvantage of this worst case approach is the objective is often exceeded by large amounts requiring a firm to hold far more capital than is needed. The method in this paper achieves the objective but is less conservative because it is designed to just satisfy the objective, not exceed it. An additional contribution of the approach in this paper is the objective is not for one firm, but is instead based on a systemic risk objective function.

The rest of the paper proceeds in four sections. Section 2 illustrates weaknesses in regulatory stress-testing that the SCSA approach is designed to address. Section 3 explains the SCSA methodology. Section 4 illustrates the SCSA methodology for some stylized trading portfolios that are exposed to interest-rate and stock market risk. A final section concludes.

2 Weaknesses in current regulatory stress testing.

This section provides more detail on why current approaches to regulatory stress-testing may fail to achieve systemic risk objectives. The missing variables problem is straightforward. Therefore, the exposition below focuses on missing directions, and the requirement that banks must also create their own bank specific scenarios.

Missing Stress-Test Directions

This section illustrates that if stress-scenarios are created without accounting for the directions in which banks take risk, the stress-testing exercise may fail to require banks to hold capital against potentially very significant risk exposures. If, in addition, the direction of missed risk is common across banks, the fact that the stress-scenario missed the risk-taking can itself be a source of systemic risk. These points are simple to illustrate in a univariate setting. For example, suppose all banks write call options on the S&P 500 stock index, and that is the only asset position that they have. In this setting banks are only subject to the risk that the stock market moves in an upward direction.

[11] An alternative approach chooses the most likely scenario that generates losses of a given amount [Glasserman, Kang, and Kang (2015)].

Because all banks have a common exposure, the chance of large upward movements represents a systemic risk. If there is only one regulatory scenario and it posits that stock prices move down, it would have chosen the wrong stress direction, and not required banks to hold additional capital despite the systemic risk of their positions.

The example of missing directions when there is only one asset is contrived and unrealistic. But, analogous problems exist in higher dimensional settings. To illustrate, I assume the banking system consists of M banks ($m = 1, \ldots M$) of equal asset size that invest in a riskfree asset with gross-return normalized to 1 and N risky assets whose net returns are denoted by the $N \times 1$ vector R. The amount each bank m invests in risky assets (its exposure) is denoted by the $N \times 1$ vector δ_m, and hence the gain or loss in value of each banks portfolio at the end of the stress horizon is $\delta_m' R$.

I assume the time horizon for the stress-test is one period. There is a single stress scenario denoted by the $N \times 1$ vector $\tilde{R}$ that specifies the returns of each asset over the horizon. By elementary linear algebra, $\tilde{R}$ has an $N-1$ dimensional null space with a matrix of basis vectors v $(= [v_1, \ldots v_{N-1}])$.[12] Furthermore, each banks exposure vector can be represented as a linear combination of $\tilde{R}$ and the vectors in its null space. For example, δ_1 can be written as

$$\begin{aligned}\delta_1 &= \theta_1 \tilde{R} + \sum_{k=1}^{N-1} \gamma_k v_k. \\ &= \theta_1 \tilde{R} + e_1,\end{aligned}$$

where $e_1 = \sum_{k=1}^{N-1} \gamma_k v_k$ represents the part of δ_1 in $\tilde{R}$'s null space.

The value of bank 1's assets at the end of the stress scenario is $\delta_1' \tilde{R}$. Using the decomposition, these losses are given by $\theta_1 \tilde{R}' \tilde{R}$. If θ_1 is negative, then bank 1 experiences losses in the stress scenario. Note that bank 1's losses in the stress scenario do not involve e_1 since that part of bank 1's exposures are in the null-space of the stress scenario vector $\tilde{R}$. Put differently, the exposure component e_1 is a missing risk direction for bank 1 since it represents bank 1's risk-taking in directions other than the stress-scenario.

To illustrate the potential consequences of the risks the stress-scenario misses for bank 1, suppose the missing risks of banks $m = 2, \ldots, M$ are in the same missing risk directions as bank 1, but scaled up, so their exposure vectors are given by:

$$\delta_m = \theta_1 \tilde{R} + (10^m) * e_1 \tag{1}$$

If each banks capital adequacy was judged on the basis of the single stress scenario $\tilde{R}$, then

[12] Vectors in $\tilde{R}$'s null space are orthogonal to $\tilde{R}$

since all of the banks will lose the same amounts in the stress scenario, they will be judged equally despite the fact their e_1 risks differ by arbitrarily large amounts as $M \to \infty$. This failure to distinguish among huge differences in banks risks represents a microprudential failure of stress-testing. Additionally, the stress-test fails to detect systemic risk. Recall that systemic risk is the probability of events that banks experience low capital together. Without loss of generality in this example, assume the bulk of bank's risk taking is captured in the second-term of the decomposition for their exposures. Because in this example, the second terms are all proportional to e_1, the commonality of their exposures are an important source of systemic risk that is missed by the stress test.

To put the results from this stress-test in perspective, recall that in CCAR regulatory stress-testing a handful of stress-scenarios are used in the trading book, but tens of thousands of variables are shocked. In this setting, the dimensionality of the risks missed by any single stress-scenario can be very high if banks exposures are not accounted for in choosing the scenario.[13]

Bank Specific Stress Tests

As noted in the introduction, a possible method to address the missing stress direction problem is to allow each bank to conduct its own tailored bank-specific stress-test based on its own exposures. Regulatory guidance encourages banks to significantly stress all of its material exposures, which can cause overconservatism. To illustrate, assume for simplicity that the return on each asset is independent and normally distributed with a mean of 0 and a variance of 1, and that each element of δ_m is 1. Under this circumstance each bank's portfolio return is distributed $\mathcal{N}(0, N)$, where N is the number of assets in the bank's portfolio. Assume a tailored stress scenario moves the return on each asset by 2 standard deviations in a direction that is unfavorable to the bank. The loss to the bank in the stress-scenario is $2N$.

The question is whether this stress scenario is too severe. To see that it probably is, note that the probability the bank experiences losses that are the same or exceed losses in the stress scenario are $\Phi(-2\sqrt{N})$, where $\Phi(.)$ is the CDF of the standard normal distribution. If we assume banks are exposed to 9 independent sources of risk, this is $\Phi(-6)$ or about 10^{-9}. This means requiring banks to stress themselves in this way against 9 independent sources of risk in the example would require the bank to hold enough capital so that the probability that losses exceed capital is one in a billion. Or if the stress horizon is a year, a stress test of this form would require banks to hold so much capital that losses would exceed capital only about once every billion years.[14] While this example is a very simple illustration of overconservatism, it is also trivial to illustrate it in more general settings.

[13] An additional concern is if banks can anticipate the specification of the scenario, they are incented to load up on risk exposures in its null space to avoid regulatory capital charges. This is a potential concern because results in Glasserman and Tangirala (2015) suggest that stress-test results and hence perhaps scenarios have been predictable from year-to-year.

[14] As N, the number of independent risk sources increases, this approach becomes even more conservative.

The main lesson is if banks stress every exposure substantially, then it is quite possible that the capital called for is excessive by any reasonable standard. For this reason, it is unlikely that banks, when choosing their own scenarios, would choose scenarios that would require them to hold this amount of capital. But, it then becomes unclear what level of capital adequacy is achieved if banks each choose their own scenario in a way to avoid excessive capital requirements. A separate issue is if banks choose their own scenarios, it is very possible that they will not focus on scenarios where banks have common exposures. Hence, the individual bank scenarios will not ensure that the banking system as a whole is well capitalized against systemic risk.

In sum, this section has illustrated in detail why the current approach to regulatory stress-testing may miss important stress-directions, and why bank specific scenarios are not a solution to the shortcomings of regulatory scenarios.

3 The SCSA methodology

As noted above, current regulatory stress-scenario selection has four main shortcomings:

1. Banks exposures are not formally used in scenario selection.
2. The wrong variables may be utilized in scenario formulation.
3. The variables may be stressed in the wrong directions.
4. The stress-scenarios are not explicitly designed to achieve a systemic risk objective.

The SCSA methodology helps to address all four problems. It is based on three principles:

Principle 1 *The value of banks positions (assets, liabilities, and derivative securities) depends on a large number of variables including interest rates, FX rates, stock returns, implied volatilities, etc. The large number of variables in turn depend on a much smaller number of underlying potentially latent economic factors.*

Principle 2 *Systemic impairment is the event that too many banks become financially distressed during the same period of time. Systemic risk is the probability that systemic impairment occurs. One way systemic impairment can occur is if banks are exposed to common economic factors and those factors move in ways that are unfavorable at the same time.*

Principle 3 *Regulatory stress scenarios should be chosen so that if the banking system is well capitalized against the stress-scenarios, then systemic risk is low with high probability.*

Principles 1 and 2 suggest that systemic impairment can occur if the common factors that banks are exposed to move against them by enough to cause financial distress at the same time. This suggests, by principle 3, that stress-scenarios should be designed based on movements in the factors.

The remaining analysis is divided in two parts: section 3.1, provides economic and statistical theory to identify the factors; section 3.2 provides details on how to use the identified factors to choose stress scenarios, and if feasible to ensure systemic risk is low.

3.1 Identifying Systemic-Risk Factors using SCSA

The goal of this section is to illustrate how to estimate common factors that affect the value of banks portfolios based on the set of tangible variables that banks included in a stress test use to model their assets, plus additional tangible variables that regulators may use to use value banks assets.[15] The total set of variables is denoted by the $1 \times N$ vector X.

