Risk Matters: Breaking Certainty Equivalence*

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August 2017

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Abstract

We compare the effects of uncertainty in the solution to an otherwise standard neoclassical macroeconomic model subject to technology shocks for different degrees of approximation. Our results show that certainty equivalence breaks in a continuous-time version of the model even to a first-order approximation, in contrast to its discrete-time version. We compare both local and global numerical methods to compute the rational expectation equilibrium dynamics and impulse response functions. We show how perturbation and collocation methods based on the Hamilton-Jacobi-Bellman (HJB) equation can be used to compute the model's equilibrium in the space of states, fully accounting for the effects of non-linearities and uncertainty. We also show how a first-order approximation is able to capture the effects of uncertainty if risk matters. We illustrate our results in models known to generate substantial risk premia: the capital adjustment cost and habit formation model, and the rare-disaster model.

Keywords: Certainty equivalence, Risk premia, Perturbation methods, Risky steady state, Continuous-time, Discrete-time.

JEL classification: C02, C61, C63, E13, E32, G12.

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*We would like to thank Johannes Pfeiffer and Alexander Meyer-Gohde for their valuable comments. We also thank the German Research Foundation (DFG), and the Center for Research in Econometric Analysis of Time Series, CREATES, funded by the Danish National Research Foundation for financial support.

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1 Introduction

Recently there has been a renewed interest in the ability of linear approximations to the solution of non-linear stochastic macroeconomic models to appropriately account for risk. These approximations belong to the class of perturbation methods introduced in Judd and Guu (1993) where a first order Taylor expansion of the model's equilibrium conditions is built around a known point. Although perturbation-based methods are valid only locally around the approximation point, authors like Judd (1998), Aruoba et al. (2006), Caldara et al. (2012) suggest that the method exhibits high levels of accuracy outside this point, comparable to those delivered by more involved global approximation techniques.

If time is assumed to be discrete and the approximation point used in the Taylor expansion is the model's deterministic steady state, then the approximated solution exhibits certainty equivalence as defined in Simon (1956) and Theil (1957). That is, up to a first order, the solution to a model where agents maximize their expected future utility is identical to the solution of the same model under the assumption of perfect foresight. The direct implication of certainty equivalence is that the solution while still depending on the mean value of the exogenous shocks affecting the economy, it becomes invariant to higher order moments. Certainty equivalence can also be found in the classical linear-quadratic optimal control problem popularized in economics by Kydland and Prescott (1982) and summarized in Anderson et al. (1996).

As discussed in Fernandez-Villaverde et al. (2016) certainty equivalence has several drawbacks: (i) it makes it difficult to talk about the welfare effects of uncertainty; (ii) the approximated solution generated under certainty equivalence cannot generate any risk premia for assets; (iii) certainty equivalence prevents from analyzing the consequences of changes in volatility. To break certainty equivalence while remaining within the class of perturbation methods, economist have restored to the computation of higher order Taylor series expansions which translates in non-linear approximations of the model's solution. Although this simple extension was already implicit in Judd and Guu (1993), it only became popular with the work of Schmitt-Grohe and Uribe (2004) for the case of second order approximations and Andreasen (2012) and Ruge-Murcia (2012) for the case of third order approximations. However, the use of higher order approximations to overcome certainty equivalence poses two further difficulties: (i) the computation of higher order perturbations for medium- and large-scale DSGE models is computationally expensive; and (ii) the estimation of the model's structural parameters requires computationally demanding non-linear estimation methods.

Contrary to the discrete-time case, certainty equivalence breaks, even up to a first order, when time is assumed to evolve continuously. This result initially shown in Judd (1996) and Gaspar and Judd (1997), allows to account for some degree of risk even in a linear world, overcoming the shortcomings of discrete-time perturbations mentioned above. The reason is that while in discrete-time the perturbation approach is built over the expectational equations that define the equilibrium of the economy, the application of Itô's lemma eliminates

In this paper we assess the ability of a first-order approximation (linear) solution to capture the effects of risk when the objective is to remain in the linear world in order to overcome the limitations mentioned above. To do so, we compare the effects of uncertainty in a standard neoclassical framework subject to technology shocks when both first and second order perturbations are used. We build the approximations both under the assumption that time evolves discretely and continuously and provide evidence that in continuous-time the certainty equivalence property breaks already in the first order approximation, a result showed initially in Judd (1996) and Gaspar and Judd (1997).

We then extend the prototypical real business cycle (RBC) model with real rigidities as in Jermann (1998), and with a small probability of a macroeconomic disaster as in Posch and Trimborn (2013) to assess the effects of risk on asset prices in approximate linear economies.

Our work relates to that of Collard and Juillard (2001), Coeurdacier et al. (2011), de Groot (2013) and Meyer-Gohde (2015) who propose an alternative way to break certainty equivalence while remaining in the discrete-time world. In particular, they suggest to build first-order approximations not around the deterministic steady state but around the risky steady state based on the fact that the non-linearities present in any macroeconomic model could imply that the center of the ergodic distribution of the endogenous variables is be away from the deterministic steady state (Juillard and Kamenik, 2005).

The rest of the paper is organized as follows. Section 2 introduces the neoclassical growth model both in discrete- and continuous-time and defines the equilibrium conditions used to build the first and second order approximations; Section 3 describes the calibration used for the different numerical exercises and shows the link that exists between the parameters of the models in discrete- and continuous-time. Section 4 summarizes the perturbation approach while Section 5 discusses the main results by comparing policy functions and impulse-response functions under both discrete and continuous-time models. Finally, Section 6 concludes.

2 Framework

Our benchmark model corresponds to a prototype real business cycle model (RBC) similar to that in Aruoba et al. (2006) and Parra-Alvarez (2017). In what follows we formulate and solve the model in both continuous-time and discrete-time.

2.1 Continuous-time RBC model

2.1.1 Technology

Consider the problem faced by a benevolent planner with a production function:

$$Y_t = A_t K_t^{\alpha}, \tag{1}$$

where A_t is the total factor productivity (TFP) and K_t is the aggregate capital stock. The capital stock increases if gross investment exceeds capital depreciation,

$$\mathrm{d}K_t = (Y_t - C_t - \delta K_t)\mathrm{d}t, \tag{2}$$

where δ is the depreciation rate. The logarithm of the total factor productivity (TFP) is described by an Ornstein-Uhlenbeck process with mean reversion parameters $\rho_A > 0$ of the form:

$$\mathrm{d}\log A_t = -\rho_A \log A_t \mathrm{d}t + \sigma_A \mathrm{d}B_{A,t} \tag{3}$$

where $B_{A,t}$ is a standard Brownian motions with volatility σ_A. An application of Ito's lemma shows that the level of the TFP follows:

$$\mathrm{d}A_t = -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t \mathrm{d}t + \sigma_A A_t \mathrm{d}B_{A,t}. \tag{4}$$

2.1.2 Households

The economy is assumed to be inhabited by a large number of identical individuals, which maximize their expected discounted life-time utility

$$U_0 \equiv \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} \mathrm{d}t \right], \tag{5}$$

subject to the dynamics in Equations (2) and (4), where C_t denotes aggregate consumption, $\rho > 0$ is the household's subjective discount rate and $\gamma > 0$ the coefficient of relative risk aversion.

2.1.3 The HJB equation and the first-order conditions

The benevolent planner chooses a path for consumption in order to maximize expected life-time utility of a representative household. Define the value of the optimal program

$$V(K_0, A_0) = \max_{\{C_t \in \mathbb{R}^+\}_{t=0}^\infty} U_0 \quad s.t. \quad (2) \quad \text{and} \quad (4) \tag{6}$$

in which $C_t \in \mathbb{R}^+$ denotes the control at instant $t \in \mathbb{R}^+$. A necessary condition for optimiality is given by the *Hamilton-Jacobi-Bellman* (HJB) equation[1]:

$$\begin{aligned}\rho V(K_t, A_t) = \max_{C_t \in \mathbb{R}^+} \Bigg\{ &\frac{C_t^{1-\gamma}}{1-\gamma} + (A_t K_t^\alpha - C_t - \delta K_t) V_K(K_t, A_t) \\ &- (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2) A_t V_A(K_t, A_t) + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AA}(K_t, A_t) \Bigg\}.\end{aligned}$$

[1] A formal derivation of the HJB equation can be found in Chang (2009).

The first-order condition for any interior solution reads:

$$C_t^{-\gamma} = V_K(K_t, A_t), \tag{7}$$

making optimal consumption a function of the state variables, $C_t = C(K_t, A_t)$. As shown in Appendix A, the Euler equation for consumption can be written as:

$$\frac{\mathrm{d}C_t}{C_t} = \left[\frac{1}{\gamma}\left(\alpha A_t K_t^{\alpha-1} - \delta - \rho\right) + \tfrac{1}{2}(1+\gamma)\left(\frac{C_A A_t}{C_t}\right)^2 \sigma_A^2\right]\mathrm{d}t + \left(\frac{C_A A_t}{C_t}\right)\sigma_A \mathrm{d}B_{A,t}. \tag{8}$$

2.1.4 Conditional deterministic system

Following Posch and Trimborn (2013), the solution to the stochastic optimal control problem faced by the social planner can be obtained from what they define as the conditional deterministic system. In general, their method delivers a non-linear solution to the HJB equation that coincides with the policy function implied by the equivalent boundary value problem.

