A Class of Time-Varying Parameter Structural VARs for Inference under Exact or Set Identification

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Boring Fed disclaimer

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To fix ideas

$$\underset{(1\times n)}{\mathbf{y}_t'}\underset{(n\times n)}{\mathbf{A}} = \mathbf{y}_{t-1}'\underset{(n\times n)}{\mathbf{F}_1} + \cdots + \mathbf{y}_{t-p}'\mathbf{F}_p + \underset{(1\times n)}{\mathbf{c}} + \underset{(1\times n)}{\varepsilon_t'}\ , \quad \varepsilon_t \sim N(\mathbf{0}, \mathbf{I}_n)$$

Define

$$\mathbf{x}_t \equiv [\mathbf{y}_{t-1}', \dots, \mathbf{y}_{t-p}', 1]' \quad \text{and} \quad \mathbf{F} \equiv [\mathbf{F}_1', \dots, \mathbf{F}_p', \mathbf{c}']'$$

Write

$$\mathbf{y}_t'\mathbf{A} = \mathbf{x}_t'\mathbf{F} + \varepsilon_t'$$

Want to infer $(\mathbf{A}, \mathbf{F})$ because they

- represent equilibrium relationships between variables
- determine response of $\mathbf{y}_t$ to the mutually orthogonal "structural" shocks in ε_t

But $(\mathbf{A}, \mathbf{F})$ don't come for free.

The identification problem

Rewriting the SVAR

$$\mathbf{y}_t' = \mathbf{x}_t'\mathbf{F}\mathbf{A}^{-1} + \varepsilon_t'\mathbf{A}^{-1} \,,$$

Likelihood for $\mathbf{y}_t$

$$p(\mathbf{y}_t|\mathbf{A},\mathbf{F},\mathbf{y}_{t-p:t-1}) = Npdf(\mathbf{y}_t|\underbrace{\mathbf{x}_t'\mathbf{F}\mathbf{A}^{-1}}_{\mu},\underbrace{(\mathbf{A}\mathbf{A}')^{-1}}_{\boldsymbol{\Sigma}})$$

But consider the alternative parameter point $(\widetilde{\mathbf{A}},\widetilde{\mathbf{F}})$

$$(\widetilde{\mathbf{A}},\widetilde{\mathbf{F}}) = (\mathbf{AQ},\mathbf{FQ}) \quad \text{for } \mathbf{Q} \in \mathcal{O}_n$$

$$\mu = \widetilde{\mathbf{F}}\widetilde{\mathbf{A}}^{-1} = \mathbf{FQ}(\mathbf{AQ})^{-1} = \mathbf{FQQ}^{-1}\mathbf{A}^{-1} = \mathbf{FA}^{-1}$$

$$\boldsymbol{\Sigma} = \widetilde{\mathbf{A}}\widetilde{\mathbf{A}}' = (\mathbf{AQ})(\mathbf{AQ})' = \mathbf{AQQ}'\mathbf{A}' = \mathbf{AA}'$$

Hence, we cannot identify $(\mathbf{A},\mathbf{F})$.

The reduced-form VAR

We can identify

$$g(\mathbf{A}, \mathbf{F}) = (\mathbf{F}\mathbf{A}^{-1}, \mathbf{A}\mathbf{A}') = (\mathbf{B}, \mathbf{\Sigma})$$

$$\mathbf{y}_t' = \mathbf{x}_t \mathbf{B} + \mathbf{u}_t', \quad \mathbf{u}_t \sim N(\mathbf{0}, \mathbf{\Sigma})$$

Key practical feature:

- Easy to estimate $(\mathbf{B}, \mathbf{\Sigma})$

Key drawback:

- $(\mathbf{\Sigma}, \mathbf{B})$ are not $(\mathbf{A}, \mathbf{F})$

Most traditional approaches to estimating $(\mathbf{A}, \mathbf{F})$ construct a one-to-one mapping from $(\mathbf{A}, \mathbf{F})$ to $(\mathbf{\Sigma}, \mathbf{B})$.

The literature since then

1. Set identification (with static VAR parameters):
   - Canova and de Nicolo (2002)
   - Uhlig (2005)
2. Coefficients that change (with exact identification)
   - Cogley and Sargent (2005)
   - Primiceri (2005)
   - Sims and Zha (2006), Sims, Waggoner and Zha (2008)

Not obvious how to coherently combine these approaches.

A Motivating Example

- Based on Baumeister and Peersman (2013, AEJ Macro)
- $\mathbf{y}_t = [\Delta p_t^{oil}, \Delta q_t^{oil}, \Delta GDP_t, \Delta p_t^{CPI}]'$
- Identify time-varying IRFs of oil supply shocks

Their method:

- Estimate Primiceri (2005) VAR-TVP-SV
- Reassemble into "reduced-form VAR" parameters t-by-t
- Find structural parameters satisfying sign-restrictions

$$\varepsilon_t^{oil,s} < 0 \Rightarrow \Delta q_{t+h}^{oil} < 0 < \Delta p_{t+h}^{oil} \quad \text{for } h = 0, ..., 4$$

- RRWZ "algorithm" applied to "reduced-form" parameters t-by-t.

