Learning and Technology Growth Regimes

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Introduction

- Markov-Switching used in a Variety of Economic Environments
  - Monetary or Fiscal Policy

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  - Monetary or Fiscal Policy
  - Variances of Shocks
  - Persistence of Exogenous Processes
- Typically Models Assume Full Information about State Variables, Regime Perfectly Observed

Do We Always Know The Regime? - Empirical Evidence for Technology Growth Regimes

- Utilization-adjusted TFP (z_t)
- Investment-specific technology (u_t) measured as PCE deflator divided by nonresidential fixed investment deflator
- Estimate regime switching process

$$z_t = \mu_z\left(s_t^{\mu z}\right) + z_{t-1} + \sigma_z\left(s_t^{\sigma z}\right)\varepsilon_{z,t}$$

$$u_t = \mu_u\left(s_t^{\mu u}\right) + u_{t-1} + \sigma_u\left(s_t^{\sigma u}\right)\varepsilon_{u,t}$$

- Two Regimes Each

Data

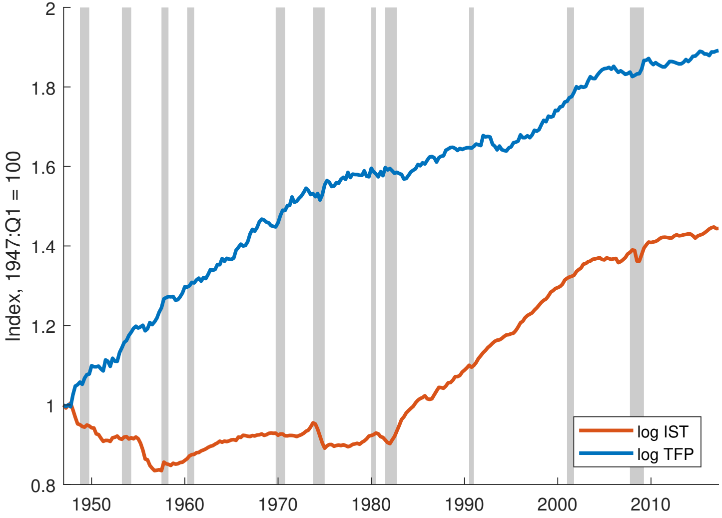
Index, 1947:Q1 = 100
2
1.8
1.6
1.4
1.2
1
0.8
1950
1960
1970
1980
1990
2000
2010
log IST
log TFP

Table: Evidence for TFP Regimes from Fernald

	Mean	Std Dev
1947-1973	2.00	3.68
1974-1995	0.60	3.44
1996-2004	1.99	2.66
2005-2017	0.39	2.35

