Rational Sunspots

Guido Ascari[a], Paolo Bonomolo[b], HedibertF.Lopes[c]
a University of Oxford, b Sveriges Riksbank[1]*, c INSPER*

Nonlinear Models in Macroeconomics and Finance for an Unstable World
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[1]The views expressed are solely the responsibility of the authors and should not to be interpreted as reflecting the views of Sveriges Riksbank.

This Paper

1. Propose a method to consider a broader class of solutions to stochastic linear models. Two generalizations:
   - A novel way to introduce sunspots that yields drifting parameters and stochastic volatility
   - Include temporary unstable solutions: we allow for determinacy, indeterminacy and instability

2. Develop an econometric strategy to verify if unstable paths are empirically relevant

3. Application:
   - Example of U.S. Great inflation (Lubik and Schorfheide, 2004, model and data)
   - U.S. inflation dynamics in the 70's are better described by unstable equilibrium paths.

Motivation

- RE generally implies multiple equilibria
  - Explosive
  - Stable
- How can we get uniqueness? (Sargent and Wallace ,1973; Phelps and Taylor, 1977; Taylor, 1977; Blanchard, 1979)
- *Stability Criterion*: Transversality conditions
  In saddle paths dynamics only one solution is stable
- This became the standard in Macroeconomics (Blanchard and Kahn, 1980)

Example: U.S. Great Inflation period

Is it appropriate to rule out unstable paths from the empirical analysis?

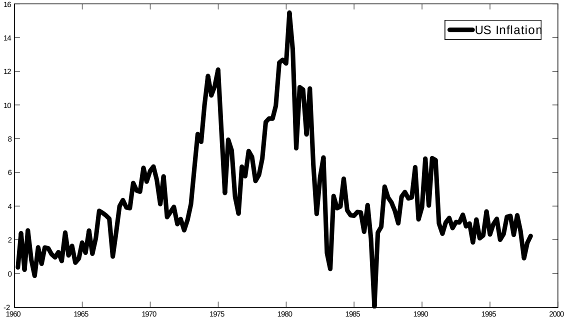

Figure: CPI inflation, quarterly data. Sample: 1960Q1 - 1997Q4

Is there any evidence that inflation is described (at least for a while) by unstable equilibria?

A simple example: multiple RE solutions

Consider the following simple one equation model:

$$y_t = \frac{1}{\theta} E_t y_{t+1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \tag{1}$$

Equation (1) has an infinite number of solutions:

$$\begin{aligned} y_{t+1} &= E_t y_{t+1} + \eta_{t+1} \\ y_{t+1} &= \theta y_t - \theta \varepsilon_t + \eta_{t+1} \end{aligned} \tag{2}$$

where $E_t \eta_{t+1} = 0$.
Assume:

$$\eta_{t+1} = (1 + M)\varepsilon_{t+1} + \zeta_{t+1}$$

where ζ_{t+1} = sunspot or non-fundamental error.

Two sources of multiplicity:
This paper considers the FIRST term: intrinsic multiplicity of RE solutions

A simple example: multiple RE solutions

Assume $\zeta_t = 0 \; \forall t$. All the solutions for y_t are described by

$$y_t = \theta y_{t-1} - \theta \varepsilon_{t-1} + (1 + M)\varepsilon_t \tag{3}$$

Degree of freedom: the solution is parameterized by $M \in (-\infty, +\infty)$

- Two famous particular cases:

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- Two famous particular cases:
  - "pure" forward-looking solution: $M = 0$ ($\eta_t = \varepsilon_t$)

$$\begin{aligned} y_t^F - \theta y_{t-1}^F &= \varepsilon_t - \theta \varepsilon_{t-1} \\ y_t^F &= \varepsilon_t \ \ \forall t \end{aligned} \tag{4}$$

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  - "pure" backward-looking solution $(M = -1)$

$$y_t^B = \theta y_{t-1}^B - \theta \varepsilon_{t-1} \tag{5}$$

The interpretation for M

- For $M \neq -1$, the expected value = an exponentially weighted average of the past observations (Muth, 1961)

$$E_t y_{t+1} = M \sum_{i=0}^{t} \left(\frac{\theta}{1+M} \right)^i y_{t+1-i}$$

Natural interpretation for M: the way agents form expectations

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Natural interpretation for M: the way agents form expectations

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- Infinite solutions = infinite way we can set that weights => how to choose?