"Reduced-form"

Primiceri (2005)

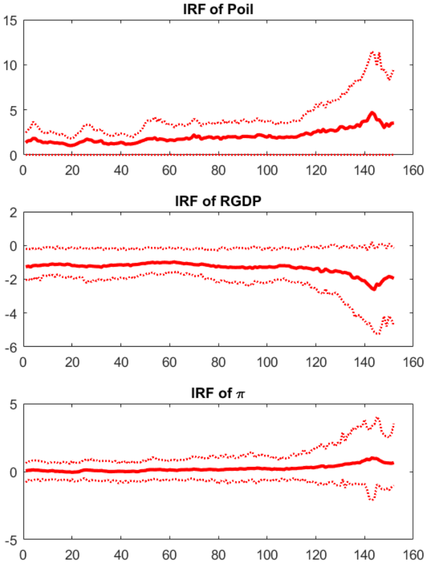
$$\mathbf{y}_t' = vec(\mathbf{B}_t)'(\mathbf{I}_n \otimes \mathbf{x}_t) + \varepsilon_t' \boldsymbol{\Xi}_t \boldsymbol{\Delta}_t^{-1}$$

where

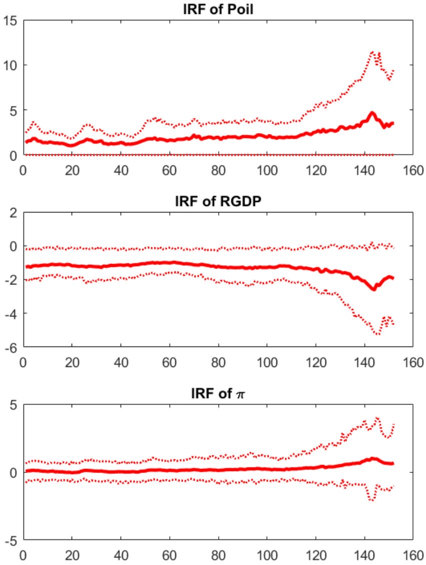
$$\boldsymbol{\Xi}_t = \begin{bmatrix} \xi_{1,t} & 0 & \cdots & 0 \\ 0 & \xi_{2,t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \xi_{n,t} \end{bmatrix}, \quad \boldsymbol{\Delta}_t = \begin{bmatrix} 1 & \delta_{12,t} & \cdots & \delta_{1n,t} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_{n-1n,t} \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \boldsymbol{\Xi}_t &= \boldsymbol{\Xi}_{t-1}\, diag(\exp(\boldsymbol{\eta}_t)), & \boldsymbol{\eta}_t &\sim N(\mathbf{0}_{n\times 1}, \boldsymbol{\Sigma}_{\boldsymbol{\eta}}) \\ \boldsymbol{\delta}_t &= \boldsymbol{\delta}_{t-1} + \boldsymbol{\zeta}_t, & \boldsymbol{\zeta}_t &\sim N(\mathbf{0}_{\frac{n(n-1)}{2}\times 1}, \boldsymbol{\Sigma}_{\boldsymbol{\zeta}}) \\ vec(\mathbf{B}_t) &= vec(\mathbf{B}_{t-1}) + \boldsymbol{\upsilon}_t, & \boldsymbol{\upsilon}_t &\sim N(\mathbf{0}_{mn\times 1}, \boldsymbol{\Sigma}_{\boldsymbol{\upsilon}}) \end{aligned}$$



- Supply shock causing $\Delta q^{oil} = -1\%$.
- "baseline" IRFs
- x-axis: time in quarters
- p_t^{oil} IRF: contemporaneous response at each t
- GDP_t and Δp_t IRFs: cumulative change over 4 quarters at each t



- Supply shock causing $\Delta q^{oil} = -1\%$.
- "baseline" IRFs
- Finding: oil demand has become increasingly inelastic

A Motivating Example

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$$\varepsilon_t^{oil,s} < 0 \Rightarrow \Delta q_{t+h}^{oil} < 0 < \Delta p_{t+h}^{oil} \quad \text{for } h = 0, ..., 4$$

- RRWZ "algorithm"

A Motivating Example Revisited

- Based on Baumeister and Peersman (2013, AEJ Macro)
- ~~$\mathbf{y}_t = [\Delta p_t^{oil}, \Delta q_t^{oil}, \Delta GDP_t, \Delta p_t^{CPI}]'$~~
  $\mathbf{y}_t = [\Delta p_t^{CPI}, \Delta GDP_t, \Delta q_t^{oil}, \Delta p_t^{oil}]'$
- Identify time-varying IRFs of oil supply shocks