Using the maximized HJB equation together with the first order condition for consumption Appendix B shows that a necessary condition for optimality is given by:

$$\frac{1}{\gamma}\left(\alpha A_t K_t^{\alpha-1} - \delta - \rho\right) C_t + \tfrac{1}{2}(1+\gamma) C_t \left(\frac{C_A A_t}{C_t}\right)^2 \sigma_A^2 = C_K(A_t K_t^{\alpha} - C_t - \delta K_t) - C_A(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t + \tfrac{1}{2} C_{AA} A_t^2 \sigma_A^2 \tag{9}$$

A system of partial differential equations (PDEs) that implies the same policy function as in Equation (9) in the absence of shocks can be constructed from:

$$\mathrm{d}C_t = \left[\frac{1}{\gamma}\left(\alpha A_t K_t^{\alpha-1} - \delta - \rho\right) C_t + \tfrac{1}{2}(1+\gamma) C_t \left(\frac{C_A A_t}{C_t}\right)^2 \sigma_A^2 \; - \tfrac{1}{2} C_{AA} A_t^2 \sigma_A^2\right]\mathrm{d}t \tag{10}$$

$$\mathrm{d}K_t = (A_t K_t^{\alpha} - C_t - \delta K_t)\mathrm{d}t \tag{11}$$

$$\mathrm{d}A_t = -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2) A_t \mathrm{d}t \tag{12}$$

together with

$$C_A = -\frac{1}{\gamma} V_K^{-\frac{1+\gamma}{\gamma}} V_{KA}, \quad C_{AA} = \frac{1+\gamma}{\gamma^2} V_K^{-\frac{1+\gamma}{\gamma}-1} V_{KA}^2 - \frac{1}{\gamma} V_K^{-\frac{1+\gamma}{\gamma}} V_{KAA}$$

such that $\mathrm{d}C_t = C_A \mathrm{d}A_t + C_K \mathrm{d}K_t$ with $\mathrm{d}C_t$, $\mathrm{d}K_t$, and $\mathrm{d}A_t$ from (10), (11), and (12), respectively, also solves the HJB equation. In other words, it is possible solve the system in the absence of shocks and still find the correct policy functions under uncertainty. The effects of risk in the optimal solution are modeled in the resulting deterministic system through the curvature term $\frac{1}{2} C_{AA} A_t^2 \sigma_A^2$ in Equation (10), which is otherwise absent in the Euler equation (8).

The system of equations in (10), (11), and (12) which is referred to as the model's conditional deterministic system and it can be solved globally using the Waveform Relaxation algorithm introduced in Posch and Trimborn (2013).

2.1.5 Equilibrium

The equilibrium in this economy is given by the sequence $\{C_t, K_t, A_t\}_{t=0}^{\infty}$ that solves the following system of equations:

$$\frac{\mathrm{d}C_t}{C_t} = \left[\frac{1}{\gamma}\left(\alpha A_t K_t^{\alpha-1} - \delta - \rho\right) + \tfrac{1}{2}(1+\gamma)\left(\frac{C_A A_t}{C_t}\right)^2 \sigma_A^2\right]\mathrm{d}t + \left(\frac{C_A A_t}{C_t}\right)\sigma_A \mathrm{d}B_{A,t} \quad \text{(Euler equation)}$$

$$\mathrm{d}K_t = (A_t K_t^{\alpha} - C_t - \delta K_t)\mathrm{d}t \quad \text{(Aggregate resource constraint)}$$

$$\mathrm{d}A_t = -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t \mathrm{d}t + \sigma_A A_t \mathrm{d}B_{A,t} \quad \text{(Total factor productivity)}$$

Together with initial conditions $K(0) = K_0$ and $A(0) = A_0$, they form a system of 3 stochastic differential equations (SDEs) in 3 variables. The solution to this system of SDEs in the time space delivers the optimal trajectories of A_t and the endogenous variables, K_t and C_t.

It is also possible to compute the solution of the model in the space of states. In this case, the equilibrium of the economy is characterized by Equation (9) associated to the conditional deterministic system which can be formally summarized by:

$$F(K, A) := f(K, A, C(K, A), C_K(K, A), C_A(K, A), C_{AA}(K, A)) = 0.$$

Whereas solving the model in the time domain or in the space domain, the solution of the model demands the capital accumulation constraint (2), the goods market equilibrium (implicitly assumed), and the optimality condition for consumption (7) to hold at every instant $t \in [0, \infty)$.

2.1.6 Analytical Results

Two values of interest are the deterministic steady-state and the conditional deterministic (or risky) steady state. The former corresponds to the limiting behavior of the economy under the assumption that the variables in the economy do not grow and agents do not anticipate the effects of future shocks. In other words, a deterministic steady state is defined as the triple $(\bar{C}, \bar{K}, \bar{A})$ that solves the dynamic system when $\mathrm{d}C_t = \mathrm{d}K_t = \mathrm{d}A_t = 0$ together with $\sigma_A = 0$. Hence, the steady state value of capital is equal to:

$$\bar{K} = \left[\frac{\alpha \bar{A}}{\rho + \delta}\right]^{\frac{1}{1-\alpha}}$$

which then implies the following value for consumption,

$$\bar{C} = \bar{A}(\bar{K})^{\alpha} - \delta \bar{K}$$

where $\bar{A} = 1$ corresponds to the stationary solution of the exogenous TFP.

Now, assume the existence of a stationary point to which the dynamic system converges in the absence of shocks, $\mathrm{d}B_{A,t} = 0$ for all $t \geq 0$ but where $\sigma_A \geq 0$ given that the agents in the economy are risk averse. We define the conditional deterministic (or risky) steady state as the triple (C^*, K^*, A^*) that solves both, the system of PDEs in Equations (10)-(12) and the maximized HJB equation when $\mathrm{d}C_t = \mathrm{d}K_t = \mathrm{d}A_t = 0$. Hence, the risky steady state corresponds to the solution to:

$$\begin{aligned} 0 &= \frac{1}{\gamma}\left(\alpha A^*(K^*)^{\alpha-1} - \delta - \rho\right) + \tfrac{1}{2}(1+\gamma)\tilde{C}_A(K^*, A^*)^2\sigma_A^2 - \tfrac{1}{2}\tilde{C}_{AA}(K^*, A^*)\sigma_A^2 \\ 0 &= A^*(K^*)^\alpha - C^* - \delta K^* \\ 0 &= \rho_A \log A^* - \tfrac{1}{2}\sigma_A^2 \end{aligned}$$

where:

$$\tilde{C}_A = -\frac{1}{\gamma}V_K^{-1}V_{KA}A^*, \quad \tilde{C}_{AA} = (1+\gamma)\tilde{C}_A - \frac{1}{\gamma}V_K^{-1}V_{KAA}\left(A^*\right)^2.$$

From the third equation it is straightforward to conclude that:

$$A^* = \exp\left(\frac{1}{2}\frac{\sigma_A^2}{\rho_A}\right)$$

but, for most parameterizations of the model, the conditional deterministic steady state values of K^* and $C^* = A^*(K^*)^\alpha - \delta K^*$ are available only numerically.

Under the assumption that the output elasticity of capital equals the reciprocal of the elasticity of intertemporal substitution (EIS), $\alpha = \gamma$, the model has an analytical solution. The optimal policy function for consumption is then given by (see e.g. Posch (2009) and Posch and Schrimpf (2012)):

$$C_t = C(K_t, A_t) = \mathbb{C}_1^{-1/\gamma}K_t, \quad \mathbb{C}_1^{-1/\gamma} = \frac{\rho + (1-\gamma)\delta}{\gamma}$$

such that $\tilde{C}_A = 0$, $\tilde{C}_{AA} = 0$ so the risky steady state for aggregate capital is given by:

$$K^* = \left[\frac{\alpha A^*}{\rho+\delta}\right]^{\frac{1}{1-\alpha}} \geq \bar{K}$$

given that $A^* \geq \bar{A}$.

2.2 Discrete-time RBC model

2.2.1 Technology

Consider again the problem of a benevolent planner with production function:

$$Y_t = A_t K_t^\alpha, \tag{13}$$

where A_t is total factor productivity and K_t is the aggregate capital stock. The capital stock increases if gross investment exceeds capital depreciation,

$$K_{t+1} = Y_t - C_t + (1-\delta)K_t, \tag{14}$$

and the logarithm of TFP follows an AR(1) process:

$$\log A_{t+1} = \tilde{\rho}_A \log A_t + \tilde{\sigma}_A \epsilon_{A,t+1} \Leftrightarrow \; A_{t+1} = A_t^{\tilde{\rho}_A} \exp(\tilde{\sigma}_A \epsilon_{A,t+1}) \tag{15}$$

where $\tilde{\rho}_A$ denote the autorregresive coefficient of the TFP process, $\tilde{\sigma}_A$ its standard deviation, and $\epsilon_{A,t}$ is an *iid* normally distributed disturbance with mean zero and unitary variance.