By principle 1, it is assumed that X is driven by a factor structure,

Assumption 1

$$X = G(F_A, F_B, U) \tag{2}$$

where F_A *and* F_B *are* $1 \times K_A$ *and* $1 \times K_B$ *vectors of potentially latent economic factors, and* U *is a* $1 \times N$ *vector of idiosyncratic risks that are independent of the factors.*

In this setting, the factors F_B denote factors that all banks hedge against, while F_A represent factors that banks remain exposed to. To avoid difficulties with missing variables, it is important not to create a stress-test based on F_B: since all banks are hedged against F_B nothing would be learned about banks systemic risk by stressing F_B. Conversely, F_A represents common factors that banks remain exposed to; by principle 2 stress-tests should be based on those factors. Each bank i's remaining idiosyncratic risk after hedging is represented by ϵ_i. Mathematically, this implies $V_i(X)$, the value of bank i's portfolio as a function of X, reduces to a function of F_A and ϵ_i,

$$V_i(X) = V_i(F_A, \epsilon_i), \tag{3}$$

which stacked across banks has form

$$V(X) = V(F_A, \epsilon). \tag{4}$$

[15] Intangible variables include a loan officers judgment.

Equations (4) and (2) are useful for thinking about the types of risks that systemic stress-tests may be designed to control. One form of risk is that banks exposures to common unhedged factors as captured by F_A in equation (4) cause them to become financially impaired at the same time. A second type of systemic risk is that banks hedging strategies may fail for some reason such as a counterparty default in which the party that is providing a hedge against factor such as F_B cannot provide it when required to do so. The second type of systemic risk is in the process of being addressed through policy reforms that control counterparty credit risk.[16] The scope of this paper is limited to the first type of systemic risk, the risk of systemic impairment due to common factor exposures, where systemic impairment is written as $SysImpair(V(X))$ because it depends on V, the vector of the values of banks net worth. Systemic impairment will also be written as $SysImpair(F_A, \epsilon)$ to emphasize its dependence on common factors and residual risks.

The relevant factors F_A for creating stress tests are F_A. The challenge is how to "identify" those factors, where formally identification of the factors means the identification of the space spanned by the factors.[17]

To identify the factors, I assume an approximation of the mapping between the variables X and the value of each banks portfolio $V_i(.)$ is known, or knowable to regulatory authorities:

Assumption 2 *Regulators have approximations of $V_i(X)$ that are sufficient to identify the factors.*

The assumption that regulators have approximations of $V_i(X)$ is increasingly realistic. For example, in the case of stress-tests for market risk, the Federal Reserve collects risk sensitivities for approximately 30,000 X variables, where each sensitivity measures how the value of the portolio changes for small to medium-size changes in individual X variables. Similarly, for positions in the banking book, the Federal Reserve receives detailed information on banks loan portfolios, including for example information on each wholesale C&I loan that has value of at least 1 million dollars. This banking book information is used to analyze how movements in economic variables are likely to affect the value of the loans. If the value of the approximations depends on the factors F_A, then under additional regularity conditions discussed below, it is also likely that the factors will be identifiable, as discussed further below.

The steps used to identify the factors are the following:

1. Draw X from its distribution.

2. Compute $V(X)$.

[16] These reforms include the migration of bilateral derivatives positions to CCPs and higher margin requirements on bilateral derivatives trades.

[17] If ψ is $K_A \times K_A$ and has full rank, then ψF_A and F_A contain same statistical information about $V(.)$. Therefore, the factors F_A can only be identified up to a rotation matrix ψ.

3. Compute $SysImpair[V(X)]$

4. Repeat steps 1-3 $Ndraws$ times.[18]

5. Use the simulated values of $SysImpair(V(X))$ and X in Sliced Inverse Regression (SIR) factor analysis to identify the space spanned by the factors F_A.

Intuition for why this approach can identify the factors F_A comes from the steps. In step 1, X depends on F_A, F_B, and U. In step 2, because F_B is hedged, $V(X)$ only depends on F_A and ϵ. Therefore, in step 3, systemic impairment is only a function of F_A and ϵ: $SysImpair[V(X)] = SysImpair(F_A, \epsilon)$. In step 5, sliced inverse regression projects the simulated values of the X variables onto the simulated values of $SysImpair(F_A, \epsilon)$. If the X variables are independent of banks remaining idiosyncratic risk ϵ, then the projected values of the X variables, $E[X|SysImpair(F_A, \epsilon)]$ will only be functions of F_A. Under certain regularity conditions described below, it will then be possible to use the fitted values to identify factors that lie within a subspace of the space spanned by F_A; under some conditions the identified factors will span the same space as F_A. Moreover, the identified factors will turn out to be principal components that are ranked by their ability to explain systemic impairment. Because the relationship between the principal component factors and the X variables can be estimated, changes in the factors can be used to find the size and direction of movements in the X variables that are most likely to contribute to the risk of systemic impairment.

Steps 1-4 provide an ideal setting to apply SIR in step 5. For step 5, the following assumptions are made to identify the space spanned by factors:

Assumption 3 *There are K_A factors F_A that affect systemic impairment. Each of the K_A factors $F_{A,k}$ is expressible as a linear combination of the X variables.*

$$F_{A,k} = X\beta_k, \qquad k = 1, \ldots, K, \tag{5}$$

where each of the β_k vectors is $N \times 1$.

Assumption 4 *The X variables are distributed independently of the vector of banks residual risks ϵ.*

Assumption 5 *For every $N \times 1$ vector b, there exist constants $c_k(b)$, $k = 0, \ldots K_A$ such that*

$$E(Xb|X\beta_1, \ldots X\beta_K) = c_0(b) + \sum_{k=1}^{K} c_k(b) X\beta_k \tag{6}$$

[18] The draws of X should be made from the conditional distribution of X viewed as appropriate for the stress-test exercise. The draws of X should be i.i.d.

Assumption 3 is equivalent to assuming that the information in the factors that generate systemic risk are expressible as linear combinations of the X variables that affect the banks.[19] It therefore follows that systemic impairment has the functional form

$$SysImpair(F_A, \epsilon) = SysImpair(X\beta_1, X\beta_2, \ldots X\beta_K, \epsilon). \tag{7}$$

Assumption 4 has the implication that idiosyncratic risk at the portfolio level (ϵ_i) for each bank i cannot be used to forecast the variables X. This assumption cannot literally be true because ϵ for each bank depends on the idiosyncratic risk of the variables X, but the assumption holds approximately since the forecasting power of the residuals approaches zero in diversified portfolios.[20] Put differently, assumption 4 should be interpreted as an assumption that the large banks to which stress-testing is applied hold diversified portfolios.

Assumption 5 states that the expected value of linear combinations of the X variables given the systemic risk factors is a linear combination of the systemic risk factors. This assumption will be satisfied if the X variables are elliptically distributed. As discussed in Li (1991), the methodology for uncovering the factors also works well even if this assumption holds approximately.

In step 5, the systemic risk factors are identified using the Sliced Inverse Regression (SIR) method of Li (1991) as refined using the Correlation Pursuit (COP) methodology of Zhong et al (2012). The main intuition for how SIR identifies the space spanned by the systemic risk factors will be presented in this subsection. Further information on SIR and COP is presented in the appendix.

SIR relies on inverse regression in which each of the simulated X variables is nonparametrically regressed on the simulated measure of systemic impairment $SysImpair(F_A, \epsilon)$, to compute the fitted value $E[X|SysImpair(F_A, \epsilon)]$. By assumption 4, the fitted value does not depend on the banks portfolios' idiosyncratic risk ϵ, it only depends on F_A. To recover the space spanned by the factors F_A, SIR performs a principal components analysis based on the fitted values.

To economize on notation below, $SysImpair(F_A, \epsilon)$ will be denoted $Y(X\beta_1, \ldots X\beta_{K_A}, \epsilon)$, or simply as Y. The factors F_A will be used interchangably with $X\beta_1, \ldots X\beta_K$. Σ_{XX} denotes the variance covariance matrix of X and $\Sigma_{E(X|Y)}$ denote the variance covariance matrix of the fitted values:

[19] This is similar in spirit to factor-mimicking portfolios that are often used in empirical asset pricing studies.

[20] For example, suppose $X = f + U$, where U is i.i.d., and bank i's portfolio has exposure of $1/N$ to each of the X variables. Then the portfolio's return, R_i is given by $R_i = f + (1/N)\sum U_i)$, where the term in parenthesis is ϵ_i. The covariance between any element of X such as X_j and ϵ_i is $(1/N)\sigma^2(U)$, which vanishes with N, showing that the residual return of the portfolio has very little power to forecast X_j. By contrast the covariance between X_j and the systematic component of the portfolio returns is $\sigma^2(f)$, which does not vanish with N. This shows the portfolio's return has power to forecast the elements of X because of the portfolios exposure to the factor risk; the idiosyncratic part of the portfolio's return, by contrast, has essentially no forecasting power.

$$\begin{aligned}\Sigma_{XX} &= E\left\{[X - E(X)]' \times [X - E(X)]\right\} \\ \Sigma_{(E(X|Y)} &= E\left\{[E(X|Y) - E(X)]' \times [E(X|Y) - E(X)]\right\}\end{aligned}$$

Sliced Inverse Regression identifies a subspace of the space spanned by the factors as the vectors Xb_k where the b_k vectors are solutions to the problem:

$$\text{Max}_{b_k}\ b_k' \Sigma_{E(X|Y)} b_k \tag{8}$$

subject to the constraint

$$b_k' \Sigma_{XX} b_k = 1,$$

and subject to the condition that the b_k vectors are orthogonal $b_k' b_j = 0$ for $j \neq k$.

When SIR is used to estimate the b_k coefficients, it does so using sample estimates of Σ_{XX} and $\Sigma_{E(X|Y)}$. The analysis in this section illustrates the information that SIR recovers about the factors when Σ_{XX} and $\Sigma_{E(X|Y)}$ are known. Distribution theory for the b_k coefficients is contained in Li(1991), Chen and Li(1998), and Zhong et al (2012).