The stability criterion (e.g., Blanchard and Kahn, 1980)

$$y_t = \theta y_{t-1} - \theta \varepsilon_{t-1} + (1+M)\varepsilon_t$$

Is the stability criterion sufficient to identify a unique path?

1. If $|\theta| > 1$ *YES* determinacy, by imposing $M = 0 =$ f.l. solution $y_t^F = \varepsilon_t$ (MSV solution)
2. If $|\theta| < 1$ *NO* indeterminacy

$=>$ *"Sunspot equilibria can often be constructed by randomizing over multiple equilibria of a general equilibrium model"* Benhabib and Farmer (1999, p.390)

Introducing sunspot equilibria

We have infinite equilibria because:

- there is an infinite number of ways of forming expectations
- parametrized by M

hence we introduce sunspots randomizing over M:

$$M_t = M_t(\zeta_t) \tag{6}$$

where ζ_t i.i.d., orthogonal to the fundamental shocks ε_s $(s = 1, 2, ...)$, and $E_t\zeta_t = 0 \ \forall t$.

Introducing sunspot equilibria: drifting parameters and unstable paths

If M_t is a stochastic process with $E_t M_{t+1} = M_t$ then

$$y_t = \alpha_t y_{t-1} - \alpha_t \varepsilon_{t-1} + (1 + M_t)\varepsilon_t \tag{7}$$

with $\alpha_t = \theta \frac{M_t}{M_{t-1}}$ (with $M_{t-1} \neq 0$ otherwise FL solution).

- Same form as $y_t = \theta y_{t-1} - \theta \varepsilon_{t-1} + (1 + M)\varepsilon_t$

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- Intuition:agents can modify in every period the expectation formation process
- Reconsidering unstable paths: $|\theta| > 1$ *and M_t temporarily different from zero*

Which process for M? RE Solutions

- With M_t random variable the forecast error becomes:

$$\eta_t = (1 + M_t)\varepsilon_t + (M_t - M_{t-1}) \left(\sum_{i=1}^{t-1} \theta^i \varepsilon_{t-i} \right) \tag{8}$$

- Under RE $E_{t-1}\eta_t = 0$, then:

1. $E_{t-1}(M_t) = M_{t-1}$ (M_t martingale)
2. $E_{t-1}\left[(1 + M_t)\varepsilon_t\right] = 0$, ($M_t$ must be uncorrelated with ε_t)

if $|\theta| < 1$: Use conditions 1 and 2

if $|\theta| > 1$: Unstable paths. Consider the role of transversality condition

Temporarily explosive paths

- All the paths corresponding to expected temporary deviations of M_t from 0 will not violate the transversality condition
- RE requires M_t martingale, but
  if $|\theta| > 1$ and M_t martingale, when $M_t \neq 0$ the economy is expected to remain on the unstable path, so transversality condition would be violated

$=>$ To allow for temporary unstable paths relax the martingale assumption (and RE)
Deviation can be minimal without practical implications

Mprocess

Example: Lubik and Schorfheide (2004) model

$$
\begin{aligned}
x_t &= E_t(x_{t+1}) - \tau(R_t - E_t(\pi_{t+1})) + g_t \\
\pi_t &= \beta E_t(\pi_{t+1}) + \kappa(x_t - z_t) \\
R_t &= \rho_R R_{t-1} + (1-\rho_R)(\psi_1 \pi_t + \psi_2(x_t - z_t)) + \varepsilon_{R,t}
\end{aligned}
$$

and

$$
g_t = \rho_g g_{t-1} + \varepsilon_{g,t}; \qquad z_t = \rho_z z_{t-1} + \varepsilon_{z,t}
$$

allow for non-zero correlation between the two shocks: ρ_{gz}

Compare two "models": M_S (stable solutions) and M_U (unstable solutions).