Estimated (Filtered) Regimes

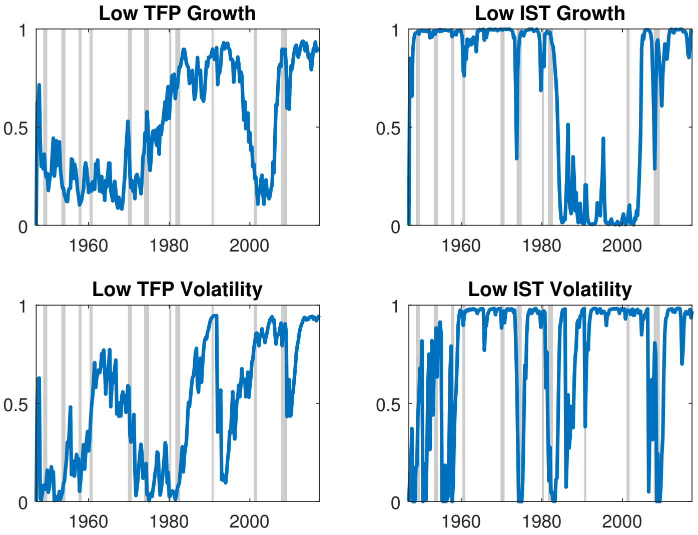

Table: Parameter Estimates

	$\mu_j\,(L)$	$\mu_j\,(H)$	$\sigma_j\,(L)$	$\sigma_j\,(H)$	$P_{LL}^{\mu j}$	$P_{HH}^{\mu j}$	$P_{LL}^{\sigma j}$	$P_{HH}^{\sigma j}$
TFP $(j = z)$	0.6872	1.8928	2.5591	3.8949	0.9855	0.9833	0.9803	0.9796
	(0.6823)	(0.6476)	(0.2805)	(0.3744)	(0.0335)	(0.0295)	(0.0215)	(0.0232)
IST $(j = u)$	0.4128	2.0612	1.1782	3.9969	0.9948	0.9839	0.9627	0.9026
	(0.1112)	(0.1694)	(0.0745)	(0.4091)	(0.0049)	(0.0162)	(0.0197)	(0.0463)

This Paper

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- Agents are Fully Rational

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- Key Issue: How do we Define the Steady State? What Point do we Approximate Around?
- Joint Approximations to Both Learning Process and Decision Rules

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- Use a RBC Model as a Laboratory

The Model

$$\tilde{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \xi \log\left(1 - l_t\right)\right]$$

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$$c_t + x_t = y_t$$

$$y_t = \exp(z_t)\, k_{t-1}^{\alpha} l_t^{1-\alpha}$$

$$k_t = (1-\delta)\, k_{t-1} + \exp(u_t)\, x_t$$

$$z_t = \mu_z\left(s_t^{\mu z}\right) + z_{t-1} + \sigma_z\left(s_t^{\sigma z}\right) \varepsilon_{z,t}$$

$$u_t = \mu_u\left(s_t^{\mu u}\right) + u_{t-1} + \sigma_u\left(s_t^{\sigma u}\right) \varepsilon_{u,t}$$

Bayesian Learning

$$\tilde{\mathbf{y}}_t = \tilde{\lambda}_{s_t}\left(\mathbf{x}_{t-1}, \varepsilon_t\right) = \left[u_t \ z_t\right]'$$

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$$\varepsilon_t = \lambda_{s_t}\left(\tilde{\mathbf{y}}_t, \mathbf{x}_{t-1}\right)$$

$$\psi_{i,t} = \frac{\overbrace{J_{s_t=i}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right)\phi^{\varepsilon}\left(\lambda_{s_t=i}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right)\right)}^{likelihood}\overbrace{\sum_{s=1}^{n_s} p_{s,i}\psi_{s,t-1}}^{prior}}{\sum_{j=1}^{n_s} J_{s_t=j}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right)\phi^{\varepsilon}\left(\lambda_{s_t=j}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right)\right)\sum_{s=1}^{n_s} p_{s,j}\psi_{s,t-1}}.$$

Bayesian Learning

- To Keep Probabilities between 0 and 1, Define

$$\eta_{i,t} = \log\left(\frac{\psi_{i,t}}{1-\psi_{i,t}}\right)$$

- $$\tilde{E}_t \mathbf{f}\left(...\right) = \sum_{s=1}^{n_s} \sum_{s'=1}^{n_s} \frac{p_{s,s'}}{1+\exp\left(-\eta_{s,t}\right)} \int \widetilde{\mathbf{f}}\left(...\right) \phi_{\varepsilon}\left(\varepsilon'\right)$$

Issues When Applying Perturbation to MS Models

- What Point Should we Approximate Around?
- What Markov-Switching Parameters Should be Perturbed?
- Best Understood in an Example - Let's Assume we are Only Interested in Approximating (a Stationary) TFP Process