2.2.2 Households

Similar to the continuous-time case we assume that a large number of individuals inhabit the economy. A representative household maximizes his expected discounted life-time utility:

$$U_0 \equiv \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right], \tag{16}$$

subject to the dynamics in Equations (14) and (15), where $\beta > 0$ denotes the households subjective discount factor and $\gamma > 0$ the coefficient of relative risk aversion.

2.2.3 The HJB Equation and the First-Order Conditions

The benevolent planner chooses the path of consumption and capital stock accumulation that maximizes the expected life-time utility of the representative household. Define the value of the optimal program

$$V(K_0, A_0) = \max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} U_0 \quad s.t. \quad (14) \quad \text{and} \quad (15) \tag{17}$$

in which $C_t \in \mathbb{R}^+$ and $K_{t+1} \in \mathbb{R}^+$ define the control variables at time $t \in \mathbb{Z}$. Then, the *Bellman* equation reads for any $t \in \{0, 1, 2, \dots\}$

$$V(K_t, A_t) = \max_{C_t \in \mathbb{R}^+} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t V(K_{t+1}, A_{t+1}) \right\} \tag{18}$$

with associated first order condition:

$$C_t^{-\gamma} = \beta \mathbb{E}_t \left[\frac{\partial V(K_{t+1}, A_{t+1})}{\partial K_{t+1}} \right], \tag{19}$$

making optimal consumption a function of the state variables, $C_t = C(K_t, A_t)$. As shown in Appendix C, the Euler equation for consumption is given by:

$$C_t^{-\gamma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\gamma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta \right) \right]. \tag{20}$$

2.2.4 Equilibrium

The equilibrium in the economy is given by the sequence $\{C_t, K_t, A_t\}_{t=0}^{\infty}$ that solves the following system of equations:

$$\begin{aligned}
C_t^{-\gamma} &= \mathbb{E}_t \left[\beta C_{t+1}^{-\gamma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta \right) \right] && \text{(Euler equation)} \\
K_{t+1} &= A_t K_t^{\alpha} - C_t + (1-\delta) K_t && \text{(Aggregate resource constraint)} \\
\log A_{t+1} &= \tilde{\rho}_A \log A_t + \tilde{\sigma}_A \epsilon_{A,t+1} && \text{(Total factor productivity)}
\end{aligned}$$

The equilibrium of the economy is characterized by a system of 3 stochastic difference equations in 3 variables that determine whose solution delivers the optimal paths of the exogenous variable A_t and the endogenous variables, K_t and C_t. This system of equations can be formally summarized by:

$$F(K_t, A_t) := \mathbb{E}_t f(C_{t+1}, C_t, K_{t+1}, K_t, A_{t+1}, A_t) = \mathbf{0}$$

where

$$\mathbb{E}_t f(C_{t+1}, C_t, K_{t+1}, K_t, A_{t+1}, A_t) = \mathbb{E}_t \begin{bmatrix} C_t^{-\gamma} - \beta C_{t+1}^{-\gamma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta \right) \\ A_t K_t^{\alpha} + (1-\delta) K_t - C_t - K_{t+1} \\ \log A_{t+1} - \tilde{\rho}_A \log A_t - \tilde{\sigma}_A \epsilon_{A,t+1} \end{bmatrix}.$$

2.2.5 Analytical results

The deterministic steady state is again defined as the the equilibrium point that prevails in the absence of uncertainty when the variables in the economy do not change over time. Formally, a deterministic steady state is given by the triple $(\bar{C}, \bar{K}, \bar{A})$ that solves the dynamic system when $C_t = C_{t+1} = \bar{C}$, $K_t = K_{t+1} = \bar{K}$ and $A_t = A_{t+1} = \bar{K}$ together with $\tilde{\sigma}_A = 0$. Hence, the steady state value of capital is equal to:

$$\bar{K} = \left(\frac{\alpha \bar{A}}{\beta^{-1} - 1 + \delta} \right)^{\frac{1}{1-\alpha}},$$

which then implies the following value for consumption $\bar{C} = \bar{A}(\bar{K})^{\alpha} - \delta \bar{K}$, where $\bar{A} = 1$ corresponds to the stationary value of the exogenous TFP.

On the other hand, the risky steady state as defined in Coeurdacier et al. (2011) corresponds to the point where agents choose to stay at a given date if they expect future risk and if the realization of shocks is zero at this date. For most parameterizations, the risky steady state can only be approximated numerically as will be shown in Section 4.2.1.

Table 1. Summary of the two modeling approaches.

	Discrete-time	Continuous-time
Objective function (U_0)	$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\frac{C_t^{1-\gamma}}{1-\gamma}\right]$	$\mathbb{E}_0\left[\int_0^{\infty}e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma}\mathrm{d}t\right]$
Market clearing	$A_tK_t^{\alpha}=C_t+I_t$	$A_tK_t^{\alpha}=C_t+I_t$
Capital dynamics	$K_{t+1}=I_t+(1-\delta)K_t$	$\mathrm{d}K_t=(I_t-\delta K_t)\,\mathrm{d}t$
TFP dynamics	$\log A_{t+1}=\tilde{\rho}_A\log A_t$ $+\tilde{\sigma}_A\epsilon_{A,t+1}$	$\mathrm{d}\log A_t=-\rho_A\log A_t\mathrm{d}t$ $+\sigma_A\mathrm{d}B_{A,t}$
Uncertainty	$\epsilon_{A,t}\sim N(0,1)$	$(B_{A,t+\Delta}-B_{A,t})\sim N(0,\Delta)$

3 Calibration

The prototype models presented in Section 2 are summarized in Table 1. The model is calibrated annually and the parameters should be interpreted accordingly. The values of the parameters are taken from the discrete-time version of the RBC model in Jermann (1998). The set of parameter can be split into two groups: (i) those whose value is independent of choice of discrete- or continuous-time; and (ii) those whose value depends on it. Table 2 summarizes the calibration.

The first group includes the coefficient of relative risk aversion, the share of capital in output and the depreciation rate. Their values are set to $\gamma=5$, $\alpha=0.36$ and $\delta=0.0963$. The second group includes the subjective discount rate, the subjective discount factor, the persistence of the TFP shocks and their volatility. For the discounting parameters, we set $\beta=0.9606$ and $\rho=0.041$. The latter value ensures the same deterministic steady state values of the capital stock in the discrete- and continuous-time models.

With regards to the TFP parameters, Jermann (1998) reports quarterly values of 99% and 1% for its persistence and volatility, respectively. These values imply an annual volatility of TFP of around 8% which is at odds with those values reported recently in literature for the U.S. Therefore, we assume quarterly values of 95% and 0.5%, in line with those reported in Aruoba et al. (2006). To ensure consistency between the parameter values across discrete- and continuous-time models we follow Christensen et al. (2016). Hence, the quarterly values imply an annual autorregresive coefficient of $\tilde{\rho}_A=0.8145$ when time is assumed to be discrete, and a value of $\rho_A=0.2052$ when time is assumed to be continuous. Finally, the annualized volatility in the continuous-time model consistent with the quarterly value is $\sigma_A=0.041$, while that for the discrete-time model is $\tilde{\sigma}_A=0.0372$[2].

With these parameter values, the resulting deterministic steady state values for aggregate capital

[2] The relation between the subjective discount factor and the subjective discount rate is given by $\rho=\frac{1}{\beta}-1$. As shown in Christensen et al. (2016), the link between the persistence parameter of the discrete- and continuous-time models is given by $\tilde{\rho}_A=1-e^{-\Delta\rho_A}$, while that between volatilities reads $\tilde{\sigma}_A=\Delta\sigma_A\sqrt{\left(1-e^{-2\rho_A\Delta}\right)/(2\rho_A)}$, where Δ denotes the observation frequency.

Table 2. Parameter values for the RBC model.

Parameter	Discrete-time	Continuous-time
Discounting, β/ρ	0.9606	0.0410
Relative risk aversion, γ	5.0000	5.0000
Depreciation rate, δ	0.0963	0.0963
Capital share in output, α	0.3600	0.3600
Persistence of TFP, $\tilde{\rho}_A/\rho_A$	0.8145	0.2052
Volatility of TFP, $\tilde{\sigma}_A/\sigma_A$	0.0372	0.0410

and aggregate consumption are $\bar{K} = 4.5077$ and $\bar{C} = 1.2854$, respectively.

4 Perturbation method

Perturbation methods approximate the solution of the stochastic optimal control problem by means of the implicit function theorem and the Taylor's series expansion theorem. The perturbed solution consists of a polynomial, or a similar function, that approximates the true solution of the problem in a neighborhood of an *a priori* know solution. In what follows, this approximation point will be the deterministic steady state computed in Section 2. Following Judd (1998), the perturbation method can be summarized by the following steps:

1. Express the problem as a continuum of problems parameterized by the added perturbation parameter η, with the $\eta = 0$ case known.

2. Differentiate the continuum of problems with respect to the state variables and η.

3. Solve the resulting equation for the implicitly defined derivatives at the known solution of the state variables and $\eta = 0$.

4. Compute the desired order of approximation by means of Taylor's theorem. In general, the order of approximation should be determined by the first non-trivial term or dominant term, that is, apply Taylor approximations until the first non-zero term is reached. Set $\eta = 1$ to recover the original model.