The first order condition for choosing b_k is:

$$\Sigma_{E(X|Y)} b_k = \lambda_k \Sigma_{XX} b_k,$$

where λ_k is the Lagrange multiplier on the constraint. Rearrangement shows b_k and λ_k are eignvectors and eigenvalues of $\Sigma_{XX}^{-1} \Sigma_{E(X|Y)}$:

$$\Sigma_{XX}^{-1} \Sigma_{E(X|Y)} b_k = \lambda_k b_k, \tag{9}$$

and that the Xb_k are therefore principal components constructed from $\Sigma_{XX}^{-1} \Sigma_{E(X|Y)}$. Because the b_k coefficients are eigenvectors, they are orthogonal, and thus the orthogonality condition does not constrain them. Following Zhong et al (2012), each b_k vector is referred to as a principal direction. The number of principal directions is the number of positive eigenvalues from equation (9).

The principal components are not the systemic factors, but they lie within a subspace of the space spanned by the factors. When the number of principal directions is equal to the number of factors, then the principal components and the factors F_A span the same space. An advantage of focusing on the principal directions for modeling systemic impairment is that the eigenvalues measure the principle components based on their ability to statistically explain systemic impairment, with the larger eigenvalues corresponding to more explanatory power.[21]

The proposition and corollary that follow show that the principal components are spanned by the factors, and when both have the same dimension they span the same space.

To illustrate that the Xb_k vectors that are identified in the maximization problem 8 are spanned by the factors, note that any Xb_k can be decomposed into its projection on the factors ($= \sum_{k=1}^{K} c_k X\beta_k$) and into a component $Xb^{\perp}$ that is orthogonal to the factors. Because the projection component is spanned by the factors, it suffices to show that b vectors that solve equation 8 cannot contain an orthogonal component $b^{\perp}$. The theorem and proof of this result is based on Li (1991).

Proposition 1 *For the b_k coefficients that satisfy equation 9, each principal component Xb_k is spanned by the factors $X\beta_k$, $k = 1, \ldots K_A$..*

Proof: See the appendix. □.

The main step in the proof shows that $\mathrm{Var}[E(Xb^{\perp}|Y)] = 0$ $(= b^{\perp\prime}\Sigma_{E(X|Y)}b^{\perp} = 0)$, or equivalently, that $E(Xb^{\perp}|Y)$ is a constant that does not vary with Y.[22] To see that it is a constant, note that the information contained in Y is ϵ and the factors $X\beta_k, k = 1, \ldots K_A$. By assumption 4, the ϵ coefficients have no power for forecasting $E(Xb^{\perp})$. By assumption 5, $E(Xb^{\perp}|X\beta_k, k = 1, \ldots K_A)$ is linear in the $X\beta_k$, but also by definition $Xb^{\perp}$ is uncorrelated with each of the $X\beta_k$. It follows that $E(Xb^{\perp}$ does not change with the $X\beta_k$, and therefore that $E(Xb^{\perp}|Y)$ does not vary with Y. This means any Xb_k that solves equation (9) is spanned by the factors.

Corollary 1 *If the number of principal directions is equal to the number of factors, then the factors and the principal components span the same space.*

Proof: Let B denote the matrix $(b_1, b_2, \ldots b_{K_A})$ and β denote the matrix $\beta_1, \ldots \beta_{K_A}$. Since b and β are nonsingular and β spans the elements of B, $B = \beta\Pi$ for some nonsingular Π. Therefore $\beta = B\Pi^{-1}$,and therefore β is also spanned by B and both span the same space. □.

The corollary shows that SIR will identify the space spanned by the factors provided that the rank of $\Sigma_{XX}^{-1}\Sigma_{E(X|Y)}$ has the same rank as the number of factors. Although SIR can be used to identify large parts of the factor space that are important to systemic risk, it is important to emphasize it can fail to identify factors in cases when $E(X|Y)$ does not change with Y, even though a factor affects Y. For example, if the factor is just X_1 and Y is a symmetric function of X_1 such

[21]See appendix B for details.

[22]If $E(Xb^{\perp}$ is a constant that does not vary with Y, then it follows that $\Sigma_{E(X|Y)}b^{\perp} = 0$. The proof then show thats any b_k that solves equation (9) must have as its $b^{\perp}$ component $b^{\perp} = 0$.

as $Y = bX_1^2$, and X_1 is standard normal, then $E(X_1|Y) = 0$ and therefore SIR could not detect X_1 as a factor in this simple example.

When using SIR, identification of the space spanned by the factors relies on estimates of the matrix $\Sigma_{XX}^{-1}\Sigma_{E(X|Y)}$ When X is high dimensional and the time series on X is short, then estimates of Σ_{XX} and its inverse are likely to be inaccurate. As noted in Zhong et al (2012), this problem, if not addressed, will reduce the accuracy of SIR when the number of potential X variables is large. The Correlation Pursuit (COP) methodology of Zhong et al (2012) is designed to address this difficulty. The underlying assumption in Zhong et al is that a relatively sparse subset of the X variables, denoted x, contains the essential information on the factors. Conditional on x the information on Y contained in the other X variables is assumed to be redundant. The COP methodology chooses the relevant variables x based on their ability to create principal components that explain Y. For this paper, the elements of x are chosen based on their ability to explain systemic impairment. COP selects the relevant x variables by starting with a candidate set of active variables $x_0 \in X$; it then scrolls through the remaining variables in X and performs variable addition and deletion steps that add (delete) variables to (from) the active set if they statistically improve (don't improve) explanatory power for Y. Zhong et al show that under a set of regularity conditions as the size of the sample of X and Y variables approaches infinity, COP consistently chooses the set of x variables that are relevant for determining the factors that explain Y. Although Zhong et al provide asymptotic theory for choosing x consistently, they emphasize that the asymptotics treat K_A as known when it is not, and also the asymptotics for adding and deleting variables make strong assumptions, and only hold asymptotically. Therefore, for finite samples they recommend choosing K_A based on the BIC criterion; and they choose the critical values for determining whether to add or delete variables using cross-validation. Further details on how to implement SIR and COP are provided in the appendix.

In summary, this subsection has illustrated an approach using SIR and COP to identify the relevant factors that explain systemic impairment given how banks hedge and it has presented an approach for identifying the variables x that explain these factors. The next section provides details on how to use the identified variables and factors to create stress tests for systemic risk.

3.2 Choosing Stress Scenarios for Systemic Risk

This section uses the principal components extracted in the last subsection to create stress scenarios and resulting capital injections to ensure the financial system is resilient against systemic risk with high probability. To implement the methodology, three elements are required. First, a measure of systemic impairment is needed to serve as the Y variable in the last section, as well as to measure systemic risk. Second, a method is needed to define stress scenarios in terms of movements in the systemic risk factors. Third, a method is needed to choose stress scenarios such that if banks are well capitalized against the scenarios considered, then they will be well capitalized with high probability against systemic impairment; i.e. systemic risk will be low. Below,each of these elements is provided in turn.

Measuring systemic impairment

Recall that systemic impairment is the event that too many banks become financially distressed together, and hence cannot provide needed financial intermediation services to the real sector. Financial distress is measured based on banks equity capital (net worth) relative to its risk. For example, if the stress horizon is one year, and at the end of that year a banks capital is low, while the volatility of capital is high, then the bank is likely to become insolvent shortly thereafter. It will therefore not be able to raise funds to intermediate loans, and hence its financial distress will be high.

The analysis on systemic impairment measurement is based on Pritsker (2013). Without loss of generality, banks are stress-tested at date 0, and the stress-test horizon is normalized to be one period. There are J financial intermediaries $j = 1, \ldots J$. At date 0, each financial intermediary has has equity E_j, and liabilities L_j that finance assets $A_j(= E_j + L_j$ by the balance sheet identity). Bank j's asset portfolio has return R_j between date 0 and date 1. Additionally, the gross return earned by its liability holders is $\bar{R}_{l,j}$. Thus, bank j's capital ratio at date 0 is $C_j(0) = \frac{E_j(0)}{A_j(0)}$, and its capital ratio at date 1 is $C_j(1) = \text{Max}[\frac{E_j(1)}{A_j(1)}, 0]$. The capital ratio for bank j at date 1 can be written as a function of its initial capital ratio and the return on its assets:

$$\begin{aligned}
C_j(1) &= \text{Max}(\frac{E_j(1)}{A_j(1)}, 0) \\
&= \text{Max}(\frac{A_j(1) - L_j(1)}{A_j(1)}, 0) \\
&= \text{Max}(1 - \frac{L_j(0)\bar{R}_{l,j}}{A_j(0)R_j}, 0) \\
&= \text{Max}(1 - \frac{(1 - C_j(0))\bar{R}_{l,j}}{R_j}, 0)
\end{aligned}$$

As a result of the stress-test conducted at date 0, banks may be required to inject more equity into the bank. I assume this equity is invested at the risk free rate, and earns a gross return of R_f. If the equity injected at date 0 is equal to a fraction CI_j of initial assets, then assets at date 1 become $A_j(0)R_j + A_j(0)CI_jR_f$. Making this substitution, bank j's capital ratio at date 1 is given by

$$C_j(1) = \text{Max}\left(1 - \frac{[1 - C_j(0)]\bar{R}_{l,j}}{R_j + CI_jR_f}, 0\right) \tag{10}$$

The volatility of bank j's date 1 capital ratio as of date 1 is denoted $\sigma(C_j(1))$. Bank j's financial distress at date 1 is modeled as a decreasing function of its capital ratio normalized by its volatility: $D_j(\frac{C_j(1)}{\sigma(C_j(1))})$. This ratio is inversely related to default likelihood after period 1, and therefore distress goes down as the ratio goes up. For convenience distress is parameterized to lie between zero and one ($D_j \in [0, 1]$).