The method:

- Estimate Primiceri (2005) VAR-TVP-SV
- Reassemble into "reduced-form VAR" parameters t-by-t
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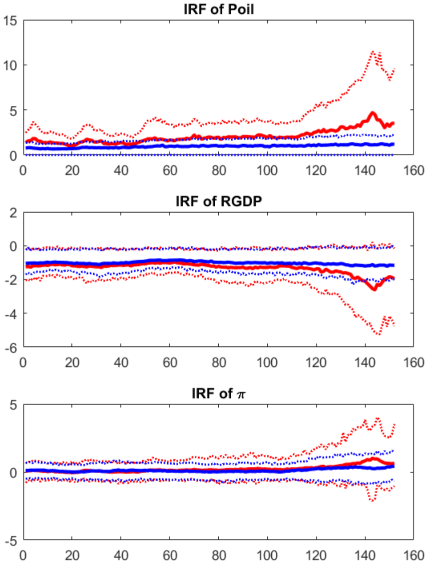
$$\varepsilon_t^{oil,s} < 0 \Rightarrow \Delta q_{t+h}^{oil} < 0 < \Delta p_{t+h}^{oil} \quad \text{for } h = 0, \ldots, 4$$

- RRWZ "algorithm"



- Supply shock causing $\Delta q^{oil} = -1\%$.
- "baseline" IRFs
- IRFs under alternative variable ordering

- **Time-variation in IRFs is gone!**
- **Would have been a different paper!**

Takeaway from the exercise

- **Not** that Baumeister Peersman are "wrong."
  (Indeed, I will find something similar them).

But

- Methodologically, the BP method is deeply problematic.
- The "reduced-form" can be sensitive to variable ordering.
- Spills over into any inference based on the "reduced-form"

Key resulting shortcomings:

1. Results driven as much by an unacknowledged modeling choice (variable ordering) as by the explicit identifying assumptions.
2. $n!$ different candidate reduced-forms.

Examining the posterior I

Let $\mathbf{S}_t = (\mathbf{A}_t, \mathbf{F}_t)$ and $\mathbf{S}_t * \mathbf{Q}_t = (\mathbf{A}_t\mathbf{Q}_t, \mathbf{F}_t\mathbf{Q}_t)$

$$p(\phi, \mathbf{S}_{0:T} | \mathbf{y}_{1:T}) \propto \underbrace{p(\phi, \mathbf{S}_0)}_{\text{prior}} \; \underbrace{p(\mathbf{S}_{1:T} | \phi, \mathbf{S}_0)}_{\substack{\text{density of the } \mathbf{S}_{1:T} \\ \text{sequence under the} \\ \text{model's law of motion}}} \; \underbrace{p(\mathbf{y}_{1:T} | \phi, \mathbf{S}_0, \mathbf{S}_{1:T})}_{\text{data density given } \mathbf{S}_{0:T}}$$

where

$$\begin{aligned} p(\mathbf{y}_{1:T} | \phi, \mathbf{S}_0, \mathbf{S}_{1:T}) &= \prod_{t=1}^{T} p(\mathbf{y}_t | \mathbf{y}_{t-p:t-1}, \mathbf{S}_t) \\ &= \prod_{t=1}^{T} p_N(\mathbf{y}_t | \underbrace{\mathbf{x}_t' \mathbf{F}_t \mathbf{A}_t^{-1}}_{\mathbf{x}_t' \mathbf{F}_t \mathbf{Q}_t \mathbf{Q}_t^{-1} \mathbf{A}_t^{-1}}, \underbrace{(\mathbf{A}_t \mathbf{A}_t')^{-1}}_{(\mathbf{A}_t \mathbf{Q}_t \mathbf{Q}_t' \mathbf{A}_t')^{-1}}) \end{aligned}$$

$\Rightarrow$ In each t, $\mathbf{S}_t * \mathbf{Q}_t$ gives same evaluation of this term as $\mathbf{S}_t$.

Examining the posterior II

$$p(\phi, \mathbf{S}_{0:T}|\mathbf{y}_{1:T}) \propto \underbrace{p(\phi, \mathbf{S}_0)}_{\text{prior}} \underbrace{\color{red}{p(\mathbf{S}_{1:T}|\phi, \mathbf{S}_0)}}_{\substack{\text{density of the } \mathbf{S}_{1:T} \\ \text{sequence under the} \\ \text{model's law of motion}}} \underbrace{p(\mathbf{y}_{1:T}|\phi, \mathbf{S}_0, \mathbf{S}_{1:T})}_{\text{data density given } \mathbf{S}_{0:T}}$$

where

$$\color{red}{p(\mathbf{S}_{1:T}|\phi, \mathbf{S}_0)} = \prod_{t=1}^{T} p(\mathbf{S}_t|\phi, \mathbf{S}_{t-1})$$

(This is the tricky part.)

This paper

- Let's try something else.