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The estimation strategy

- The model has stochastic volatility, then the likelihood distribution is not Gaussian
- Common practice in non linear DSGE literature: use particle filter to approximate the likelihood in MCMC (Fernandez-Villaverde and Rubio-Ramirez, 2007)
- **Different approach**: Particle filter to approximate the posterior distribution of the parameters and M_t
- Sequential Learning on the parameters: how inference evolves over time gives additional information about the role of sunspots and unstable paths

Particle filter

1. Marginalization. I: all latent states different from M; y: data

$$p(I, M|y) = \underbrace{p(I|y, M)}_{\text{Kalman Filter}} \underbrace{p(M|y)}_{\text{Particle Filter}}$$

2. Parameter learning, combining:
    1. Particle Learning by Carvalho, Johannes, Lopes and Polson (2010):
    2. Liu and West (2001)

See also Chen, Petralia and Lopes (2010)

Priors and Distributions: same as in LS

Table 1: Prior Distributions

Parameter	Density	Mean	Standard Deviation
ψ_1	Gamma	1.1	0.5
ψ_2	Gamma	0.25	0.15
ρ_R	Beta	0.5	0.2
π^*	Gamma	4	2
r^*	Gamma	2	1
κ	Gamma	0.5	0.2
τ^{-1}	Gamma	2	0.5
ρ_g	Beta	0.7	0.1
ρ_z	Beta	0.7	0.1
γ	Beta	0.8	0.15
σ_R	Inverse Gamma	0.31	0.16
σ_ζ	Inverse Gamma	0.1	0.05
Variance Covariance	**Density**	**Scale**	**Degrees of freedom**
Σ_{gz}	Inverse Wishart	$3\begin{bmatrix} 0.38^2 & 0 \\ 0 & 1 \end{bmatrix}$	8

▸ Models

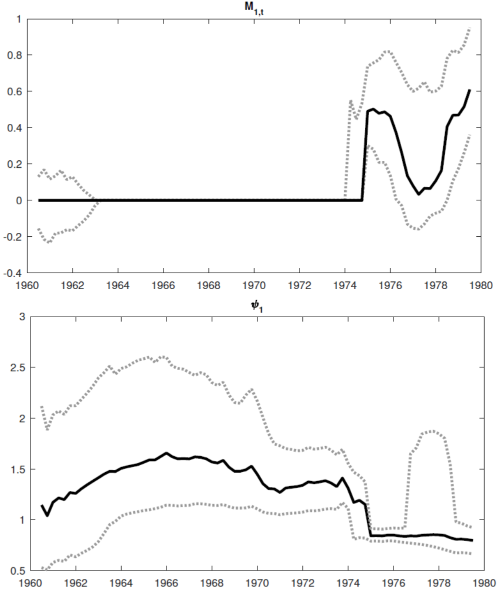
Estimates Great Inflation sample

	Pre- Volcker 1960:I - 1979:II		
Parameter	M_S	M_U	LS
ψ_1	0.80 [0.66 0.92]	0.76 [0.61 0.91]	0.77 [0.64 0.91]
ψ_2	0.16 [0.11 0.20]	0.20 [0.16 0.34]	0.17 [0.04 0.30]
ρ_R	0.68 [0.65 0.71]	0.60 [0.53 0.68]	0.60 [0.42 0.78]
π^*	1.90 [1.62 2.25]	1.73 [1.31 2.47]	4.28 [2.21 6.21]
r^*	1.41 [1.29 1.58]	1.23 [0.93 1.74]	1.13 [0.63 1.62]
κ	0.14 [0.10 0.18]	0.10 [0.07 0.14]	0.77 [0.39 1.12]
τ^{-1}	3.41 [2.65 4.51]	3.02 [2.46 3.74]	1.45 [0.85 2.05]
ρ_g	0.64 [0.59 0.69]	0.68 [0.63 0.74]	0.68 [0.54 0.81]
ρ_z	0.76 [0.72 0.80]	0.75 [0.67 0.81]	0.82 [0.72 0.92]
ρ_{gz}	0.26 [0.19 0.37]	0.16 [0.06 0.25]	0.14 [−0.4 0.71]
γ	–	0.96 [0.85 0.99]	–
σ_R	0.22 [0.2 0.26]	0.19 [0.16 0.22]	0.23 [0.19 0.27]
σ_g	0.35 [0.3 0.4]	0.31 [0.24 0.37]	0.27 [0.17 0.36]
σ_z	1.11 [0.97 1.29]	1.00 [0.85 1.31]	1.13 [0.95 1.30]
σ_ς	0.08 [0.07 0.1]	0.06 [0.05 0.08]	0.20 [0.12 0.27]