5. If possible write the results in unit-free terms such as elasticities and shares.

In what follows we define the perturbation parameter η in terms of the amount of uncertainty introduced into the model. In particular, the added parameter η will control the amount of volatility of the disturbances in discrete-time models, while it will control the absolute magnitude of their variance in continuous-time models. Hence, a model with $\eta = 0$ is equivalent to a model with perfect foresight (no uncertainty) for which a solution can be easily computed. In the same fashion, a model with $\eta = 1$ is equivalent to the original model. Alternative definitions of the perturbation parameter can be used to study a wide range of problems in economics, e.g Kogan and Uppal (2001), Chacko and Viceira (2005) and Hansen et al. (2008).

4.1 Continuous-time setup

As discussed in Parra-Alvarez (2017), a solution to the continuous-time stochastic optimal control problem in Section 2 is characterized by the policy function for aggregate consumption, $C_t = C(K_t, A_t)$. Perturbation methods allow us to building an approximation to the (extended) policy function $C(K_t, A_t; \eta)$ around the deterministic steady state $(K_t, A_t, \eta) = (\bar{K}, \bar{A}, 0)$. Thus, a second order approximation to the extended problem[3]:

$$F(K_t, A_t; \eta) := f(K, A, C(K, A; \eta), C_K(K, A; \eta), C_A(K, A; \eta), C_{AA}(K, A; \eta); \eta) = 0.$$

where:

$$\begin{aligned} f(\cdot) \;=\; & \frac{1}{\gamma}\left(\alpha A_t K_t^{\alpha-1} - \delta - \rho\right) C(K, A; \eta) + \tfrac{1}{2}(1+\gamma) C(K, A; \eta) \left(\frac{C_A(K, A; \eta) A_t}{C(K, A; \eta)}\right)^2 \eta \sigma_A^2 \\ & - C_K(K, A; \eta)\left(A_t K_t^{\alpha} - C(K, A; \eta) - \delta K_t\right) + C_A(K, A; \eta)\left(\rho_A \log A_t - \tfrac{1}{2}\eta\sigma_A^2\right) A_t \\ & - \tfrac{1}{2} C_{AA}(K, A; \eta) A_t^2 \eta \sigma_A^2 \end{aligned}$$

is given by:

$$\begin{aligned} C(K_t, A_t; \eta) \approx\; & \overline{C} + \overline{C_K}\left(K_t - \bar{K}\right) + \overline{C_A}\left(A_t - \bar{A}\right) + \overline{C_\eta}\eta \\ & + \overline{C_{KA}}\left(K_t - \bar{K}\right)\left(A_t - \bar{A}\right) + \overline{C_{K\eta}}\left(K_t - \bar{K}\right)\eta + \overline{C_{A\eta}}\left(A_t - \bar{A}\right)\eta \\ & + \frac{1}{2}\left(\overline{C_{KK}}\left(K_t - \bar{K}\right)^2 + \overline{C_{AA}}\left(A_t - \bar{A}\right)^2 + \overline{C_{\eta\eta}}\eta^2\right) \end{aligned} \quad (21)$$

where the $\overline{C_{ij}} = C_{i,j}\left(\bar{K}, \bar{A}; 0\right)$ for $i, j = \{K, A, \eta\}$ denote variables evaluated at their deterministic steady state. Higher order approximation can be immediately written down.

The approximation in Equation (21) requires the computation of 10 constants. From the calculation of the deterministic steady state we instantly obtain $\overline{C} = C\left(\bar{K}, \bar{A}; 0\right)$. The remaining 9 constants still need to be computed. To do so we exploit the fact that $F(K_t, A_t; \eta) = 0$ and hence all of its partial derivatives must also be zero. In a first step, we partitioning the set of constants into two groups. The first one, called the deterministic component, groups all the constants of the form $F_{K^i A^j}(K_t, A_t; \eta) = 0$ for $i, j = 0, 1, 2$, while the second one, called the stochastic component, collects all the constants of the form $F_{K^i A^j \eta^l}(K_t, A_t; \eta) = 0$ for $i, j, l = 0, 1, 2$.

Once defined both groups, we start the approximation by computing the first order terms related to the deterministic components and evaluating them at the deterministic steady state. As shown in Gaspar and Judd (1997) and Judd (1998), the resulting system of equations in the unknown constants $\overline{C_K}$, $\overline{C_A}$ and $\overline{C_\eta}$ correspond to a Ricatti equation with r roots, where r is the number of equilibrium paths. Once the stable path is chosen, we proceed to the computation of the first order terms related to the stochastic components which corresponds to a linear equation. Once the first order approximation is completed, the computation of the remaining second order terms reduces

[3] Recall that the extended equilibrium condition is given in Equation (9).

to the solution of a system of linear equations. Appendix D shows how the computation of the first order terms is done manually for the model in Section 2.

As a general rule, Judd and Guu (1993) showed that the computation of an n-th order approximation of the policy function that solves a stochastic optimal control model in continuous-time requires an $(n+2)$-th order deterministic approximation. Hence, a second order approximation to the policy function requires a fourth deterministic approximation and a second order stochastic approximation.

At this point is important to emphasize that perturbation methods applied to continuous-time models do not exhibit certainty equivalence as suggested in Judd (1996). As an illustration, consider the first order stochastic component of the RBC model from Section 2, which is formally derived in Appendix D:

$$\overline{C_\eta} = -\left(\overline{C_K}\right)^{-1}\left[\frac{1}{2}\left(1+\gamma\right)\overline{C}\left(\frac{\overline{C_A}\bar{A}}{\overline{C}}\right)^2\sigma_A^2 - \frac{1}{2}\overline{C_A}\sigma_A^2\bar{A} - \frac{1}{2}\overline{C_{AA}}\bar{A}^2\sigma_A^2\right]. \tag{22}$$

A standard result for first order discrete-time perturbations is that $\overline{C_\eta} = 0$ implying that certainty equivalence always holds up to a first order irregardless on the modeling assumptions. However, as Equation (22) shows the same is not true for first-order continuous-time perturbations. Certainty equivalence will hold only in very particular cases that are of no or little interest for most applications in macroeconomics. E.g. whenever (i) the volatility of the disturbances is zero, $\sigma_A = 0$; (ii) the volatility of the state variables is constant; (iii) the utility function is quadratic and/or (iv) the production function is linear.

4.2 Discrete-time setup

Following the notation introduced in Schmitt-Grohe and Uribe (2004), let us define $\boldsymbol{y}_t = C_t$ to be the vector of control variables and $\boldsymbol{x}_t = \{K_t, A_t\}$ the vector of state variables. The latter can be partitioned into endogenous states and exogenous states:

$$\boldsymbol{x}_t = \left[\boldsymbol{x}_{1,t}'; \boldsymbol{x}_{2,t}'\right]'$$

where $\boldsymbol{x}_{1,t} = K_t$ is the vector of the endogenous state variable, and $\boldsymbol{x}_{2,t} = A_t$ is the vector of the exogenous state variable. If η denotes the perturbation parameter, the solution to the extended problem:

$$F(\boldsymbol{x}_t;\eta) := \mathbb{E}_t f(\boldsymbol{y}_{t+1}, \boldsymbol{y}_t, \boldsymbol{x}_{t+1}, \boldsymbol{x}_t; \eta) = \mathbf{0} \tag{23}$$

is given by the (extended) policy functions $C_t = \boldsymbol{g}\left(K_t, A_t; \eta\right)$, $K_{t+1} = \boldsymbol{h}_1\left(K_t, A_t; \eta\right)$, and $A_{t+1} = \boldsymbol{h}_2\left(K_t, A_t; \eta\right) + \eta\tilde{\sigma}_A\epsilon_{A,t+1}$ satisfying:

$$\boldsymbol{y}_t = \boldsymbol{g}(\boldsymbol{x}_t;\eta) \quad \text{and} \quad \boldsymbol{x}_{t+1} = \boldsymbol{h}(\boldsymbol{x}_t;\eta) + \eta\boldsymbol{Q}\boldsymbol{\epsilon}_{t+1}, \tag{24}$$

where $\boldsymbol{g} : R^2 \times R^+ \to R$, $\boldsymbol{h} : R^2 \times R^+ \to R^2$, $\boldsymbol{Q} = [0, \tilde{\sigma}_A]'$, $\boldsymbol{\epsilon}_{t+1} = \epsilon_{A,t+1}$.