This paper

- Let's try something else.
- I define a class of models with laws of motion for $\mathbf{S}_t$ such that:

1. whole sequences of $\mathbf{S}_{0:T}$ have densities invariant to orthogonal rotations
2. yield a shared reduced-form

Key benefits

- Time-varying parameter model amenable to identification driven by RRWZ conditions/algorithms.
- (Also, more straightforward to estimate.)

Outline

Extending the SVAR

$$\mathbf{y}_t'\mathbf{A}_t = \mathbf{x}_t'\mathbf{F}_t + \varepsilon_t' , \quad \varepsilon_t \sim N(\mathbf{0}, \mathbf{I}_n)$$

Law of motion and stochastic processes for $(\mathbf{A}_t, \mathbf{F}_t)$:

$$(\mathbf{A}_t, \mathbf{F}_t) \sim p(\mathbf{A}_{t-1}, \mathbf{F}_{t-1}, \phi)$$

Extending the SVAR

$$\mathbf{y}_t'\mathbf{A}_t = \mathbf{x}_t'\mathbf{F}_t + \varepsilon_t' \,, \quad \varepsilon_t \sim N(\mathbf{0}, \mathbf{I}_n)$$

Law of motion for $(\mathbf{A}_t, \mathbf{F}_t)$:

$$\mathbf{A}_t = \beta^{-1/2}\,\mathbf{A}_{t-1}\mathbf{\Omega}_t$$
$$\mathbf{F}_t = \mathbf{F}_{t-1}\mathbf{A}_{t-1}^{-1}\mathbf{A}_t + \mathbf{\Theta}_t \;.$$

Shocks:

$$\mathbf{\Omega}_t = \mathbf{L}_t h(\mathbf{\Gamma}_t)\mathbf{R}_t \;, \qquad \mathbf{\Gamma}_t \sim B_n(\beta/(2(1-\beta)), 1/2)$$
$$\mathbf{\Theta}_t \sim MN_{m,n}(\mathbf{0}, \mathbf{W}, \mathbf{I}_n)$$

where

$$\beta \in [(n-1)/n, 1]$$
$$\mathbf{L}_t, \mathbf{R}_t \in \mathcal{O}_n$$

Detour: alternate form of SVAR

$$\mathbf{y}_t' = \mathbf{x}_t'\mathbf{B}_t + \varepsilon_t'\mathbf{Q}_t' h(\mathbf{H}_t)^{-1}, \quad \varepsilon_t \sim N(\mathbf{0}, \mathbf{I}_n)$$

Law of motion for $(\mathbf{A}_t, \mathbf{F}_t) = (\mathbf{B}_t, \mathbf{H}_t, \mathbf{Q}_t)$:

$$\begin{aligned} h(\mathbf{H}_t)\mathbf{Q}_t &= \beta^{-1/2}\, h(\mathbf{H}_{t-1})\mathbf{Q}_{t-1}\mathbf{\Omega}_t \\ \mathbf{B}_t h(\mathbf{H}_t)\mathbf{Q}_t &= \mathbf{B}_{t-1} h(\mathbf{H}_{t-1})\mathbf{Q}_t + \mathbf{\Theta}_t \\ \mathbf{Q}_t &= p(\mathbf{Q}_t | \mathbf{B}_t, \mathbf{H}_t) \end{aligned}$$

Shocks:

$$\begin{aligned} &\mathbf{\Omega}_t = h(\mathbf{\Gamma}_t) \qquad \mathbf{\Gamma}_t \sim B_n(\beta/(2(1-\beta)), 1/2) \\ &\mathbf{\Theta}_t \sim MN_{m,n}(\mathbf{0}, \mathbf{W}, \mathbf{I}_n) \end{aligned}$$

where

$$\beta \in [(n-1)/n, 1]$$

Some notation

A Dynamic SVAR (call it DSVAR) denoted:

$$\mathcal{S}^{U}_{0:T}(\mathbf{L}_{1:T}, \mathbf{R}_{1:T})$$

and let

$$\boldsymbol{\phi} = (\beta, \mathbf{W})$$

Key result

Theorem (Theorem 1)

Let $\mathcal{S}^U_{0:T}(\mathbf{L}_{1:T}, \mathbf{R}_{1:T})$ *have prior* $p(\phi, \mathbf{S}_0)$ *for which* $p(\phi, \mathbf{S}_0) = p(\phi, \mathbf{S}_0 * \mathbf{P})$ *for any* $\mathbf{P} \in \mathcal{O}_n$*.*