90% credibility interval in brackets

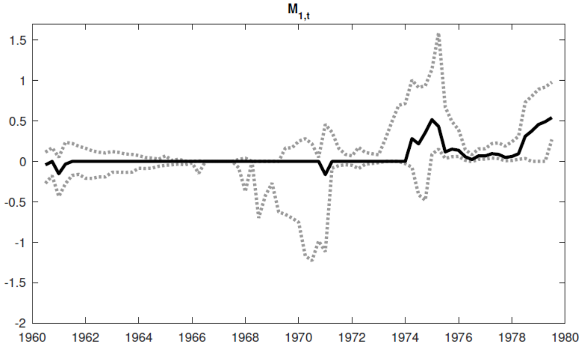
Stable Model, Great Inflation:

Estimated path for M_t and sequential inference on the parameter ψ_1.



Unstable Model, Great Inflation

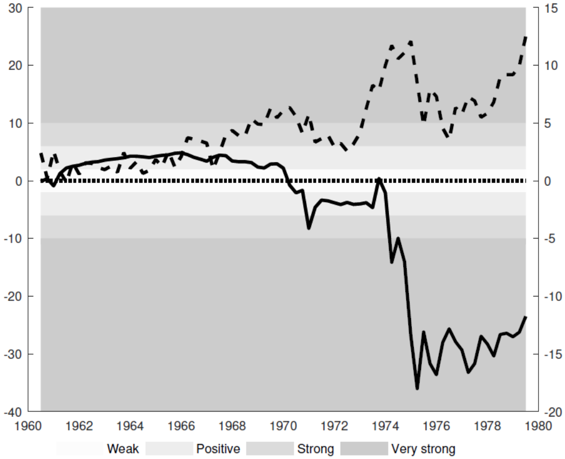
The behavior of M_t



▸ IRFs ▸ Models

Comparing the relative fit of Ms/Mu

Cumulative Bayes Factor: $2\ln(W_t)$ and the inflation rate



The Bayes Factor strongly favours the unstable model

Estimates Great Moderation sample

	Post-82 1982:IV - 1997:IV		
Parameter	M_S	M_U	LS
ψ_1	2.07 [1.50 2.81]	2.12 [1.07 3.65]	2.19 [1.38 2.99]
ψ_2	0.16 [0.06 0.39]	0.51 [0.19 1.38]	0.30 [0.07 0.51]
ρ_R	0.86 [0.81 0.90]	0.85 [0.80 0.89]	0.84 [0.79 0.89]
π^*	3.18 [2.73 3.74]	3.22 [2.74 3.74]	3.43 [2.84 3.99]
r^*	2.65 [1.87 3.56]	2.81 [2.17 3.60]	3.01 [2.21 3.80]
κ	0.28 [0.20 0.40]	0.64 [0.38 1.09]	0.58 [0.27 0.89]
τ^{-1}	2.52 [1.90 3.41]	1.28 [0.85 1.88]	1.86 [1.04 2.64]
ρ_g	0.77 [0.68 0.84]	0.81 [0.74 0.87]	0.83 [0.77 0.89]
ρ_z	0.71 [0.62 0.80]	0.78 [0.70 0.84]	0.85 [0.77 0.93]
ρ_{gz}	0.03 [0.00 0.07]	0.04 [0.01 0.08]	0.36 [0.06 0.67]
γ	–	0.05 [0.02 0.11]	–
σ_R	0.16 [0.13 019]	0.16 [0.14 0.2]	0.18 [0.14 0.21]
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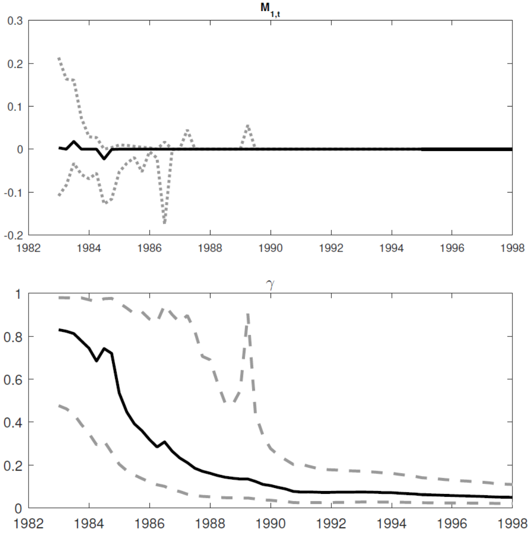
Unstable Model, Great Inflation