Equilibrium Conditions

- Equilibrium Conditions

$$\begin{bmatrix} \mu\left(s_t\right) + \sigma\left(s_t\right)\varepsilon_t - z_t \\ \log\left(\frac{\frac{1}{\sigma(1)}\phi_\varepsilon\left(\frac{z_t-\mu(1)}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,1}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}{\frac{1}{\sigma(2)}\phi_\varepsilon\left(\frac{z_t-\mu(2)}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,2}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}\right) - \eta_{1,t} \\ \log\left(\frac{\frac{1}{\sigma(2)}\phi_\varepsilon\left(\frac{z_t-\mu(2)}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,2}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}{\frac{1}{\sigma(1)}\phi_\varepsilon\left(\frac{z_t-\mu(1)}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,1}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}\right) - \eta_{2,t} \end{bmatrix} = 0$$

Steady State

- Steady State of TFP Independent of σ_z
- (In RBC Model We Rescale all Variables to Make Everything Stationary)
- The First Equation Would Also Appear in a Full Information Version of the Model
- So Under Full Information Perturbing σ_z is not Necessary and Leads to Loss of Information - *Partition Principle* of Foerster, Rubio, Waggoner & Zha
- What About Learning?

Steady State

- Steady State with Naive Perturbation

$$\begin{bmatrix} \bar{\mu} - z_{ss} \\ \log\left(\dfrac{\frac{1}{\overline{\sigma}}\phi_{\varepsilon}\left(\frac{z_{ss}-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{p_{1,1}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{p_{2,1}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}{\frac{1}{\overline{\sigma}}\phi_{\varepsilon}\left(\frac{z_{ss}-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{p_{1,2}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{p_{2,2}}{1+\exp\left(-\eta_{2,ss}\right)}\right)} \right) - \eta_{1,ss} \\ \log\left(\dfrac{\frac{1}{\overline{\sigma}}\phi_{\varepsilon}\left(\frac{z_{ss}-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{p_{1,2}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{p_{2,2}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}{\frac{1}{\overline{\sigma}}\phi_{\varepsilon}\left(\frac{z_{ss}-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{p_{1,1}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{p_{2,1}}{1+\exp\left(-\eta_{2,ss}\right)}\right)} \right) - \eta_{2,ss} \end{bmatrix} = 0$$

Steady State

- Steady State with Naive Perturbation

$$\begin{bmatrix} \bar{\mu} - z_{ss} \\ \log\left(\frac{\frac{1}{\overline{\sigma}}\phi_{\varepsilon}\left(\frac{z_{ss}-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{p_{1,1}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,1}}{1+\exp(-\eta_{2,ss})}\right)}{\frac{1}{\overline{\sigma}}\phi_{\varepsilon}\left(\frac{z_{ss}-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{p_{1,2}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,2}}{1+\exp(-\eta_{2,ss})}\right)} \right) - \eta_{1,ss} \\ \log\left(\frac{\frac{1}{\overline{\sigma}}\phi_{\varepsilon}\left(\frac{z_{ss}-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{p_{1,2}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,2}}{1+\exp(-\eta_{2,ss})}\right)}{\frac{1}{\overline{\sigma}}\phi_{\varepsilon}\left(\frac{z_{ss}-\overline{\mu}}{\overline{\sigma}}\right)\left(\frac{p_{1,1}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,1}}{1+\exp(-\eta_{2,ss})}\right)} \right) - \eta_{2,ss} \end{bmatrix} = 0$$

- $\eta_{2,ss} = \eta_{1,ss}$ if Probability of Staying in a Regime is the Same Across Regimes
- in general $\eta_{j,ss} = f(P)\ \forall j = 1,2$

- Steady State with Partition Principle

$$\begin{bmatrix} \bar{\mu} - z_{ss} \\ \log\left(\frac{\frac{1}{\sigma(1)}\phi_\varepsilon\left(\frac{z_{ss}-\overline{\mu}}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{p_{2,1}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}{\frac{1}{\sigma(2)}\phi_\varepsilon\left(\frac{z_{ss}-\overline{\mu}}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{p_{2,2}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}\right) - \eta_{1,ss} \\ \log\left(\frac{\frac{1}{\sigma(2)}\phi_\varepsilon\left(\frac{z_{ss}-\overline{\mu}}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{p_{2,2}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}{\frac{1}{\sigma(1)}\phi_\varepsilon\left(\frac{z_{ss}-\overline{\mu}}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{p_{2,1}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}\right) - \eta_{2,ss} \end{bmatrix} = 0$$