For any $\mathbf{Q}_{0:T}$ *such that each* $\mathbf{Q}_t \in \mathcal{O}_n$*, the model* $\mathcal{S}^U_{0:T}(\widetilde{\mathbf{L}}_{1:T}, \widetilde{\mathbf{R}}_{1:T})$ *defined by* $(\widetilde{\mathbf{L}}_t, \widetilde{\mathbf{R}}_t) = (\mathbf{Q}'_{t-1}\mathbf{L}_t, \mathbf{R}_t\mathbf{Q}_t)$ *is such that, for every point* $\mathbf{S}_{0:T}$*, the point* $\widetilde{\mathbf{S}}_{0:T} = \mathbf{S}_{0:T} * \mathbf{Q}_{0:T}$ *satisfies*

$$\begin{aligned} p(\phi, \mathbf{S}_{0:T} | \mathbf{y}_{1:T}, \mathcal{S}^U_{0:T}(\mathbf{L}_{1:T}, \mathbf{R}_{1:T})) \\ = p\left(\phi, \widetilde{\mathbf{S}}_{0:T} | \mathbf{y}_{1:T}, \mathcal{S}^U_{0:T}(\widetilde{\mathbf{L}}_{1:T}, \widetilde{\mathbf{R}}_{1:T})\right) . \end{aligned}$$

Theorem 1: restatement and implications

For

1. any realization of the data,
2. any dynamic structural VAR,
3. and any $\mathbf{Q}_{1:T}$

there exists an alternative model with the "same posterior" as the original model, but with each point rotated by $\mathbf{Q}_{1:T}$.

- Set of equivalent models does not depend on $\mathbf{y}_{1:T}$
- $\Rightarrow$ All structural models in the class are observationally equivalent.

Outline

Reduced-form VAR with TVP-SV

Define $(\mathbf{H}_t, \mathbf{B}_t) = g(\mathbf{S}_t) = (\mathbf{A}_t \mathbf{A}_t', \mathbf{F}_t \mathbf{A}_t^{-1})$

$$\mathbf{y}_t' = \mathbf{x}_t' \mathbf{B}_t + \mathbf{u}_t', \quad \mathbf{u}_t \sim N(\mathbf{0}, \mathbf{H}_t^{-1})$$

Laws of motion for $(\mathbf{B}_t, \mathbf{H}_t)$:

$$\mathbf{H}_t = \frac{1}{\beta} h(\mathbf{H}_{t-1})' \, \mathbf{\Gamma}_t \, h(\mathbf{H}_{t-1})$$
$$\mathbf{B}_t = \mathbf{B}_{t-1} + \mathbf{V}_t$$

distributions of shocks $(\mathbf{\Gamma}_t, \mathbf{V}_t)$

$$\mathbf{\Gamma}_t \sim Beta_n(\beta/(2(1-\beta)), 1/2)$$
$$\mathbf{V}_t \sim MN_{m,n}(\mathbf{0}, \mathbf{W}, \mathbf{H}_t^{-1})$$

A short history of the reduced-form

The reduced-form model is "a known quantity."

- Uhlig (1994, 1997) – the stochastic volatility part
- Mike West and coauthors – "dynamic linear model with discounted Wishart stochastic volatility," (DLM-DWSV)

Why does this work?

Suppose I've estimated the reduced-form $\mathbf{H}_{0:T}$.
Shocks rationalizing movement from $\mathbf{A}_{t-1}$ to $\mathbf{A}_t$ satisfy,

$$\beta \mathbf{A}_{t-1}^{-1} \underbrace{\mathbf{H}_t}_{\mathbf{A}_t \mathbf{A}_t'} \mathbf{A}_{t-1}^{-1'} = \mathbf{\Gamma}_t$$

Suppose instead my identification scheme said that in $t-1$, $\widetilde{\mathbf{A}}_{t-1} = \mathbf{A}_{t-1}\mathbf{Q}_{t-1}$.
Shocks rationalizing movement to $\mathbf{H}_t$:

$$\beta \mathbf{A}_{t-1}^{-1} \mathbf{H}_t \mathbf{A}_{t-1}^{-1'} = \mathbf{Q}_{t-1} \mathbf{\Gamma}_t \mathbf{Q}_{t-1}' = \tilde{\mathbf{\Gamma}}_t$$

Critical thing: $\mathbf{\Gamma}_t$ and $\tilde{\mathbf{\Gamma}}_t$ have the same density!

A property of the multivariate Beta distribution:
Srivastava (2003) Corollary 4.1,

$$p(\mathbf{\Gamma}_t) = p(\mathbf{Q}_t \mathbf{\Gamma}_t \mathbf{Q}_t')$$

Estimation of reduced-form

Need to characterize

$$p(\beta, \mathbf{W}, \mathbf{B}_{0:T}, \mathbf{H}_{0:T} | \mathbf{y}_{1:T}) \, .$$

- Can't characterize it analytically.
- Can construct an MCMC algorithm.