To model systemic impairment, I make the following assumptions:

Assumption 6 *1. Each banks maximal financial intermediation capacity is proportional to its assets: $FICapacity(j) = \gamma A_j$ with a constant of proportionality γ that is the same for all banks.*

2. The fraction of a banks maximal intermediation capacity that is lost in a scenario is proportional to its distress in that scenario:

$$\textit{Loss of j's capacity} = D_j\left(\frac{C_j(1)}{\sigma(C_j(1))}\right)\gamma A_j.$$

3. Systemic impairment occurs when the fraction of the economy's maximal intermediation capacity that is lost exceeds a threshold ζ.

These assumptions capture the ideas the larger banks, measured by the size of their balance sheets, have more intermediation capacity, and that therefore more intermediation capacity is lost when larger banks are more financially distressed. It is assumed that when a little intermediation capacity is lost, other banks can step in and fill the capacity that is lost. But, when too much maximal capacity is lost, it becomes too large for others to fill in, resulting in systemic impairment.

Under assumption 6, the fraction of maximal intermediation capacity that is lost given a realization of banks return vector $R_1, ...R_J$, and given the Capital Injections received by banks, is denoted System Assets in Distress, abbreviated SAD:

$$SAD(R_1, \ldots R_J, CI_1, \ldots CI_J) = \frac{\sum_{j=1}^{J} D_j(\frac{C_j(1)}{\sigma(C_j(1))})\gamma A_j}{\sum_{j=1}^{J} \gamma A_j} = \sum_{j=1}^{J} w_j D_j(.) \tag{11}$$

Note, that the arguments of the capital ratios that are made explicit on the left hand side of the expression for SAD are for simplicity suppressed on the right hand side, and will be suppressed whenever it is convenient to do so.

The constant of proportionality γ drops out of the expression for SAD. As a result, it reduces to a weighed average of each banks distress function where each banks weight is its assets as a fraction of all banks assets.

Systemic risk is a function of the distribution function of systemic impairment. For simplicity, in this paper systemic risk, denoted $\psi(0,T)$ is defined as the probability that systemic impairment occurs at the end of the time horizon T of the stress-test:

$$\psi = \text{Prob}[SAD(T) \geq \zeta].$$

Creating Stress Scenarios

A stress scenario specifies values for all of the X variables. As noted in the introduction, choosing an appropriate scenario to achieve regulatory objectives is difficult when X is high dimensional. To reduce dimensionality, this paper defines stress-scenarios in terms of the identified factors, and then sets the X variables in the scenario to their expected values given the factors.

Definition 1 *A Systemically Chosen Stress Scenario is a specification of realizations for the systemic risk factors F_A and a specification for the expected realizations of the other relevant X variables for determining the value of banks conditional on F_A.*

To compute the expected value of the X variables, consistent with assumption 5, the X variables are modeled as a linear function of the factors, and take a form that can be estimated by OLS through regressing the simulated values of the X variables on the simulated value of the factors[23,24]
:

$$X_i = \alpha_i + F_A\theta_i + \epsilon_i \tag{12}$$

where α_i is the $N \times 1$ regression intercept, θ_i is $K_A \times 1$ vector of regression coefficients and ϵ_i is an $N \times 1$ residual.

Using the definition and equation (12), if the chosen factor realizations are $\tilde{F_A}$, then the stress-scenario is given by:

$$X_i = \alpha_i + \tilde{F_A}\theta_i, \quad X_i \in \mathbf{X} \tag{13}$$

To assure that banks have enough capital on the basis of a stress test, it is also necessary to know how each banks financial distress is related to the factors. To model this, for simplicity I

[23]Assumption 5 implies $E(X|F_A)$ is a linear function of F_A. This is consistent with the OLS regression specification in equation 12.

[24]Recall the identified risk factors are $F_{A,k} = xb_k, k = 1, \ldots K_A$. Because the X variables and the x variables are simulated and the b_k's are estimated, the simulated risk factors are "observable" in the simulation, as are the X variables. This makes it possible to estimate the relationship between the X variables and the risk factors.

assume banks liabilities are unaffected by stress, that each bank's asset returns are linearly related to the X variables with $\beta_{i,j}$ representing the sensitivity of R_j to X_i, and that the realizations of the X variables fully explain banks returns.[25] With this formulation, for each bank j, R_j can be expressed as a linear function of the systemic factors, and a residual term that will be correlated across banks because many banks are exposed to common $\mathbf{X}$ variables:

$$
\begin{aligned}
R_j &= \alpha_{0,j} + \sum_i \beta_{i,j} X_i \\
&= \alpha_{0,j} + \sum_i \beta_{i,j}(\alpha_i + F_A \theta_i + \epsilon_i) \\
&= \alpha_j + F_A \theta_j + \epsilon_j
\end{aligned}
\tag{14}
$$

Formulating Stress Scenarios based on a Systemic Risk Objective

The main result in the paper is if SAD is approximated by $ASAD$, a variant of SAD that linearizes the relationship between SAD and $R_j + CI_j R_f$, then if regulators objective function is defined in terms of $ASAD$, then there is a stress-scenario and resulting capital injections that assures systemic risk is low. This is formally stated in the following proposition:

Proposition 2 *If SAD is linearly approximated by $ASAD$[26]:*

$$ASAD = C_0 + \sum_j D_j (R_j + CI_j R_f), \tag{15}$$

and the return on each bank j's portfolio, R_j, satisfies equation (14)*, and if regulators systemic risk objective is to ensure that*

$$Prob(ASAD \geq \xi) \leq \psi,$$

then there is systemic risk factor shock F_A^* such that when the stress scenario is $X_i = \alpha_i + F_A^* \theta_i$ for all X_i, and banks inject capital equal to the present value of their losses in the stress scenario, then after the capital is injected, $Prob(ASAD \geq \xi) \leq \psi$.

Proof: See the appendix.

[25] These assumptions can be relaxed to allow the liabilities to be affected by X, to allow the X variables to nonlinearly affect banks asset returns and liabilities, and to allow the value of assets and liabilities to fluctuate for reasons other than the X variables. However, relaxing these assumptions significantly complicates the modeling.

[26] The parameters of the linear approximation to SAD, C_0 and the D_j's, can be derived from a first-order Taylor series for SAD, or they can be estimated by linearly regressing simulated values for SAD on simulated values of $(R_j + CI_j R_f)$.

To provide intuition for the proposition note that equation (15) for $ASAD$ and equation (14) together imply that $ASAD$ has a linear stochastic component that depends on the factors F_A and non-factor risks, and a linear component in terms of the capital injected by banks:

$$\begin{aligned} ASAD &= C_0 + \sum_j D_j(\alpha_j + F_A\theta_j + \epsilon_j + CI_jR_f) & (16) \\ &= C_0 + \alpha + F_A\theta + \epsilon + CIE, & (17) \end{aligned}$$

where D_j is negative since more capital reduces banks financial distress. The expression shows the magnitude of $ASAD$ can be controlled by capital injections, summarized by the CIE term. The first part of the proof finds the least negative value of CIE, denoted CIE^*, that just satisfies regulators objective for systemic risk.[27]

$$CIE^* = CIE : \text{Prob}(C_0 + \alpha + F_A\theta + \epsilon + CIE \geq \xi) = \psi.$$

The second part of the proof finds values of the systemic factor F_A^* that satisfy the condition that if the stress-scenario is

$$X_i = \alpha_i + F_A^*\theta_i$$

for all i, then if banks inject enough equity capital to cover their losses (measured from their net returns), then the resulting capital injections ensure $CIE = CIE^*$, thus achieving the systemic risk objective.[28] The condition for F_A^* is the equation

$$F_A^*\theta = -CIE^* - \alpha + \sum_j D_j. \tag{18}$$

When θ is a nonzero scalar this equation has one solution. When θ is a vector, then there are multiple solutions for F_A^*, which means there is room to choose F_A* to satisfy equation (18), while also satisfying other side criteria. Two criteria are considered here, maximum likelihood and minimum cost.

The maximum likelihood criterion chooses the value of $F_{\underset{A}{*}}$ to satisfy equation (18) and have maximal likelihood. To solve for the maximum likelihood F_A^*, note that F_A has mean 0 since each element of X is normalized to have mean 0, and F_A has variance I since F_A is a matrix of principal component factors. Under the auxiliary assumption that F_A is also multivariate Gaussian, then it

[27]Solving for CIE^* requires knowledge of the CDF of $F_A\theta + \epsilon$, denoted $H(.)$. Finding this CDF is relatively straightforward because although $H(.)$ is not known, it is relatively easily estimated since $F_A\theta + \epsilon$ is a single random variable, simulated values of F_A are available from COP/SIR, and simulated values of ϵ can be constructed using estimated versions of equations (12), and (14), with the D_j coefficients from equation (15).