Estimated path for M_t and sequential inference on the parameter γ.

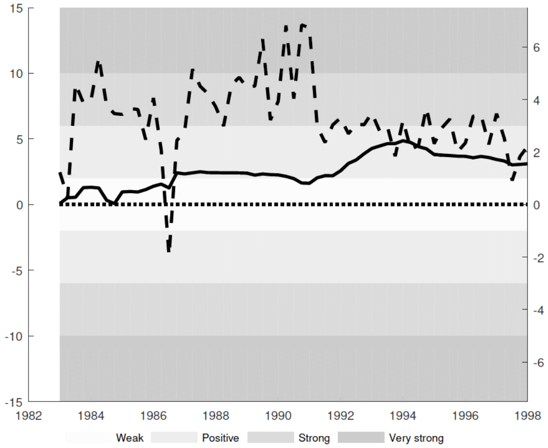


Model M_U selects the unique stable solution

▸ Models

Comparing the relative fit of Ms/Mu

Cumulative Bayes Factor: $2\ln(W_t)$ and the inflation rate



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  - Drifting parameters and stochastic volatility
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- Our methodology allows the data to choose between different possible alternatives: determinacy, indeterminacy and instability
- When the data are allowed this possibility, they unambiguously select the unstable model to explain the stagflation period in the ‘70s
- **Temporary unstable paths** can be empirically relevant and should not be excluded a priori

EXTRA

Which process for M? Temporary unstable paths

To guarantee the transversality condition relax the martingale assumption (and RE):

$$M_t = N_t A_{t-1}$$

- N_t martingale
- $A_t \in \{0, 1\}$ non increasing random sequence

Indicate with $\bar{T}$ random variable: $\bar{T} = \inf\{t : A_t = 0\}$

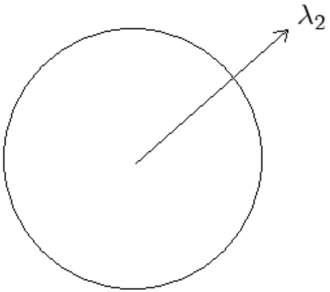
Properties of M_t :

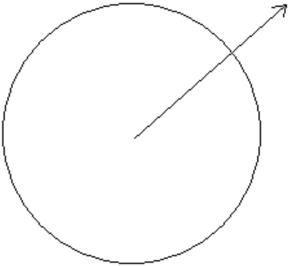
1. $$E_t(M_{t+1}) = M_t \quad \text{for } t < \bar{T} \text{ and } t > \bar{T}$$
   $$E_t(M_{t+1}) = 0 \quad \text{for } t = \bar{T}$$

   In general $E_t(\eta_{t+1}) = 0$.

2. $$\lim_{h\to\infty} E_t M_{t+h} = 0$$

   The transversality condition holds ▸ backl


λ_2



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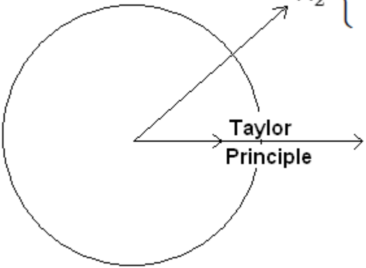
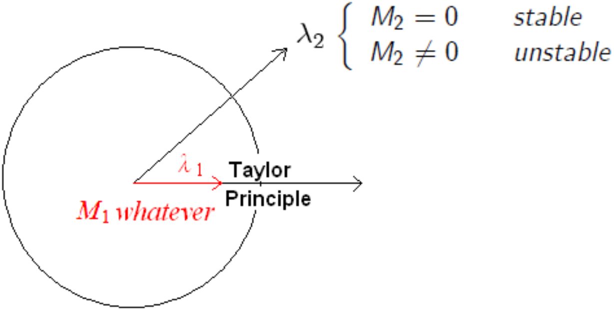
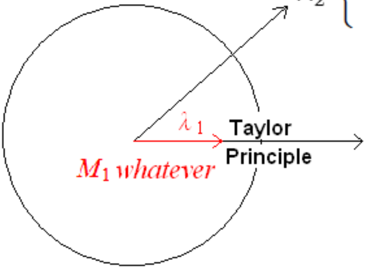