Gibbs Sampler

- **Block 1.** $p(\mathbf{W} | \mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T} | \mathbf{y}_{1:T}, \mathbf{W})$

Gibbs sampler: block 1

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$

- Super easy.
- If prior is $\mathbf{W} \sim IW(\mathbf{\Psi}_0, \nu_0)$,

$$\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T} \sim IW(\bar{\mathbf{\Psi}}, \bar{\nu})$$

where

$$\bar{\mathbf{\Psi}} = \mathbf{\Psi}(\mathbf{y}_{1:T}, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}) + \mathbf{\Psi}_0$$
$$\bar{\nu} = Tn + \nu_0$$

Gibbs sampler: block 2

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$

- Factor joint density as

$$p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W}) = \underbrace{p(\beta|\mathbf{y}_{1:T}, \mathbf{W})}_{\text{Block 2a}} \cdot \underbrace{p(\mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \beta, \mathbf{W})}_{\text{Block 2b}}$$

Gibbs sampler: block 2

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2a.** $p(\beta|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2b.** $p(\mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \beta, \mathbf{W})$

Gibbs sampler: block 2a

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2a.** $p(\beta|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2b.** $p(\mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \beta, \mathbf{W})$

Random-walk Metropolis-Hastings,

- "Propose" a $\beta^* \sim q(\beta^*|\beta^{(i-1)}) = Npdf(\beta^{(i-1)}, \sigma^2_\beta)$
- Set $\beta^* = \beta^{(i)}$ with probability

$$\alpha(\beta^*|\mathbf{y}_{1:T}, \mathbf{W}) = \min\left\{ \frac{\overbrace{p\left(\beta^*, \mathbf{W}^{(i)}|\mathbf{y}_{1:T}\right)}^{\propto p(\beta^*, \mathbf{W}^{(i)}) \cdot p(\mathbf{y}_{1:T}|\beta^*, \mathbf{W}^{(i)})}}{p\left(\beta^{(i-1)}, \mathbf{W}^{(i)}|\mathbf{y}_{1:T}\right)},\ 1 \right\}$$

Gibbs sampler: block 2a

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2a.** $p(\beta|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2b.** $p(\mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \beta, \mathbf{W})$

Evaluating $\alpha(\beta^*|\mathbf{y}_{1:T}, \mathbf{W})$ requires pointwise evaluation of

$$p(\mathbf{y}_{1:T}|\beta^*, \mathbf{W}^{(i)}) = \int_{(\mathbf{H}_{0:T}, \mathbf{B}_{0:T})} p(\mathbf{y}_{1:T}|\beta^*, \mathbf{W}^{(i)}, \mathbf{H}_{0:T}, \mathbf{B}_{0:T}) p(\mathbf{H}_{0:T}, \mathbf{B}_{0:T}) d(\mathbf{H}_{0:T}, \mathbf{B}_{0:T})$$

Block 2a: evaluating $p(\mathbf{y}_{1:T}|\beta^*, \mathbf{W}^{(i)})$

Distribution of Interest	Distributional Family	Parameters and Supporting Computations
Step 1 – Prior for $\mathbf{D}_t$ given $\mathbf{y}_{1:t-1}$		
		$(d_{t-1\|t-1}, \boldsymbol{\Psi}_{t-1\|t-1}, \overline{\mathbf{B}}_{t-1\|t-1}, \mathbf{C}_{t-1\|t-1})$ given from iteration $t-1$
$(\mathbf{H}_t\|\mathbf{y}_{1:t-1}, \boldsymbol{\phi})$	$W\left(d_{t\|t-1}, \boldsymbol{\Psi}^{-1}_{t\|t-1}\right)$	$d_{t\|t-1} = \beta d_{t-1\|t-1}$ $\boldsymbol{\Psi}_{t\|t-1} = \beta \boldsymbol{\Psi}_{t-1\|t-1}$
$(\mathbf{B}_t\|\mathbf{y}_{1:t-1}, \boldsymbol{\phi}, \mathbf{H}_t)$	$N\left(\overline{\mathbf{B}}_{t\|t-1}, \mathbf{C}_{t\|t-1}, \mathbf{H}_t^{-1}\right)$	$\overline{\mathbf{B}}_{t\|t-1} = \mathbf{G}\overline{\mathbf{B}}_{t-1\|t-1}$ $\mathbf{C}_{t\|t-1} = \mathbf{G}\mathbf{C}_{t-1\|t-1}\mathbf{G}' + \mathbf{W}$
Step 1.5 – Forecast density of $\mathbf{y}_t$		
$(\mathbf{y}_t\|\mathbf{y}_{1:t-1}, \boldsymbol{\phi})$	$T_{\zeta_t}\left(\overline{\mathbf{y}}_{t\|t-1}, \boldsymbol{\Sigma}_{\mathbf{y}_t}\right)$	$\zeta_t = d_{t\|t-1} - n + 1$ $\overline{\mathbf{y}}_{t\|t-1} = \overline{\mathbf{B}}'_{t\|t-1}\mathbf{x}_t$ $q_t = \mathbf{x}_t'\mathbf{C}_{t\|t-1}\mathbf{x}_t + 1$ $\boldsymbol{\Sigma}_{\mathbf{y}_t} = (q_t/\zeta_t)\boldsymbol{\Psi}_{t\|t-1}$
Step 2 – Posterior for $\mathbf{D}_t$ after observing $\mathbf{y}_{1:t}$		
$(\mathbf{H}_t\|\mathbf{y}_{1:t}, \boldsymbol{\phi})$	$W\left(d_{t\|t}, \boldsymbol{\Psi}^{-1}_{t\|t}\right)$	$d_{t\|t} = d_{t\|t-1} + 1$ $\mathbf{e}_t = \mathbf{y}_t - \overline{\mathbf{y}}_{t\|t-1}$ $\boldsymbol{\Psi}_{t\|t} = \boldsymbol{\Psi}_{t\|t-1} + \frac{1}{q_t}\mathbf{e}_t\mathbf{e}_t'$
$(\mathbf{B}_t\|\mathbf{y}_{1:t}, \boldsymbol{\phi}, \mathbf{H}_t)$	$N\left(\overline{\mathbf{B}}_{t\|t}, \mathbf{C}_{t\|t}, \mathbf{H}_t^{-1}\right)$	$\mathbf{K}_t = \mathbf{C}_{t\|t-1}\mathbf{x}_t/q_t$ $\overline{\mathbf{B}}_{t\|t} = \overline{\mathbf{B}}_{t\|t-1} + \mathbf{K}_t\mathbf{e}_t'$ $\mathbf{C}_{t\|t} = \mathbf{C}_{t\|t-1} - \mathbf{K}_t\mathbf{K}_t' q_t$