[28]Banks capital injections equal the present value of their losses in the stress-scenario discounted at the risk-free rate. Since the capital is assumed to be invested riskfree, it produces enough capital to cover losses at date 1 in the stress scenario.

is straightforward to show that the maximum likelihood value for F_A^*, denoted $F_A^*(Maxlik)$, is:

$$F_A^*(MaxLik) = \frac{(-CIE^* - \alpha + \sum_j D_j)\theta'}{\theta'\theta}. \tag{19}$$

An alternative criterion for choosing F_A^* is to find the stress scenario that minimizes banks capital costs while at the same time satisfying the constraint in equation (18). F_A^* is chosen inefficiently if the ensuing capital requirements from the stress scenario inject large amounts of capital into banks for which the marginal systemic risk benefits are small, while injecting too little capital in banks with high marginal benefits. The following problem for choosing F_A minimizes the costs of injecting capital while choosing a stress scenario that satisfies the constraint in (18), where λ is the marginal cost of injecting equity capital.

$$F_A^*(effic) = Argmin_{F_A} - \lambda \sum_j A_j \text{Min}(-1 + F_A'\theta_j + \alpha_j, 0) \tag{20}$$

such that

$$\sum_j D_j Min(-1 + F_A'\theta_j + \alpha_j, 0) = -CIE^*$$

Note, in this optimization problem the "Min" operator on the right hand side of equation (20) rules out negative capital injections. These are ruled out because if they are allowed, then the optimization problem becomes a linear objective function with linear constraints, and thus would not have a bounded solution.

To the best of my knowledge, this is one of very few papers (the only ?) that have derived stress-tests with the explicit goal that the resulting capital injections satisfy an explicit systemic risk objective, and that moreover are designed to guarantee that if the banking system is well capitalized against the scenario, it is well capitalized against other plausible scenarios with a high likelihood that is chosen by the regulator.

Several qualifiers regarding proposition 2 are important. First, because ASAD approximates systemic impairment, satisfying regulators systemic risk objectives based on $ASAD$ will not necessarily satisfy the objectives based on SAD. This suggests using the $ASAD$ approximation with side conditions (equations (19) and (20)) to find the direction in which to move the systemic risk factor vector. Then, the amount by which the factors are moved in the appropriate direction is chosen until the resulting stress scenario achieves a systemic risk objective based on SAD.[29] As

[29] For example, if there are two systemic risk factors $F_A(1)$ and $F_A(2)$, the chosen direction might specify that $F_A(1)$ should increase 1.5 times as quickly as $F_A(2)$. Given this direction, increases in $F_A(1)$ and $F_A(2)$ can be solvved for

shown in the next section, this often works well in practice.

Second, there are circumstances when the capital injections required to achieve a systemic risk objective are unattainable on the basis of a single stress scenario. As a simple illustration of such circumstances, if banks only asset holdings are positions in a stock index, then the index is the factor F_A. If half the banks have identical long positions in the factor, while the other half have identical short positions, then absent capital injections, too much systemic impairment can occur if the factor takes very high values or very low values. This example is an extreme failure in which $ASAD$ is approximated as a linear function of the factor but SAD is strongly non-linear. The consequence of the failure is that a single stress scenario in which the index drops can ensure that half the banks are well capitalized against that scenario, but, it cannot ensure that the other half of the banks are also well enough capitalized against stocks rising. Hence, in the example a single stress scenario is not sufficient to attain the objective. An advantage of the approach advocated in this paper is that part of choosing the stress scenario would entail checking whether the SAD objective is attainable, and indicating when it is not. This is a major advance over current stress testing because it would indicate whether its objectives are attainable. Moreover, it would indicate methods that might help to attain the objective through for example requiring that banks are well capitalized against more than one stress-scenario.

In addition to these qualifications, there are areas where there is scope to further extend the SCSA approach. These include:

- Modeling the effects of interbank credit exposures.[30,31]
- Incorporating the modeling of banks liabilities and income as part of the analysis.
- Using the methodology for other measures of impairment, such as Systemic Expected Shortfall.

The next section of the paper illustrates the use of the SCSA methodology when applied to interest-rate positions.

4 SCSA applied to rates positions

The analysis in this section is preliminary. To study how well SCSA performs in generating systemic risk stress-scenarios, in this section it is applied for 6 hypothetical banks whose assets for simplicity

such that the resulting capital injections in the stress scenario achieve the systemic risk objective for SAD.

[30]The importance of accounting for interbank credit may be dimininshing if CCP clearing requirements cut such risks enough, but could become very important if CCPs get into trouble.

[31]For potential directions to incorporate them See Ota (2013),Pritsker (2014).

solely consist of positions in the zero coupon bonds of 8 countries (Australia, Canada, Germany, Japan, Sweden, Switzerland, Great Britain, and the United States). The tenors used in the analysis ranged from short maturity to 15 years for all countries, but was as long as 30 years for the US.[32] Data on zero coupon yields for all countries other than the US was generously provided by the International Finance Division at the Federal Reserve Board. Zero coupon yield data for the U.S. is based on the methodology from Gurkaynak et. al.(2006). The data spans the period from February 2006 to October 2013.

In the preliminary analysis, ten sets of portfolios are simulated for the hypothetical banks. For each set of portfolios, each banks exposures to the yield curves are chosen randomly from the same distribution function. This generates a set of portfolio weights in the assets that are i.i.d. among zero coupon yields and among the 6 hypothetical banks. In the current simulations, two distribution functions for exposures are considered. In the first, banks exposures to holdings of each zero can be short or long, but they are biased towards long holdings. In the second set of simulations, exposures can be long or short and are not biased to be long or short on average.[33]

To implement the SCSA methodology, for each set of banks portfolios, it is necessary to simulate the systemic impairment measure over the stress horizon and then select variables to create factors that covary with systemic impairment. For simplicity, the stress horizon is one-month. Systemic Impairment over this horizon is measured by SAD. The shocks to SAD are changes in zero coupon yields. For simplicity in this version of the paper the shocks are simulated using simple historical simulation. With this methodology, it is assumed the distribution of future shocks to the yield curve is the unconditional distribution based on history. The shocks are the time series of 165 non-overlapping monthly changes in 83 zero-coupon yields that occurred over the time span of the data. Because banks exposures are to yield curves with different currencies, logically international fixed income positions are sensitive to both interest-rate risk and foreign exchange risk. In this exercise, I for now assume foreign exchange risk has been hedged out, and that interest-rate risk can be treated separately from foreign exchange risk.

Recall that SAD is the weighted average of banks distress functions. In equation (11), the functional form of banks distress functions is left unspecified. In the analysis we used distress functions that have a logit form for each bank j.

[32] For the United States the yield tenors in years are 1,2,3,4,5,7,8,10,12,15,20,25,30. For the other countries the tenors in years are 0.25,.5,.75,1,2,3,5,7,10,15.

[33] Using the first choice of distribution functions, the exposures are randomly simulated by choosing DV01s for each yield from a $N(-.5, 1)$ distribution, where DV01's are measured as change in dollar value of an exposures due to a one-basis point increase in yield. After DV01s for each currency and yield curve are chosen, the DV01's are inverted to produce the amount invested in each zero coupon bond to produce its corresponding DV01. These investments are then converted into portfolio weights that sum to 1. For the second choice of distribution function, the exposures are randomly simulated by choosing DV01s for each yield from a $N(0, 1)$ distribution, where DV01's are measured as change in dollar value of an exposures due to a one-basis point increase in yield.

$$D_j(.) = \frac{1}{1 + \text{Exp}(a_j + b_j C_j(1)/\sigma(C_j(1)))}, \tag{21}$$

where $C_j(1)$ is bank j's capital ratio at date 1 and $\sigma(C_j(1))$ is the standard deviation of the capital ratio as of date 1. The expression $C_j(1)/\sigma(C_j(1))$ is bank capital divided by a measure of risk of the capital, and hence can be interpreted as risk weighted capital since for a given capital ration, the greater is the risk the lower is the effective risk-weighted capital.[34]

The distress function measures the fraction of a banks' maximal intermediation capacity that is lost when the bank becomes undercapitalized. The parameters a_j and b_j should ideally be calibrated so that the distress function for each bank roughly captures the relationship between the banks intermediation capacity and its risk-weighted capital. For the purposes of this paper the parameters are not calibrated; instead they are set so that $a_j = 0$ and $b_j = .95$ for all j. Because capital ratios span from 0 to 1, with this specification, distress for each bank has a lower bound of 0, which is approached as the volatility of capital goes to 0. The upper bound for distress is 0.5 which is approached as capital goes to 0. This means an insolvent bank loses half of its maximal financial intermediation capacity. the other parameter that needed to be calibrated as part of the analysis was the choice of banks initial capital ratios. For the analysis here this ratio was chosen so that banks have a two-percent chance of bankruptcy over the stress horizon. This is not a realistic choice, but was made for convenience and should be altered in the future.[35]

Given the distress functions, and banks portfolio weights in the bonds, for each set of randomly generated portfolios, SIR and COP are used to perform dimension reduction by choosing which variables best explain SAD, and by summarizing the information in those variables through the creation of systemic risk factors that are created based on their ability to explain SAD. The number of relevant systemic risk-factors is determined by sequential hypothesis testing: beginning with no risk factors: additional risk factors are added sequentially until it is determined that the explanatory power from adding additional factors cannot be distinguished from random noise (Li, 1991). The number of factors identified in the simulations when systemic impairment is measured by SAD is equal to one in the most recent simulations. However, as discussed below we expect more systemic factors may be identified when the approach for choosing banks portfolio weights is modified.

For the 10 sets of random portfolios (labeled Simulation 1 - Simulation 10 in the figures) when the exposures are long on average, SAD and correlation pursuit uncover a very strong relationship between SAD and a single systemic risk factor. In the case of Simulation 1, a scatter plot of SAD

[34] An alternative approach could use Basel risk-weights. An advantage of the approach in this paper is that Basel risk-weights for simplicity assume risk scales linearly, but in fact this is not descriptive of how risk scales.

[35] This seems like a high but defensible choice of unconditional insolvency risk over a year, but not over a month. It is set to be high in the analysis so that the choice of capital can be calibrated on the basis of the historical bond returns during our short sample of 165 months. A more realistic default probability per month is perhaps .0017, or a bit less than two in a thousand. A sample of 165 months cannot be used to calibrate the amount of capital needed to cover losses with such a low probability. Instead a parametric model of returns in the tail would need to be fit to the data.