- Steady State with Partition Principle

$$\begin{bmatrix} \bar{\mu} - z_{ss} \\ \log\left(\frac{\frac{1}{\sigma(1)}\phi_\varepsilon\left(\frac{z_{ss}-\bar{\mu}}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,1}}{1+\exp(-\eta_{2,ss})}\right)}{\frac{1}{\sigma(2)}\phi_\varepsilon\left(\frac{z_{ss}-\bar{\mu}}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,2}}{1+\exp(-\eta_{2,ss})}\right)}\right) - \eta_{1,ss} \\ \log\left(\frac{\frac{1}{\sigma(2)}\phi_\varepsilon\left(\frac{z_{ss}-\bar{\mu}}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,2}}{1+\exp(-\eta_{2,ss})}\right)}{\frac{1}{\sigma(1)}\phi_\varepsilon\left(\frac{z_{ss}-\bar{\mu}}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,1}}{1+\exp(-\eta_{2,ss})}\right)}\right) - \eta_{2,ss} \end{bmatrix} = 0$$

- $\eta_{2,ss} \neq \eta_{1,ss}$, but only because of σ_z and P - steady state model probabilities not account for differences in means of regimes
- Note that we can generally solve for the steady state of those variables that appear in the full information version of the model *independently* of model probabilities

Partition Principle Refinement

$$\begin{bmatrix} \bar{\mu} - z_{ss} \\ \log\left(\frac{\frac{1}{\sigma(1)}\phi_\varepsilon\left(\frac{z_{ss}-\mu(1)}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,1}}{1+\exp(-\eta_{2,ss})}\right)}{\frac{1}{\sigma(2)}\phi_\varepsilon\left(\frac{z_{ss}-\mu(2)}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,2}}{1+\exp(-\eta_{2,ss})}\right)} \right) - \eta_{1,ss} \\ \log\left(\frac{\frac{1}{\sigma(2)}\phi_\varepsilon\left(\frac{z_{ss}-\mu(2)}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,2}}{1+\exp(-\eta_{2,ss})}\right)}{\frac{1}{\sigma(1)}\phi_\varepsilon\left(\frac{z_{ss}-\mu(1)}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp(-\eta_{1,ss})}+\frac{p_{2,1}}{1+\exp(-\eta_{2,ss})}\right)} \right) - \eta_{2,ss} \end{bmatrix} = 0$$

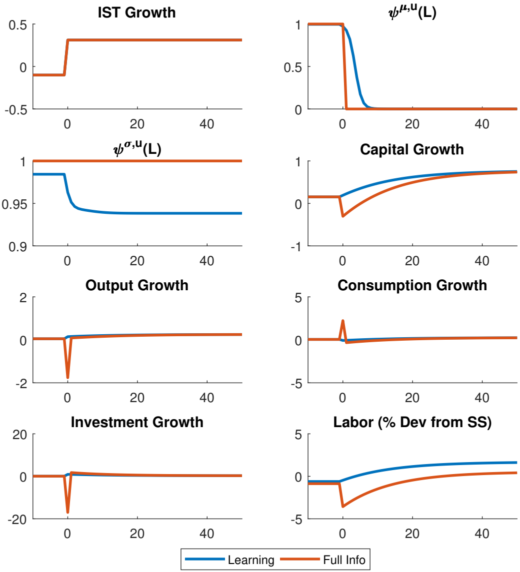
- We Find Model Probabilities That are Consistent With The Full Information Steady State Under the Partition Principle,but Take Into Account All Differences Between the Parameters of the Regimes
- Then we Apply the Methods from Foerster, Rubio, Waggoner & Zha

Back To The RBC Model

Table: Accuracy Check of Simple RBC Model

Method	MSE of Beliefs	Euler Eqn Error
Partition Principle		
First Order	0.3989	−3.0298
Second Order	0.3753	−2.4253
Third Order Order	0.3770	−2.2715
Refinement		
First Order	0.3200	−4.3101
Second Order	0.0511	−3.4995
Third Order Order	0.0519	−3.6722
Policy Function Iteration	0	−4.3042

- Why is First Order Doing so Well in Terms of EE? Expectations are Computed Using Different Model Probabilities for Each Order.