Perturbation methods for stochastic optimal control problems in discrete time approximate the unknown functions $\boldsymbol{g}$ and $\boldsymbol{h}$ by means of the Taylor theorem around the deterministic steady state $\bar{\boldsymbol{y}} = g(\bar{\boldsymbol{x}}; 0)$ and $\bar{\boldsymbol{x}} = \boldsymbol{h}(\bar{\boldsymbol{x}}; 0)$. Inserting Equation (24) into Equation (23) leads to:

$$F(\boldsymbol{x}_t; \eta) := \mathbb{E}_t f[\boldsymbol{g}(\boldsymbol{h}(\boldsymbol{x}_t; \eta) + \boldsymbol{Q}\eta\boldsymbol{\epsilon}_{t+1}; \eta), \boldsymbol{g}(\boldsymbol{x}_t; \eta), \boldsymbol{h}(\boldsymbol{x}_t; \eta) + \boldsymbol{Q}\eta\boldsymbol{\epsilon}_{t+1}, \boldsymbol{x}_t; \eta] = 0. \tag{25}$$

Similar to the continuous-time case, the approximation is constructed by exploiting the fact that if $F(\boldsymbol{x}_t; \eta) = \boldsymbol{0}$ for any $\boldsymbol{x}_t$ and η, then its partial derivatives must be also zero. Thus, a first order approximation of the functions $\boldsymbol{g}$ and $\boldsymbol{h}$ around the deterministic steady state is then given by:

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{x}_t; \eta) &= \bar{\boldsymbol{y}} + \boldsymbol{g}_{\boldsymbol{x}}(\bar{\boldsymbol{x}}; 0)(\boldsymbol{x}_t - \bar{\boldsymbol{x}}) + \boldsymbol{g}_\eta(\bar{\boldsymbol{x}}; 0)\eta \\ \boldsymbol{h}(\boldsymbol{x}_t; \eta) &= \bar{\boldsymbol{x}} + \boldsymbol{h}_{\boldsymbol{x}}(\bar{\boldsymbol{x}}; 0)(\boldsymbol{x}_t - \bar{\boldsymbol{x}}) + \boldsymbol{h}_\eta(\bar{\boldsymbol{x}}; 0)\eta \end{aligned} \tag{26}$$

where $\boldsymbol{g}_{\boldsymbol{x}}(\bar{\boldsymbol{x}}; 0)$, $\boldsymbol{h}_{\boldsymbol{x}}(\bar{\boldsymbol{x}}; 0)$, $\boldsymbol{g}_\eta(\bar{\boldsymbol{x}}; 0)$ and $\boldsymbol{h}_\eta(\bar{\boldsymbol{x}}; 0)$ are the solutions to the system of equations formed by the partial derivatives of $F(\boldsymbol{x}_t; \eta)$ when evaluated at $\boldsymbol{x_t} = \bar{\boldsymbol{x}}$ and $\eta = 0$. A particular feature of the system of equations formed with the first order terms is that that the constants $\boldsymbol{g}_\eta(\bar{\boldsymbol{x}}; 0)$ and $\boldsymbol{h}_\eta(\bar{\boldsymbol{x}}; 0)$ correspond to the solution of a sub-system of linear and homogeneous equations which imply that $\boldsymbol{g}_\eta(\bar{\boldsymbol{x}}; 0) = \boldsymbol{h}_\eta(\bar{\boldsymbol{x}}; 0) = 0$ (see Fernandez-Villaverde et al. (2016)). Therefore, the first order perturbation reduces to:

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{x}_t; \eta) &= \bar{\boldsymbol{y}} + \boldsymbol{g}_{\boldsymbol{x}}(\bar{\boldsymbol{x}}; 0)(\boldsymbol{x}_t - \bar{\boldsymbol{x}}) \\ \boldsymbol{h}(\boldsymbol{x}_t; \eta) &= \bar{\boldsymbol{x}} + \boldsymbol{h}_{\boldsymbol{x}}(\bar{\boldsymbol{x}}; 0)(\boldsymbol{x}_t - \bar{\boldsymbol{x}}) \end{aligned}$$

implying that up to a first order, the linear approximation exhibits certainty equivalence, in other words, the solution of the model is identical to the solution of the same model when $\eta = 0$.

As an illustration, the first order approximation of the policy function for consumption reads:

$$C_t^{(1)} = \overline{C} + \overline{g_K}(K_t - \bar{K}) + \overline{g_A}(A_t - \bar{A}) \tag{27}$$

where $\overline{g_K}$ and $\overline{g_A}$ belong to the Jacobian matrix of $\boldsymbol{g}$. First order approximations for the aggregate capital stock and the TFP process can be computed analogously.

The second order approximation to $\boldsymbol{g}\,(\boldsymbol{x}_t; \eta)$ and $\boldsymbol{h}\,(\boldsymbol{x}_t; \eta)$ and the identification of the coefficients follows the same approach as before. Here, the second order derivatives of $F(\boldsymbol{x}_t; \eta)$ with respect to $\boldsymbol{x}_t$ and η, respectively, have to be taken into account. A second order approximation to the policy functions for consumption is given by:

$$C_t^{(2)} = C_t^{(1)} + \frac{1}{2}\left(\overline{g_{KK}}(K_t - \bar{K})^2 + 2\overline{g_{KA}}(K_t - \bar{K})(A_t - \bar{A}) + \overline{g_{AA}}(A_t - \bar{A})^2 + \overline{g_{\eta\eta}}\eta^2\right) \tag{28}$$

$$K_{t+1}^{(2)} = K_{t+1}^{(1)} + \frac{1}{2}\left[\overline{h_{1KK}}(K_t - \bar{K})^2 + 2\overline{h_{1KA}}(K_t - \bar{K})(A_t - \bar{A}) + \overline{h_{1AA}}(A_t - \bar{A})^2 + \overline{h_{1\eta\eta}}\eta^2\right] \tag{29}$$

$$A_{t+1}^{(2)} = A_{t+1}^{(1)} + \frac{1}{2}\left[\overline{h_{2KK}}(K_t - \bar{K})^2 + 2\overline{h_{2KA}}(K_t - \bar{K})(A_t - \bar{A}) + \overline{h_{2AA}}(A_t - \bar{A})^2 + \overline{h_{2\eta\eta}}\eta^2\right] \tag{30}$$

where all the constants with a line above their value are elements of the Hessian matrices of $\boldsymbol{g}$ and $\boldsymbol{h}$.

4.2.1 Risky steady state in discrete-time

Following de Groot (2013), it is possible to approximate the risky steady state values of the model variables using the second order approximations for TFP in Equation (30) and for aggregate capital in Equation (29). In particular, by setting the current value of the disturbances to zero, $\epsilon_{A,t} = 0$, and given that $h_{2KK} = h_{2KA} = 0$ due to the exogenous nature of TFP, the second order approximation for TFP can write as:

$$A_{t+1}^{(2)} = \overline{A} + \overline{h_{2A}}(A_t - \bar{A}) + \frac{1}{2}\left[\overline{h_{2AA}}(A_t - \bar{A})^2 + \overline{h_{2\eta\eta}}\eta^2\right].$$

The risky steady state of TFP is defined as the value A_{rss} that satisfies $A_{t+1} = A_t = A_{rss}$ and thus solves the quadratic equation:

$$A_{rss} = \overline{A} + \overline{h_{2A}}(A_{rss} - \bar{A}) + \frac{1}{2}\left[\overline{h_{2AA}}(A_{rss} - \bar{A})^2 + \overline{h_{2\eta\eta}}\eta^2\right].$$

Applying the same procedure to the second order approximation of capital in Equation (29):

$$\begin{aligned} K_{t+1} &= \overline{K} + \overline{h_{1K}}(K_t - \bar{K}) + \overline{h_{1A}}(A_t - \bar{A}) \\ &+ \frac{1}{2}\left[\overline{h_{1KK}}(K_t - \bar{K})^2 + 2\overline{h_{1KA}}(K_t - \bar{K})(A_t - \bar{A}) + \overline{h_{1AA}}(A_t - \bar{A})^2 + \overline{h_{1\eta\eta}}\eta^2\right]. \end{aligned}$$

we define K_{rss} satisfying $K_{t+1} = K_t = K_{rss}$ that solves the quadratic equation:

$$\begin{aligned} K_{rss} &= \overline{K} + \overline{h_{1K}}(K_{rss} - \bar{K}) + \overline{h_{1A}}(A_{rss} - \bar{A}) \\ &+ \frac{1}{2}\left[\overline{h_{1KK}}(K_{rss} - \bar{K})^2 + 2\overline{h_{1KA}}(K_{rss} - \bar{K})(A_{rss} - \bar{A}) + \overline{h_{1AA}}(A_{rss} - \bar{A})^2 + \overline{h_{1\eta\eta}}\eta^2\right]. \end{aligned}$$

Appendix E shows a detailed derivation of the risky steady state values. Finally, the risky steady state value of aggregate consumption is given by the identity $C_{rss} = A_{rss}K_{rss}^{\alpha} - \delta K_{rss}$.[4] Under the calibration in Table 2, the risky steady state values of the model variables are $A_{rss} = 1.0004$, $K_{rss} = 3.5701$, and $C_{rss} = 1.1655$.

[4]Alternatively, we could calculate C_{rss} from its second order approximation in Equation (28) as $C_{rss} = \overline{C} + \overline{g_K}K_\Delta + \overline{g_A}A_\Delta + \frac{1}{2}\left[\overline{g_{KK}}(K_\Delta)^2 + 2\overline{g_{KA}}K_\Delta A_\Delta + \overline{g_{AA}}(A_\Delta)^2 + \overline{g_{\eta\eta}}\eta^2\right]$, which delivers the same numerical result.

Table 3. Comparison of steady states values.

Variable	Determ.	Risky				
		Discrete-time		Continuous-time		
		First	Second	First	Second	Global
A	1.0000	1.0000	1.0037	1.0041	1.0041	1.0041
K	4.5077	4.5077	4.7110	4.7130	4.7200	4.7198
C	1.2854	1.2854	1.2999	1.3006	1.3009	1.3009

Table 4. Comparison of steady states values for $\alpha = \gamma = 0.36$.