(labeled DV01 SAD, or DV01 for short) versus the factor shows a strong monotone relationship between SAD and the factor (Figure 1). The Correlation Pursuit and Sliced Inverse Regression Statistical procedures identify the factors (the space spanned by the factors), but do not identify the precise relationship between SAD and those factors. Nonparametric regression of SAD on the factors is utilized to explore the relationship between the expected value of SAD and the factors, and also to give an eyeball view of how much SAD deviates from its expected value conditional on the factors (Figure 2).[36] The regressions show that the relationship between expected SAD and the factor (the red curve labeled kernel) is slightly nonlinear (as it must be since SAD has a range of 0 to 0.5). In addition, the nonparametric fit is not excellent, but it is very good.

While the nonparametric regression curve is suggestive of how much information is likely to be statistically captured by the factor, it does not establish how SAD will actually respond to a stress-test based on the factor because that relationship depends on how the factor is mapped to returns and then to SAD as part of the stress-testing procedure. Those mapping procedures are not accounted for as part of the nonparametric regressions; if the mapping procedures are not appropriate, then a stress-test based on the factor may fail to capture SAD well even if the factor is strongly statistically related to SAD. The extent to which mapping errors are an issue can be examined by stressing the factor, applying the mapping used in the stress test, and then comparing the mapped values to the nonparametric estimates and to the scatter plots. The results in the cases considered shows that SAD that results from stress tests based on the factor (labeled S.T. and plotted in green) track the nonparametric estimates of SAD fairly well (Figure 3) when the portfolio exposures are on average long. The results using all 10 simulated sets of long-on-average portfolios are similar (Figure 4).

The results for the 10 sets of portfolios appear to be very encouraging for finding systemic-risk factors that can be successfully used as the basis for a stress test. However, a more reasonable interpretation is that the method should work quite well for the case considered because in the case considered interest-rate factors should have significant explanatory power for SAD. To see why note that for the method used to construct banks portfolio weights expected DVO1s at each tenor on each yield curve are positive and the same. If banks portfolios are not too far from what is expected, their expected portfolios are sensitive to parallel yield curve shifts, which is known to be an important factor in yield curve modeling. Put differently, it is encouraging that the method appears to find a factor when it should. However, a more stringent test of the usefulness of the approach would be if banks portfolio compositions were more heterogeneous. The analysis of this case has only been done so far for portfolios that are on average neither long nor short. Intuitively, a portfolio that has long and short positions will have exposure to changes in the slope of the yield curve as well as changes in the level, and if portfolios differ in their mix of exposures to the two types of factors, then a one-factor model will not fit as well when modeling systemic impairment.

[36] The regressions use Nadaraya-Watson kernel regression with a Gaussian kernel. The kernel bandwidth h was chosen as h = $\sigma(f)N^{-1/5}$, where N is the number of time-series observations of f. h was not optimally chosen.

This intuition is borne out from the simulations for all neither long nor short portfolios (Figure 5). Although the fit is worse, it is still very good. More detailed analysis is need to examine this issue further.

4.1 Solving for CIE and stress-scenarios

CIE and the stress-scenarios were investigated in two settings. The first used linear approximation. By construction, the stress-scenarios and capital injections solved for based on linear approximation satisfy the ASAD objective function exactly. It is appropriate to investigate whether they also satisfy the objective function of keeping SAD low with high probability. The answer in the cases investigated so far is that choosing stress scenarios using the expression for ASAD and equation (30) does not work well for SAD as parameterized because SAD is too nonlinear be sufficiently well approximated along the length of its range. As an example for simulated portfolio 1, a QQ-plot of SAD versus ASAD shows that ASAD is well below SAD for high values of SAD (Figure 6) This implies that the capital injection required to reduce ASAD to the target level chosen by the regulator can often be too small to reduce SAD by enough to achieve regulatory targets for SAD. In other words, the size of the stress scenario based on ASAD will tend to be too small, and therefore the size of the stress scenario needed to achieve the systemic risk objective needs to be solved for by other means. The approach I have pursued in the empirical analysis is just a slight modification of the linear approach with multiple factors. In particular, from equation (19), F^* is a scalar multiple of θ, which is written below as:

$$F^*(MaxLik) = \kappa\Theta.$$

Let $CI(\kappa)$ be banks required capital injections if the stress scenario is $\kappa\Theta$. Then I solve for κ such that if the required capital injections are $CI(\kappa)$, then the $\text{Prob}(SAD(CI(\kappa)) \geq \zeta) \leq \psi)$. This step is not difficult since it simply involves simulating SAD with different levels of capital injections. For our preliminary analysis of 10 simulations of banks interest rate portfolio positions, this approach usually succeeded in generating a reasonable stress scenario and resulting capital injections that satisfied the systemic risk objective that $P(SAD > .05) \leq .05$ An example of the scenarios is provided in figure 7. In this case, only one systemic risk factor was identified, and hence changes in yields are proportional to that factor. In this case the movements in stress scenario can best be described as an upward shift in the yield curve that increases curvature.

Although in most of the portfolios considered a single reasonable stress scenario was sufficient to ensure the banks were sufficiently capitalized, in some portfolios the method could not achieve the systemic risk objective using a single stress scenario. Recall that this could occur if half the banks were very distressed for high values of a factor, and the other half very distressed for low values of a factor, since in that setting a scenario with high and low values of the factor were needed to ensure all banks were well capitalized against movements in the factor. An advantage of the methodology in this paper is that it also helps to identify when multiple stress-scenarios are needed to achieve a systemic risk objective, and provides guidance on how such scenarios should be chosen if needed.

4.2 Supplemental Analysis

The analysis above illustrates the potential for using SCSA to choose scenarios. The analysis in this subsection investigates three supplementary topics:

1. Out of Sample-Fit of Correlation Pursuit and Sliced Inverse Regression.
2. Additional evidence for the importance of choosing scenarios using exposure data.
3. How symmetry biases factor identification in SIR, and how to fix the bias.

Out of Sample Fit

A potential issue with Correlation Pursuit and SIR is that it may choose variables and factors that over-fit the in-sample data and consequently fit poorly out-of sample. The out of sample fit could be poor because of over-fitting or because the distribution of the variables being simulated is mis-specified. The latter problem is not due to SIR or COP. To abstract from the latter problem, I assume that the true distribution of interest-rates in 10 countries are known, and follows the DTSM model of Wright (2011) as modified in the next subsection. Given this process, 5,000 two-year sample paths of returns were generated, and banks portfolios were constructed to be perfectly correlated with one of the first three principal components of yield curve changes. Then, using these portfolio weights, COP/SIR were used to identify the factors in-sample. Then, using additional observations from the same DGP, COP/SIR was used to again identify the factors in the out of sample data. If the data is severely over-fit in either sample, then the extracted factors in the two subsamples will not be highly correlated, but in fact they are pretty highly correlated, providing preliminary evidence that over-fitting is not a severe problem (Figure 8).[37] That said, the magnitude of overfitting should depend on the complexity of banks portfolios and the number of observations used to identify the factors in the two samples. A more complete investigation of this topic will be in the next version of the paper.

Additional Evidence for the Importance of choosing scenarios using exposure data.

This subsection further examines the importance of using exposure data to choose stress-scenarios. Recall that section 2 presented special cases in which stress-scenarios that are not chosen while

[37]The out-of-sample data used 200, 500, or 1,000 observations.

accounting for positions can miss substantial risks in banks' portfolios. This section illustrates the same underlying idea by examining how the factors that SCSA identifies as important in stress-testing change as portfolio composition changes. To examine this question, the analysis contrasts 6 banks with identical portfolios that only invest in the bond-market with 6 banks that are identical and invest 50% of their assets in the bond market and the rest in the stock market. For all of the analysis in this part, the return series for banks portfolios are modeled via historical simulation, in which it is assumed that the distribution of returns in the future on bonds and stocks is the same as was experienced based on past history.

The effect on the extracted factors is measured by how the factor shocks affect yield curves and stock returns. In the case of interest rate portfolio, SIR only extracts a single interest-rate factor. One standard deviation shocks to this factor primarily change the curvature of the yield curve for most of the 8 countries analyzed in the historical simulation analysis, suggesting that this particular randomly chosen set of 6 bond portfolios (one for each firm) is mostly exposed to a curvature factor (Figure 9). For the portfolio that represents a 50-50 mixture of stocks and zero coupon bonds, two factors are extracted. The most important factor (factor 1) steepens most yield curves and depresses stock market returns (Figure 10). The second factor, which is much less important, appears to capture yield curve curvature. These results together illustrate, not surprisingly that the important scenarios to consider for systemic risk purposes depend on banks portfolio composition, and therefore it is important to account for this composition when deriving stress scenarios to achieve systemic risk objectives.

Symmetry and SIR

As illustration in section 3.1, if the Y variables used in SIR are a symmetric function of the risk-factors, then SIR and COP may fail to detect the factors. To investigate this issue when Y is the SAD function, I created examples in which there are 6 banks and the banks have portfolios whose stochastic components are long or short the same portfolio of zero coupon bonds. In this setting the factor is the return on the portfolio, and if it happens that 3 banks are long and 3 banks are short, then SAD is a symmetric function of the factors. This is a situation when SIR/COP should have a difficult time identifying the factors. Figure 11 illustrates the problem. The figure is analogous to Figure 5 except that there is symmetry. As a result, the single factor is sometime poorly identified. For example, in the second case of simulated banks, there are 3 long and 3 short banks, which is near perfect symmetry. As a result, the factor does a poor job of tracking SAD, and the stress-scenario created based on the factor also tracks SAD very poorly. Because SAD is constructed as a sum of each banks distress function, it is possible to identify the factor by pre-analyzing banks individual distress functions. For example, if scatter plots of banks simulated distress functions against each other, show that some banks are highly positively correlated while others appear to be highly negatively correlated the factors can be identified by performing SAD and COP on the banks whose distress functions are posively correlated. Then the factor can be used to examine how well it tracks SAD across all banks, as well as how stress-scenarios based on the factor would track true SAD. Figure 12 performs this analysis for the banks in simulation 2 from Figure 11. The figure shows that this approach has potential to overcome some of the problems that symmetry can pose when using SIR and COP.