Notes: The table summarizes results given in Prado and West (2010).

Block 2b: simulation smoother

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2a.** $p(\beta|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2b.** $p(\mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \beta, \mathbf{W})$

Analogous to Kalman smoother.

Distribution to be sampled	Distributional Family	Parameters and Supporting Computations
		$(d_{t\|t}, \mathbf{\Psi}_{t\|t}, \overline{\mathbf{B}}_{t\|t}, \mathbf{C}_{t\|t}, \overline{\mathbf{B}}_{t+1\|t}, \mathbf{C}_{t+1\|t})$ given from forwards filter
$(\mathbf{H}_t\|Y_t, \boldsymbol{\phi}, \mathbf{H}_{t+1})$	$\mathbf{H}_t = \beta\mathbf{H}_{t+1} + \mathbf{\Upsilon}_t$ $\mathbf{\Upsilon}_t \sim W(d^*_{t\|t+1}, \mathbf{\Psi}^{-1}_{t\|t})$	$d^*_{t\|t+1} = (1-\beta)d_{t\|t}$
$(\mathbf{B}_t\|Y_t, \boldsymbol{\phi}, \mathbf{H}_{t+1}, \mathbf{B}_{t+1})$	$N(\overline{\mathbf{B}}_{t\|t+1}, \mathbf{C}_{t\|t+1}, \mathbf{H}_t^{-1})$	$\widetilde{\mathbf{K}}_t = \mathbf{C}_{t\|t}\mathbf{G}'\mathbf{C}^{-1}_{t+1\|t}$ $\overline{\mathbf{B}}_{t\|t+1} = \overline{\mathbf{B}}_{t\|t} + \widetilde{\mathbf{K}}_t(\mathbf{B}_{t+1} - \overline{\mathbf{B}}_{t+1\|t})$ $\mathbf{C}_{t\|t+1} = \mathbf{C}_{t\|t} - \widetilde{\mathbf{K}}_t\mathbf{C}_{t+1\|t}\widetilde{\mathbf{K}}'_t$

Note: the distribution of $\mathbf{B}_t$ corrects a typo in Prado and West (2010).

Outline

From reduced-form back to structural

Given

1. restriction regions $\mathcal{R}_t$ for each t
2. and posterior samples $\{\mathbf{H}_{0:T}^{(i)}, \mathbf{B}_{0:T}^{(i)}, \phi^{(i)}\}_{i=1}^{Nsim}$

one can

1. construct a sequence of arbitrary $(\mathbf{A}_{0:T}^{(i)}, \mathbf{F}_{0:T}^{(i)})$ consistent with $(\mathbf{H}_{0:T}^{(i)}, \mathbf{B}_{0:T}^{(i)})$ period-by-period
2. t-by-t, find $\mathbf{Q}_t^{(i)} \in \mathcal{O}_n$ such that $(\mathbf{A}_t^{(i)}\mathbf{Q}_t^{(i)}, \mathbf{F}_t^{(i)}\mathbf{Q}_t^{(i)}) \in \mathcal{R}_t$.
3. Set $(\widetilde{\mathbf{A}}_t^{(i)}, \widetilde{\mathbf{F}}_t^{(i)}) = (\mathbf{A}_t^{(i)}\mathbf{Q}_t^{(i)}, \mathbf{F}_t^{(i)}\mathbf{Q}_t^{(i)})$