IST Growth
0.5
0
-0.5
0
20
40
$\psi^{\mu,u}$(L)
1
0.5
0
$\psi^{\sigma,u}$(L)
0.95
0.9
Capital Growth
-1
Output Growth
2
-2
Consumption Growth
5
-5
Investment Growth
20
-20
Labor (% Dev from SS)
Learning
Full Info

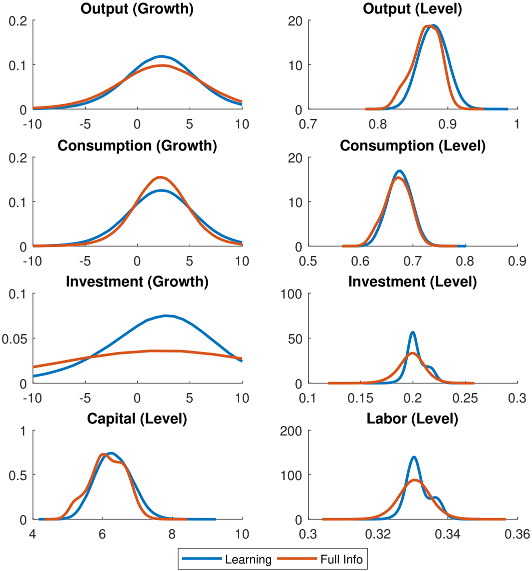

Output (Growth)
0.2
0.1
0
-10 -5 0 5 10
Output (Level)
20
10
0
0.7 0.8 0.9 1
Consumption (Growth)
0.2
0.1
0
-10 -5 0 5 10
Consumption (Level)
20
10
0
0.5 0.6 0.7 0.8 0.9
Investment (Growth)
0.1
0.05
0
-10 -5 0 5 10
Investment (Level)
100
50
0
0.1 0.15 0.2 0.25 0.3
Capital (Level)
1
0.5
0
4 6 8 10
Labor (Level)
200
100
0
0.3 0.32 0.34 0.36
Learning
Full Info

Table: Economic Effects of Learning

	Full Info		Learning		Pct Difference	
	Mean	Std Dev	Mean	Std Dev	Mean	Std Dev
Growth Variables						
Output $(400 \cdot \Delta \log y)$	2.2630	4.3014	2.2630	3.5405	0	-17.69
Consumption $(400 \cdot \Delta \log c)$	2.2630	2.8258	2.2630	3.3344	0	17.99
Investment $(400 \cdot \Delta \log x)$	2.2630	13.6136	2.2630	6.2904	0	-53.79
Detrended Variables						
Output $(\tilde{y})$	0.8689	0.0204	0.8789	0.0203	1.15	-0.49
Consumption $(\tilde{c})$	0.6703	0.0249	0.6746	0.0231	0.64	-7.23
Investment $(\tilde{x})$	0.1986	0.0131	0.2037	0.0094	2.57	-28.24
Labor (l)	0.3305	0.0047	0.3316	0.0036	0.33	-23.40
Capital $(\tilde{k})$	6.1430	0.5151	6.3156	0.5138	2.81	-0.25

Conclusions

- A Framework for the Nonlinear Approximation of Limited Information Rational Expectations Models
- Example Provides a Lower Bound for the Importance of Learning: No Feedback Effects (Think About an Endogenous Variable Multiplying a Regime-Dependent Coefficient)
- Opens up Avenues to Think About Multiple Equilibria in Learning Models
- Could be Extended to Allow for Disperse Beliefs Across Agents