5 Conclusions

Current supervisory stress-testing has been criticized for being too microprudentially oriented, for having the potential to miss important directions of banks risk taking, and for not ensuring the financial system will be well capitalized against other plausible stress-scenarios. This paper has presented a new approach for choosing stress scenarios based on dimenstion reduction techniques. The approch has the potential to improve supervisory stress-tests along all of these dimensions. First, the paper chooses stress-scenarios to ensure that a measure of systemic risk is low. Hence, the approach is macro-prudential by design, and could be used to help make regulatory stress-testing more macro-prudential. Second, the paper uses information on banks risk exposures when constructing stress scenarios. This reduces the likelihood that important directions of banks risk-taking will be missed in constructing the stress-scenario. Finally, the methodology in the paper is designed to ensure the banking system is well capitalized against a wide variety of stress-scenarios, and not just the scenarios used in stress-testing.

It is important to emphasize that the main innovations in this paper, using simulations to identify risk factors based on their ability to explain systemic risk, and then using the identified factors to develop systemic risk stress-tests are both new ideas. There is tremendous scope to further refine methods for how these ideas can be applied in practice.

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A Proofs

Proposition 1:*For the b_k coefficients that satisfy equation 9, each principal component Xb_k is spanned by the factors $X\beta_k$, $k = 1, \dots K_A$..*

Proof*Suppose b is an eigenvector that satisfies equation* (9) *with positive eigenvalue λ. b can be written as the sum $b^{span} + b^{\perp}$, where b^{span} is from Xb's projection onto the factors ($Xb^{span} = \sum_{k=1}^{K_A} c_k X\beta_k$) and $b^{\perp}$ where $Cov(Xb^{\perp}, X\beta_k) = b^{\perp\prime}\Sigma_{XX}\beta_k = 0$ for all $k = 1, \dots K_A$. Equation* (9) *then implies that*

$$\Sigma_{XX}^{-1}\Sigma_{E(X|Y)}[b^{span} + b^{\perp}] = \lambda[b^{span} + b^{\perp}] \tag{22}$$

If $\Sigma_{E(X|Y)}b^{\perp} = 0$, then

$$\Sigma_{XX}^{-1}\Sigma_{E(X|Y)}[b^{span}] = \lambda[b^{span} + b^{\perp}], \tag{23}$$

which implies $b^{span} + b^{\perp}$ cannot be an eigenvector as in equation (9) *unless $b^{\perp} = 0$. Therefore, to show solutions to* (9) *are spanned by the factors $X\beta_k, k = 1, \dots K_A$ it suffices to show:*

$$\Sigma_{E(X|Y)}b^{\perp} = 0. \tag{24}$$

If $[E(X|y) - E(X)]b^{\perp} = 0$, then $b^{\perp\prime}[E(X|y) - E(X)]'[E(X|y) - E(X)] = 0$ and $E(b^{\perp\prime}[E(X|y) - E(X)]'[E(X|y) - E(X)]) = b^{\perp\prime}\Sigma_{E(X|y)} = \Sigma_{E(X|Y)}b^{\perp} = 0$. Therefore, it suffices to show $E(Xb^{\perp}|y) - E(Xb^{\perp}) = 0$.

By the law of iterated expectations and the arguments of Y,

$$E(Xb^{\perp} - E(Xb^{\perp})|y) = E\left(E\{[Xb^{\perp} - E(Xb^{\perp})]|X\beta_1, \dots X\beta_{KA}, \epsilon\}|y\right).$$

By assumption 4 ϵ is independent of X, thus it suffices to show

$$E\left(E\{[Xb^{\perp} - E(Xb^{\perp})]|X\beta_1, \dots X\beta_{KA}\}|y\right) = 0.$$

It therefore is sufficient to show $E\{[Xb^{\perp} - E(Xb^{\perp})]|X\beta_1, \dots X\beta_{K_A}\} = 0$, or equivalently to show

$$E\left(E\{[Xb^{\perp} - E(Xb^{\perp})]|X\beta_1, \dots X\beta_{K_A}\}\right)^2 = 0$$

By assumption 5, $E(Xb^{\perp}|X\beta_1, \dots X\beta_{KA}) = \alpha_0 + \sum_{k=1}^{K_A}\alpha_k X\beta_k$. Therefore,

$$E\left(E\{[Xb^{\perp} - E(Xb^{\perp})]|X\beta_1, \dots X\beta_{K_A}\}\right)^2 =$$

$$
\begin{aligned}
&= E[b^{\perp\prime}(X - E(X))'(\alpha_0 + \sum_{k=1}^{K_A} \alpha_k X \beta_k)] \\
&= b^{\perp\prime}\alpha_0 E(X - E(X))' + \sum_{k=1}^{K} \alpha_k b^{\perp\prime} \Sigma_{XX} \beta_k \\
&= 0. \qquad (25)
\end{aligned}
$$

Therefore, $b^{\perp} = 0$, *and* b *that satisfy equation (9) are spanned by* $X\beta_1, \ldots X\beta_{K_A}$. $\square$

Proposition 2:*If SAD is linearly approximated by ASAD:*

$$
ASAD = C_0 + \sum_j D_j(R_j + CI_j R_f) \qquad (26)
$$

where R_j *satisfies equation (14), and if regulators objective function takes the form*

$$
Prob(ASAD \geq \xi) \leq \psi
$$

, then there is systemic risk factor shock F_A^* *such that when the stress scenario is* $X_i = \alpha_i + F_A^* \theta_i$ *for all* X_i*, and banks inject capital equal to the present value of their losses in the stress scenario, then after the capital is injected* $Prob(ASAD \geq \xi) \leq \psi$.

Proof: *Under the conditions of the proposition*

$$
\begin{aligned}
ASAD &= C_0 + \sum_j D_j(R_j + CI_j R_f) \\
&= C_0 + \sum_j D_j(alpha_j + F_A\theta_j + \epsilon_j) + \sum_j D_j CI_j R_f \qquad (27) \\
&= C_0 + \alpha + F_A\theta + \epsilon + CIE,
\end{aligned}
$$

In the first line of equation (27), SAD is approximated as the sum of a constant C_0 *and linear sensitivities* D_j *to* $(R_j + CI_j R_f)$*. The* D_j *coefficients are negative since when the banks asset portfolio has higher returns, the banks networth increases and its distress goes down. In the last line,* α, ϵ, *and CIE (capital injection equivalents) group terms involving* α_j, ϵ_j, *and* CI_j *respectively. In the above expression,* $F_A\theta + \epsilon$ *is a single random variable. Let* $H(.)$ *denote its CDF.*

The rest of the proof has two steps. The first step solves for the minimum CIE*, denoted* CIE^* *such that if banks inject enough capital to achive* CIE^**, then* $Prob(SAD \geq \xi) \leq \psi$*. The second step solves for a stress scenario that requires banks to inject enough capital to achieve* CIE^*.

Step 1: Solve for CIE^*.

Manipulation of equation (27) *shows*

$$Prob(SAD \geq \xi) = 1 - H(\xi - C_0 - \alpha - CIE). \tag{28}$$

The CIE^* *that solves*

$$1 - H(\xi - C_o\alpha - CIE^*) = \psi,$$

will be the smallest CIE that satisfies the condition $Prob(SAD \geq \xi) \leq \psi$. *Manipulating the above equation, it is given by:*

$$CIE^* = C_0 + \alpha - \xi - H^{-1}(1 - \psi). \tag{29}$$

Note that since $D_j(.) < 0$ *in equation 27, if more capital needs to be injected into the banking system, then* $CIE^* < 0$.

Step 2: Solve for F_A *and the stress-scenario.*

If the systemic risk factors are set to value F_A^*, *then in the resulting stress-scenrario, each banks gross return is* $R_j^* = \alpha_j + F_A^*\theta_j$. *The amount of capital banks need to raise in the scenario is equal to their losses* $-A_j(R_j^* - 1)$ *less any excess capital that was previously held. The excess capital just adds constant terms to the analysis. For simplicity, excess capital is assumed to be zero. As a fraction of its initial assets* A_j, *each bank requires additional date 1 capital* $\Delta Cap_j = -(R_j^* - 1)$. *Substituting the date 1 capital raised as a fraction of assets into equation* (27) *for* CI_jR_f *shows the capital raised alters SAD at date 1 by the amount*

$$\begin{aligned} \Delta SAD &= -\sum_j D_j(R_j^* - 1) \\ &= -(\alpha + F_A^*\theta - \sum_j D_j) \end{aligned}$$

If F_A^* *is chosen so that* $\Delta SAD = CIE^*$, *then the resulting stress scenario will require that banks inject enough capital for date 1 so that SAD at date 1 is reduced by the amount* CIE^*. *Therefore,* F_A^* *must be chosen so that*

$$F_A^*\theta = -CIE^* - \alpha + \sum_j D_j. \tag{30}$$

If there is only a single systemic risk factor, then θ *is a scalar, and*

$$F_A^* = \frac{-CIE^* - \alpha + \sum_j D_j}{\theta}.$$

If there are several systemic risk factors, then there are an infinite number of solutions for F_A^*, all of which satisfy equation (30)*.*

To verify that this is the correct solution, note that as a result of the stress scenario, at date 0 *each bank j will be required to inject capital $CI_j = \frac{-(\alpha_j + F_A^* \theta_j - 1)}{R_f}$ as a fraction of its date 0 assets. Since the capital is invested in risk free assets, it will grow to $-(\alpha_j + F_A^* \theta_j - 1)$ at date 1. Plugging into SAD and summing across j, SAD is changed by the amount*

$$\begin{aligned}
\Delta SAD &= -\sum D_j(\alpha_j + F_A^* \theta_j - 1) \\
&= -(\alpha + F_A^* \theta - \sum_j D_j) \\
&= -\left[\alpha + \left(\frac{-CIE^* - \alpha + \sum_j D_j}{\theta}\right)\theta - \sum_j D_j\right] \\
&= CIE^*
\end{aligned}$$

B Sliced Inverse Regression (SIR) and Correlation Pursuit (COP)

The purpose of the following two subsections is to provide additional information on the statistical interpretation of SIR, and details on the implementation of SIR and COP. A full description of the SIR methodology is described in Li (1991), Chen and Li (1998), and COP is described in Zhong et al (2012).