Variable	Determ.	Risky					
		Discrete-time		Continuous-time			
		First	Second	First	Second	Global	True
A	1.0000	1.0000	1.0037	1.0041	1.0041	1.0041	1.0041
K	4.5077	4.5077	4.5341	4.5367	4.5367	4.5366	4.5367
C	1.2854	1.2854	1.2929	1.2937	1.2937	1.2937	1.2937

4.2.2 Comparison of steady states

Table 3 summarizes both the deterministic and the risky steady state values for aggregate consumption, aggregate capital stock and total factor productivity. By construction, the deterministic steady state values for the discrete- and continuous-time model are identical. However, the risky steady states differ since they are not available in closed form and hence can only be numerically approximated. The table reports the risky steady state under both discrete- and continuous-time. For the latter it reports the approximation to the risky steady state values obtained from a first and a second order perturbation as well from a global numerical approximation.

Table 4 compares deterministic and stochastic steady state values under the assumption $\alpha = \gamma$, which has a closed-form solution in continuous time. For this calibration, the approximations of the steady states values obtained from first and second order perturbation as well as from a global solution method can be compared to the 'true' solution that is given in closed-form. In continuous-time already the approximation based on first-order perturbation is equal to the true solution, while in discrete-time even the approximation resulting from second-order perturbation differs from the true solution.

5 Results

5.1 Approximated policy functions

Using the calibration from Section 3, Figure (1) plots the approximated policy function for consumption for the discrete- and continuous-time models along a discretized grid containing $n_K = 1001$ values for the capital stock around the interval $K \in \left[0.5\bar{K}, 1.5\bar{K}\right]$, while keeping the TFP level at its deterministic steady state. We plot the first and second order policy functions.

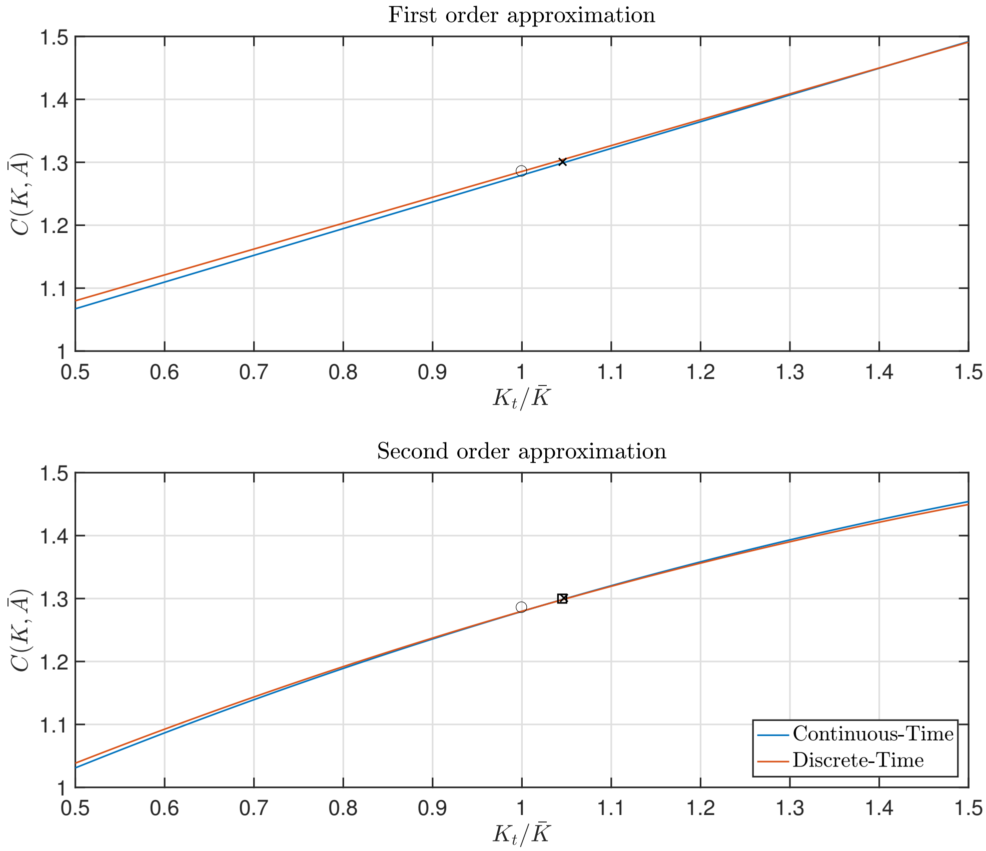

Figure 1. Policy functions for consumption. The graph plots the first order approximation (top panel) and the second order approximation (bottom panel) to the policy function for aggregate consumption along the capital lattice while keeping productivity at its determinisitic steady state, $C\left(K,\bar{A}\right)$. A circle denotes the deterministic steady state, a star denotes the risky steady state approximated from the continuous-time model, and a square the risky steady state approximated from the discrete-time model

Figure (2) plots compares the first and second order approximations to the policy functions for consumption for both the discrete- and the continuous-time model.

The first order approximated policy functions for consumption are given by[5]:

$$\begin{aligned} C_t^{(1),CT} &= 1.2854 + 0.0942\left(K_t - \bar{K}\right) + 0.4232\left(A_t - \bar{A}\right) - \mathbf{0.0059} \\ C_t^{(1),DT} &= 1.2854 + 0.0912\left(K_t - \bar{K}\right) + 0.4883\left(A_t - \bar{A}\right). \end{aligned}$$

Note that $C_t^{(1),DT}$ is invariant to the volatility of the TFP shock σ_A. In particular, this implies that this policy function is identical to a policy function that would result from a model with perfect foresight. Hence, $C_t^{(1),DT}$ is certainty equivalent. In contrast, in continuous-time the first-order approximated policy function is not certainty equivalent since the last term in bold in $C_t^{(1),CT}$ depends on σ_A, as already shown in equation (22). This term is a constant correction term for risk, which in discrete-time only appears in second order, as we will see.

[5]The discrete-time approximations to the state-variables are $K_{t+1}^{(1),DT} = 4.5077 + 0.9498(K_t - \bar{K}) + 1.2313(A_t - \bar{A})$ and $A_{t+1}^{(1),DT} = 1 + 0.8145(A_t - \bar{A}) + \tilde{\sigma}_A \epsilon_{A,t+1}$.

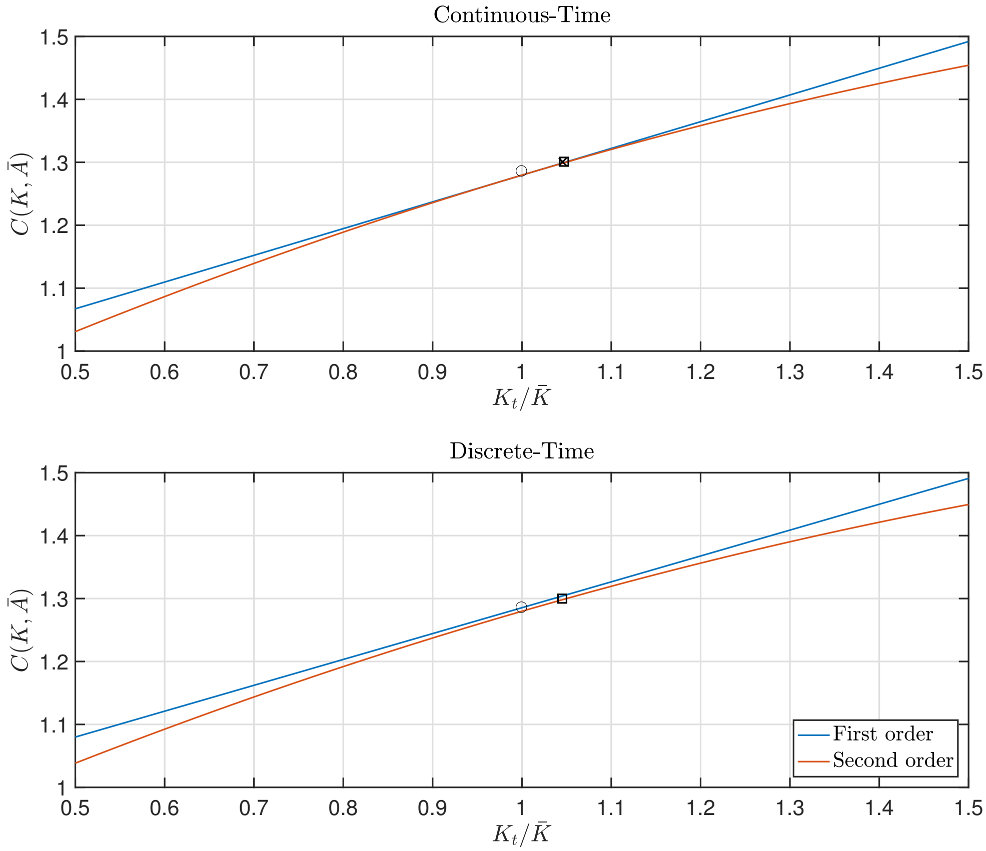

Figure 2. Policy functions for consumption. The graph plots the first and second order approximation to the policy function for aggregate consumption along the capital lattice while keeping productivity at its deterministic steady state, $C\left(K,\bar{A}\right)$ for the RBC model in continuous-time (top panel) and the model in discrete-time (bottom panel). A circle denotes the deterministic steady state, a star denotes a first order approximation to the risky steady state approximated, while a square denotes a second order approximation to the risky steady state.