λ2
M2 = 0 stable
M2 ≠ 0 unstable
λ 1
Taylor Principle
M1 whatever



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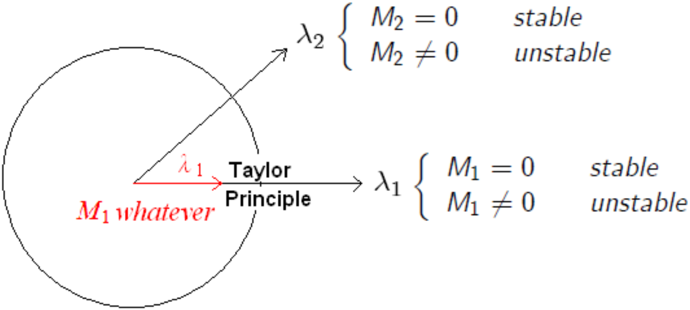
$$\lambda_1 \begin{cases} M_1 = 0 & \textit{stable} \\ M_1 \neq 0 & \textit{unstable} \end{cases}$$



Compare two "models": M_S (stable solutions) and M_U (unstable solutions).

▸ Model

Assumption on M

- UNDER STABILITY M_S
  If Taylor principle respected (determinacy) $=> M_t = 0 \; \forall t$
  If Taylor principle **not** respected (indeterminacy): $M_t = M_{t-1} + \zeta_t$

- UNDER INSTABILITY M_U

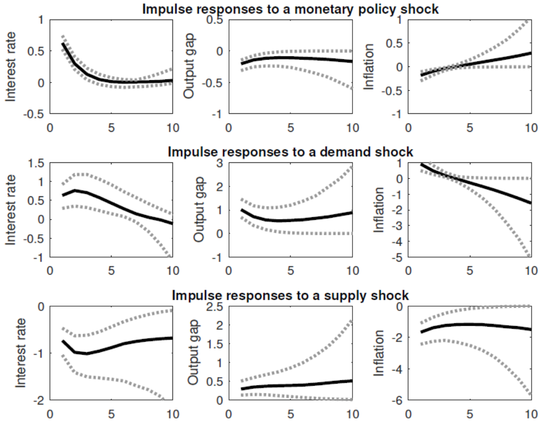
$$M_t = N_t A_{t-1}$$

with

$$N_t = \begin{cases} N_{t-1}/\gamma + \zeta_t & \text{with probability } \gamma \\ 0 & \text{with probability } 1-\gamma \end{cases}$$

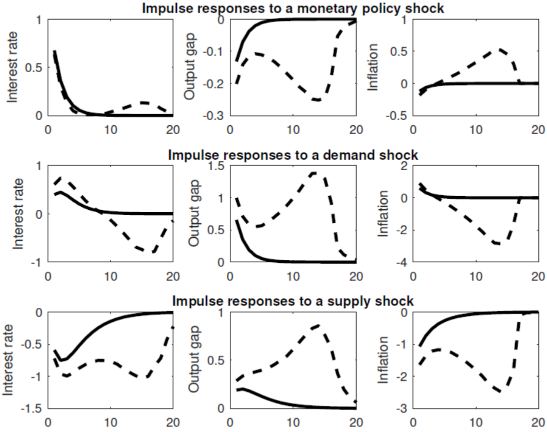
Model MuGI MuGM priors

Unstable Model, Great Inflation



Transmission mechanism of structural shocks: GIRF in the M_U model

Unstable Model, Great Inflation



Transmission mechanism of sunspot shock: GIRF in the M_U model: solid line: $M = 0$, dashed line: $M = 0.52$

back