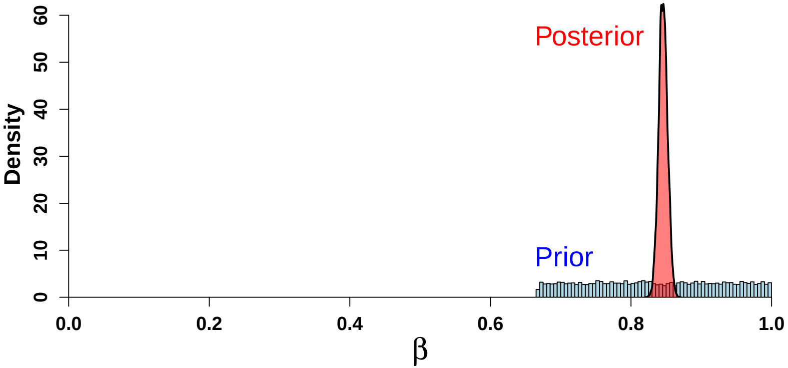
Note, $\mathbf{Q}_t^{(i)}$ can be constructed via:

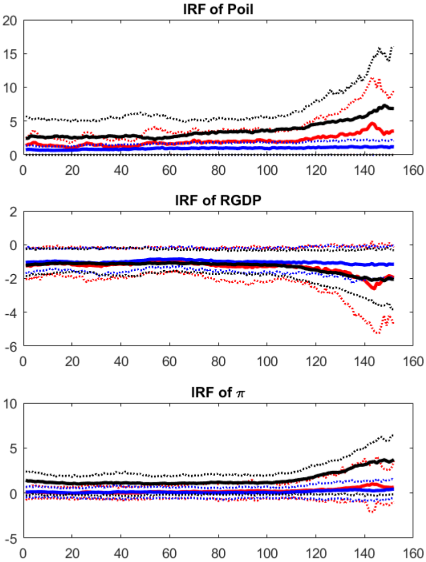
- Algorithm 1 of RRWZ (exact id), or
- Algorithm 2 of RRWZ (set id)

Outline

1. A new SVAR with dynamic parameters
2. Reduced-form Representation
3. Structural Inference Revisited
4. Revisiting the time-varying oil demand elasticity

Prior vs. Posterior: β





- Supply shock causing $\Delta q^{oil} = -1\%$.
- "baseline" IRFs
- IRFs under alternative variable ordering
- Results from **my model**.

Concluding Remarks

Main contributions:

1. Developed a new class of SVAR with time-varying parameters amenable to a variety of identification methods.
   - All models in the class have the same reduced-form representation.
2. Developed an MCMC algorithm for the fully-Bayesian estimation of the reduced-form model.
3. Applied to set identification of a time-varying object of interest about the effect of oil supply shocks.

Appendix

Outline

Dynamic parameters

Now suppose

$$(\mathbf{A}_t, \mathbf{F}_t) \sim p(\phi, \mathbf{A}_{t-1}, \mathbf{F}_{t-1})$$

We lose everything.

1. No easy "reduced-form" to estimate or analyze.
2. (Without part 1 who cares?).

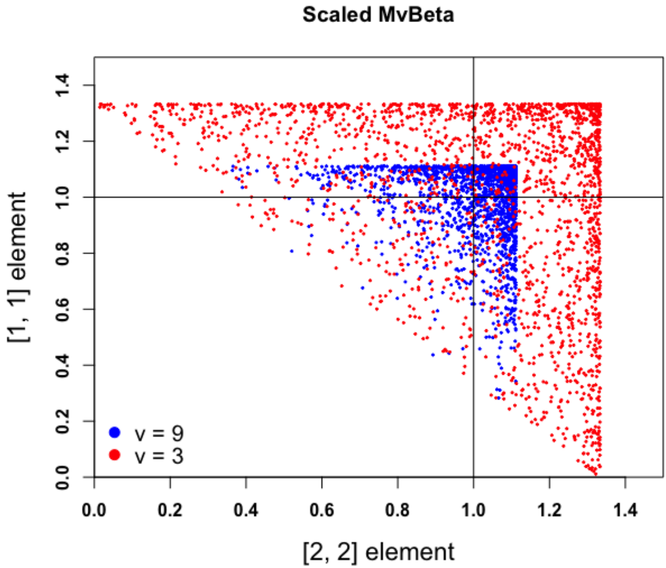
But most importantly, the same basic approach isn't on the table anymore. Why?

Lack of observational equivalence between alternative rotated sequences of structural parameters.

Some notatation before we go on:

$$\mathbf{S}_t = (\mathbf{A}_t, \mathbf{F}_t)$$
$$\mathbf{S}_t * \mathbf{Q}_t = (\mathbf{A}_t\mathbf{Q}_t, \mathbf{F}_t\mathbf{Q}_t)$$

Multivariate Beta



Chol(Multivariate Beta)