The *R* package *dr* contains a module for computing sliced inverse regression and related methods [Weisberg (2014).[38] An *R* package for COP can be downloaded from "http://cran.r-project.org/web/packages/COP [Zhong et al (2012)].

B.1 Sliced Inverse Regression

Section A shows that the principal components identified by SIR span a subspace of the space spanned by the systemic risk factors. The purpose of this subsection is to illustrate the relationship between the uncovered principal components and systemic risk. The exposition closely follows Chen and Li (1998). They show SIR can be interpreted as solving for a first pincipal direction b_1, that maximize the squared correlation between $X'b_1$ and a possibly nonlinear transformation $T(Y)$ of

[38] See http://cran.r-project.org/web/packages/dr/vignettes/overview.pdf

the Y variables:

$$\max_{T(Y),b_1} Corr(T(Y), X'b_1)^2.$$

For given b_1,the transformation of Y that is most correlated with $X'b_1$ is $E(X'b_1|Y)$. The squared correlation is given by

$$\left(\frac{Cov\{[E(X'b_1)|Y], Xb_1\}}{\sqrt{b_1'\Sigma_{XX}b_1 \times Var[E(X'b_1|y)]}} \right)^2 = \frac{b_1'\Sigma_{E(X|Y)}b_1}{b_1'\Sigma_{XX}b_1}$$

The correlation is homogenous of degree 0 in b_1. Therefore, restricting b_1 so that $b_1'\Sigma_{XX}b_1 = 1$ is without loss of generality. Solving for optimal b_1 subject to the restriction then implies $\Sigma_{E(X|Y)}b_1 = \lambda_1\Sigma_{XX}b_1$. Substituting in for $\Sigma_{E(X|Y)}b_1$, the squared correlation can be written as

$$Corr(E(Xb_1|Y), Xb_1)^2 = \frac{b_1'\Sigma_{XX}b_1\lambda_1}{b_1'\Sigma_{XX}b_1} = \lambda_1.$$

Therefore Xb_1 has maximal squared correlation λ_1 with a transformation of systemic impairment. Each additional principal direction b_k is orthogonal to the preceding principal directions, and maximizes $Corr^2(T(Y), X'b_k)$.

To perform SIR, it is necessary to compute $E(X|Y)$. As noted by Li (1991), this could be done by nonparametric regression, but to ease the computational burden, it is simply done by binning the data and taking sample averages within bins. More specifically, let Z be an $N \times P+1$ partitioned matrix of the data ($Z = [Y, X]$ where Y is $N \times 1$ and X is $N \times P$) that has been sorted by Y and partitioned into M bins that each have S rows, with $Z_m = (Y_m, X_m)$. The function $E(X|Y)$ is approximated by the sample average of the X's in each bin:

$$\bar{X}_m = \frac{1}{S}\sum_{s=1}^{S} X_m(s, .).$$

In addition, the unconditional expected value of X is estimated by $\bar{X} = \frac{1}{M}\bar{X}_m$, and Var($E(X|Y)$ by

$$\hat{\Sigma}_{E(X|Y)} = \frac{1}{M}\sum(\bar{X}_m - \bar{X})'(\bar{X}_m - \bar{X}),$$

and Var(X) is estimated by

$$Sig\hat{m}a_{XX} = \frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})'(X_i - \bar{X}).$$

The principal component factors are found by plugging the sample estimates into equation (9) and then solving for the eigenvalues and b_k coefficients.

Li (1991) shows that SIR's ability to identify the space spanned by the factors is robust to the size of the bin-size S. However, to choose the number of variables to use in SIR, Zhong et al (2012) finds the number of slices matter for performance. They find that choosing $S = 20$ performs well. For this reason in the analysis in this paper chooses $S = 20$ observations per slice.

B.2 Correlation Pursuit

Correlation pursuit chooses the variables to include in SIR to create factors. The universe of variables considered is denoted X where X is an $N \times P$ matrix of N realizations of P random variables. COP assumes that the number of factors is K. How K is chosen will be discussed below.

COP begins with a randomly chosen set of $K+1$ variables $A \in X$. The variables that are not in A are denoted A^C. To find the variables that are most suitable for creating factors to explain $T(Y)$, COP then scrolls through all P variables, and in doing so performs either a variable addition step, in which a variable is added to the set A, or a variable deletion step in which a variable is taken away. For a given K, the procedure continues until no more variables can be added or deleted. The final set of variables $A(K)$ is the set of variables that is chosen under the assumption that there are K factors.

Variable adddition step: To perform the variable addition step, let t denote a candidate X variable being considered for addition to A. To determine if the variable should be added, COP creates a test based on the scaled improvement in each of the K eigenvalues associated with the principal components when variable t is added.

The scaled improvement in the i'th eigenvalue is denoted COP_i^{A+t} given by

$$COP_i^{A+t} = N\frac{(\lambda_i^{A+t} - \lambda_i^A)}{1 - \lambda_i^A}, \tag{31}$$

where the superscripts A and $A+t$ denote the sets of variables used in computing the eigenvalues. Because the λ_i coefficients have the interpretation the R^2 from using the i'th factor to explain $T(Y)$, the COP_i^{A+t} statistics resemble F-tests for whether the addition of the variable t statistically improves the predictability attributable to a factor. The sum of these statistics is denoted $COP_{1:K}^{A+t}$ $(= \sum_{i=1}^K COP_i^{A+t})$. The statistic

$$\overline{COP}_{1:K} = \max_{t \in A^C} COP_{1:K}^{A+t};$$

and $X_{\bar{t}}$ is a variable that attains this maximum if added to A. The variable $X_{\bar{t}}$ is added to A if $\overline{COP}_{1:K} > c_e$, where c_e is a critical value for determining whether a variable should be added.

Variable deletion step: The variable deletion step is analogous to the the addition step. The scaled deterioration in the explanatory power of factor i from deleting the variable t from the set

A is given by

$$COP_i^{A-t} = N\frac{(\lambda_i^A - \lambda_i^{A-t})}{1-\lambda_i^A}. \tag{32}$$

The statistic $COP_{1:K}^{A-t} = \sum_{i=1}^K COP_i^{A-t}$ measures the deterioration in fit from deleting the variable t. The statistic $\underline{COP}_{1:K}$ denotes:

$$\underline{COP}_{1:K} = \min_{t\in A} COP_{1:K}^{A-t}, \tag{33}$$

and variable $X_{\underline{t}}$ attains this minimum. If $COP_{1:K} < c_d$, then variable $X_{\underline{t}}$ is deleted from A.

Zhong et al (2012) derive conditions under which as N goes to infinity with slice size (= bin size) fixed, COP consistently chooses X variables that should be used for SIR and consistently deletes variables that should not be used. When using COP in finite samples, they recommend choosing the critical values c_e and c_d through five-fold cross-validation.

The above shows how to select the variables for a given number of factors K. To choose K, Zhong et al (2012) used a BIC type information criterion based on Zhu et al (2006).

C Simulation of Interest Rate Changes Based on Wright (2011).

To be completed...

Figure 1: Scatter Plot of SAD versus extracted Systemic, Factor Long on Average Portfolio

Figure 2: Nonparametric Regression of SAD versus extracted Systemic Factor, Long on Average Portfolio

Figure 3: Stress Tests Based on Systemic Factor: Relation to SAD, Long on Average Portfolio

Figure 4: Stress Tests Based on Systemic Factor: Relation to SAD, Simulated Portfolio Sets 1 - 10, Long on Average Portfolio

Figure 5: Stress Tests Based on Systemic Factor: Relation to SAD: Simulated Portfolio Sets 1 - 10, Neither Long Nor Short Portfolio

Figure 6: QQ-Plot of Approximate SAD vs SAD: Simulated Portfolio 1

Figure 7: Optimal Stress-Scenario: Simulated Portfolio 1

Figure 8: Can SIR/COP detect the factors out of sample

For banks whose portfolios are highly correlated with one of the first 3 principal components of simulated yield curve changes, the figure examines if the factor that COP/SIR identifies using one subsample of the data with 5,000 observations (True F) is highly correlated with the COP/SIR factor estimated using only 500 observations from the same data generating process (F-COP).

Figure 9: The effect of one-standard deviation factor shock for a random bond portfolio

Figure 10: The effect of one-standard deviation factor shock for a random portfolio of stocks and bonds

Figure 11: SIR Simulations with occasional symmetry

Figure 12: SIR Simulation with symmetry: Scenario 2 – Corrected