The second order approximated policy functions are[6]:

$$\begin{aligned} C_t^{(2),CT} &= 1.2854 + (0.0942 - \mathbf{0.0003})\left(K_t - \bar{K}\right) + (0.4232 - \mathbf{0.0021})\left(A_t - \bar{A}\right) - \mathbf{0.0059} \\ &- 0.0054\left(K_t - \bar{K}\right)\left(A_t - \bar{A}\right) + \frac{1}{2}\left[-0.0146\left(K_t - \bar{K}\right)^2 - 0.2458\left(A_t - \bar{A}\right)^2 + \mathbf{4.0734 \times 10^{-5}}\right] \\ C_t^{(2),DT} &= 1.2854 + 0.0912\left(K_t - \bar{K}\right) + 0.4883\left(A_t - \bar{A}\right) \\ &+ \frac{1}{2}\left[-0.0141(K_t - \bar{K})^2 - 2 \times 0.0055(K_t - \bar{K})(A_t - \bar{A}) - 0.2523(A_t - \bar{A})^2 - \mathbf{0.0112}\right] \end{aligned}$$

The numbers in bold highlight the corrections for risk for each model. In discrete-time, we obtain a constant correction term of magnitude -0.0056 in second-order, which is close to but still smaller than what we get in continuous-time in first-order, -0.0059. In the second-order approximation in

[6] The discrete-time approximations to the state-variables are $K_{t+1}^{(2),DT} = 4.5077 + 0.9498(K_t - \bar{K}) + 1.2313(A_t - \bar{A}) + \frac{1}{2}\left[-0.0054(K_t - \bar{K})^2 + 2 \times 0.1428(K_t - \bar{K})(A_t - \bar{A}) + 0.2523(A_t - \bar{A})^2 + 0.0112\right]$ and $A_{t+1}^{(2),DT} = 1 + 0.8145(A_t - \bar{A}) + \frac{1}{2}\left[-0.1511(A_t - \bar{A})^2 + 0.0014\right] + \tilde{\sigma}_A \epsilon_{A,t+1}$.

continuous-time, we get further corrections, which also affect the linear coefficients.

5.2 Euler equation errors

To assess the quality of the approximations Judd (1998) introduces the normalized Euler equation errors as a measure of the goodness of fit along the state space. Following Aruoba et al. (2006), the Euler equation errors associated to the approximation of order $i = 1, 2$, is given in the discrete-time framework by:

$$EE^i_{DT}\left(K_t, A_t\right) = 1 - \frac{\left[\beta \mathbb{E}_t \left(C^i_{t+1}\right)^{-\gamma} \left(\alpha A^i_{t+1} \left(K^i_{t+1}\right)^{\alpha-1} + 1 - \delta\right)\right]^{-1/\gamma}}{C^i_t}, \tag{31}$$

while Judd and Guu (1993) introduce the Euler equation errors in continuous-time models based on the conditionally deterministic system in Equation (9) as:

$$\begin{aligned} EE^i_{CT}(K_t, A_t) = \frac{1}{\rho C(\bar{K}, \bar{A})} \Bigg[& \frac{1}{\gamma} \left(\alpha A_t K_t^{\alpha-1} - \delta - \rho\right) C^i_t \\ & + \tfrac{1}{2}(1+\gamma) C^i_t \left(\frac{C^i_A A_t}{C^i_t}\right)^2 \eta \sigma_A^2 - C^i_K \left(A_t K_t^{\alpha} - C^i_t - \delta K_t\right) \\ & + C^i_A \left(\rho_A \log A_t - \tfrac{1}{2}\eta\sigma_A^2\right) A_t - \tfrac{1}{2} C^i_{AA} A_t^2 \eta \sigma_A^2 \Bigg]. \end{aligned} \tag{32}$$

The partial derivatives of the policy function are computed using the approximated policy functions. For the benchmark calibration in Section 2, Figure 3 compares the Euler equation errors between discrete and continuous-time for each of the orders of approximation along the aggregate capital lattice while keeping the productivity at its deterministic steady state value. Appendix F displays Euler equation errors by type of approximation independently for the discrete- and continuous-time model.

5.3 Impulse Response Functions

This section compares the response of the endogenous variables of the discrete- and continuous-time models to a temporary shock on the level of total factor productivity.

Figure 4 plots the impulse response functions (IRFs) for the levels of aggregate consumption, aggregate capital, and aggregate output, and future productivity when the discrete-time economy is subject to a one standard deviation shock in the TFP. The blue lines plot the IRFs based on the first order approximation, and the red lines those based on the second order approximation to the policy functions (see Section 5.1). For the sake of comparison, we assume that before the shock hits, the economy rests in its deterministic steady state. Up to a first order approximation, the model converges to the deterministic steady state as time passes. However, when the model is approximated to a first order, the economy converges instead to its (approximated) risky steady state. Note that compared to the deterministic steady state, the risky steady state TFP level is

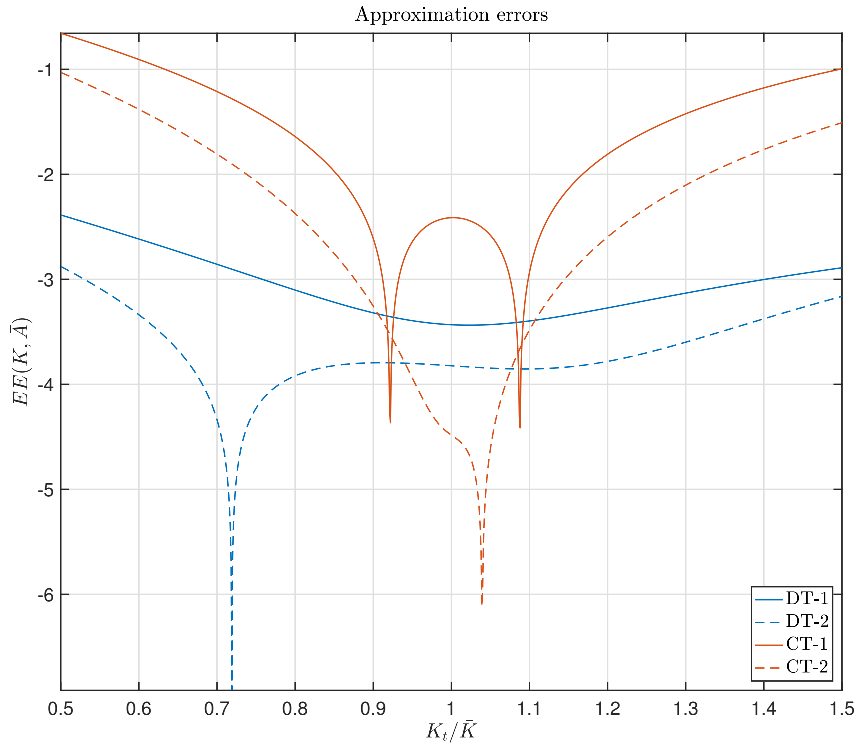

Figure 3. Euler equation errors. The graph plots the log10 of the absolute value of the Euler equation errors for the first order approximation and the second order approximation to the policy function of aggregate consumption along the capital lattice while keeping productivity at its determinisitic steady state, $EE\left(K, \bar{A}\right)$. Continuous lines denote approximation errors for the first order approximations, and dotted lines denote approximation errors for the second order approximations.

higher, which leads to a higher capital stock and higher output in the second order approximation over the whole time span. In contrast, the reaction of consumption in second-order is lower during the first 8 periods than in first-order although TFP, capital stock, and output are higher. The reason is that the first-order approximation ignores risk (certainty equivalent), which is accounted for in the second-order. Once we accounting for risk, consumption increases less in response to the TFP shock and converges to the higher risky steady state value. All in all, consumption is more volatile up to a first order, while being smoother in the case of a second order approximation.

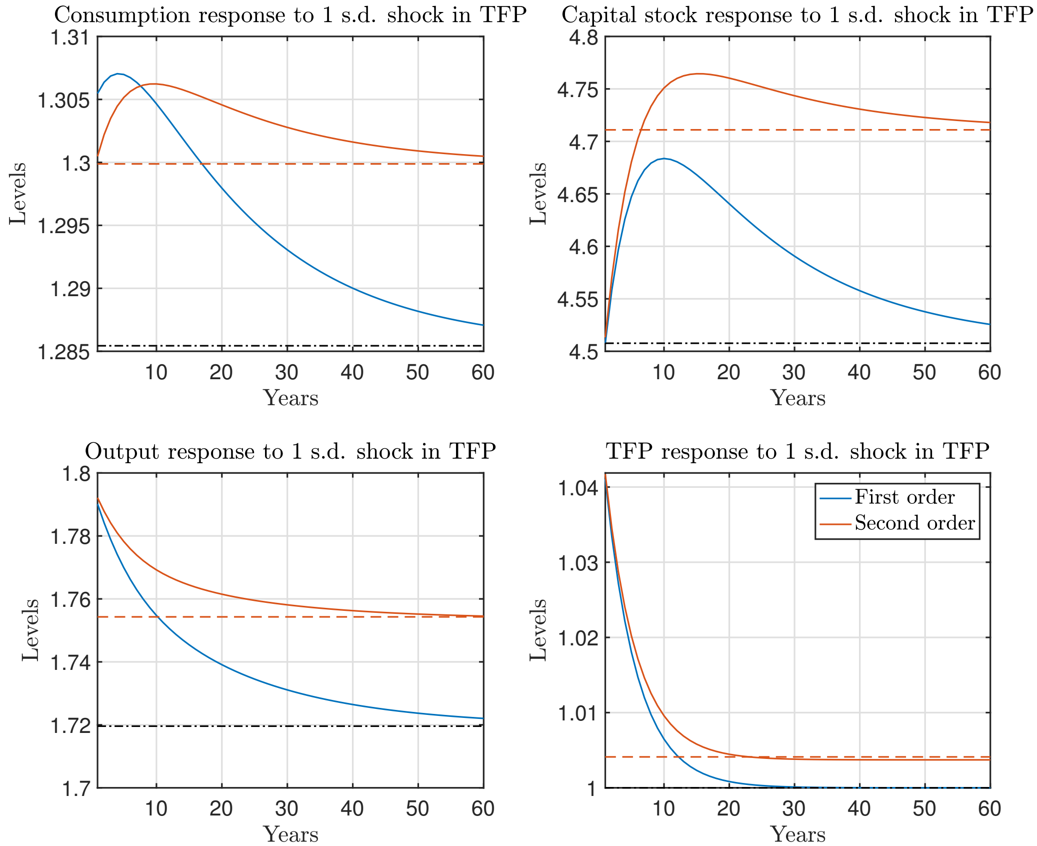

Figure 4. Impulse-Response function to a TFP shock: Discrete-time. The graph plots the impulse response functions (IRFs) for the levels of aggregate consumption, aggregate capital, and aggregate output when time in the economy is assumed to be discrete. The economic system is assumed to be in its deterministic steady state before the shock hits. The dash-dotted line corresponds to the deterministic steady state, and the dashed line to the approximated risky steady state.

Figure 5 plots the responses of aggregate consumption, aggregate capital stock, aggregate output and future productivity to the same TFP shock when time in the economy is assumed to be continuous. The blue lines plot the IRFs based on the first order approximation, and the red lines those based on the second order approximation to the policy functions. The IRF's are computed by iterating forward, from the deterministic steady state, the system of equations consisting of Euler-discretized versions of the stochastic process for the state variables and the approximated consumption function. As opposed to discrete-time, all variables in the economy converge to the (approximated) risky steady states. The differences across IRFs correspond to numerical differences in the approximation of the policy function. The response of aggregate consumption shows how certainty equivalence is broken even in the first order approximation.

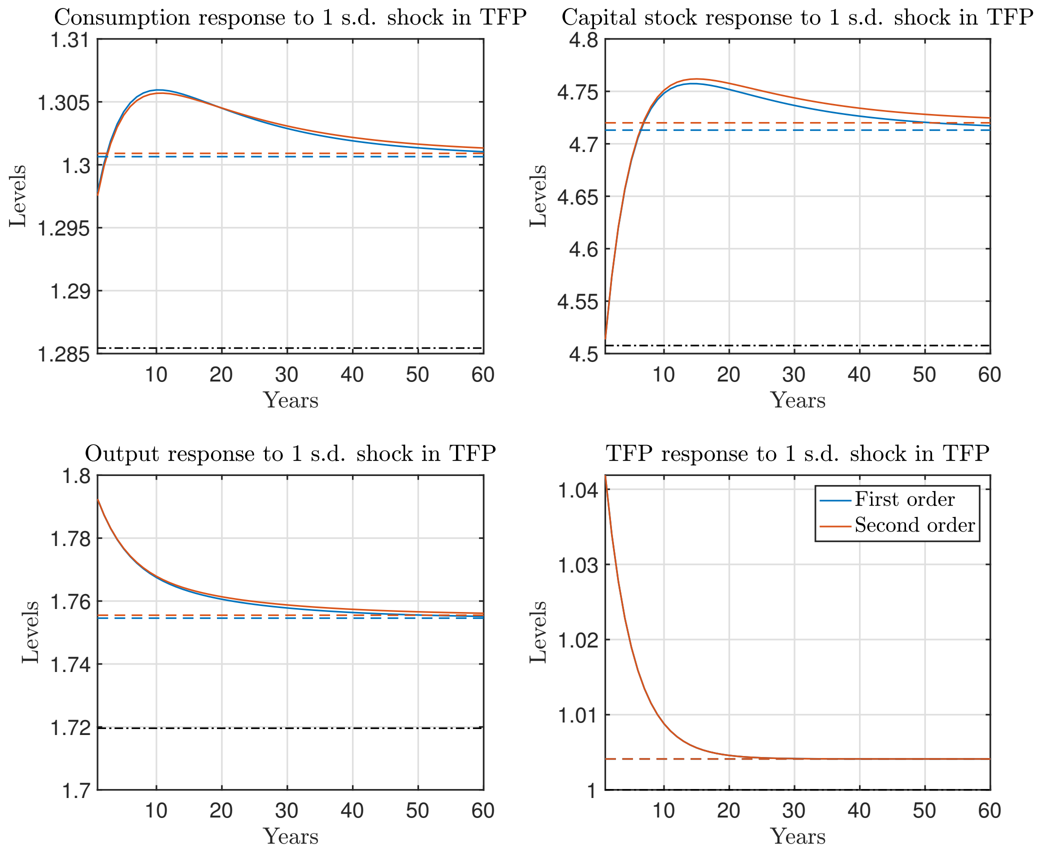

Figure 5. Impulse-Response function to a TFP shock: Continuous-time. The graph plots the impulse response functions (IRFs) for the levels of aggregate consumption, aggregate capital, and aggregate output when time in the economy is assumed to be continuous. The economic system is assumed to be in its deterministic steady state before the shock hits. The dash-dotted line corresponds to the deterministic steady state, and the dashed lines to the approximated risky steady states.

For completeness, Figures 6 and 7, compare the effect of the timing assumption on the IRFs for each type of approximation.

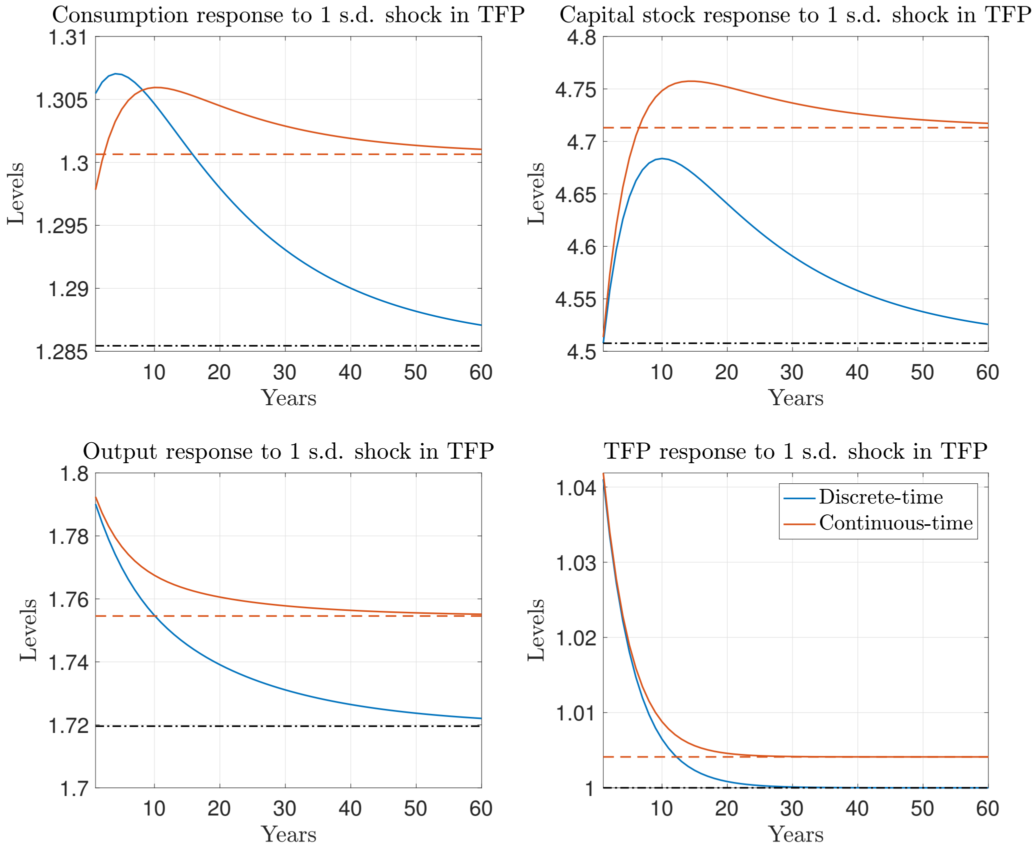

Figure 6. First order Impulse-Response function to a TFP shock: Discrete-time vs. Continuous-time. The graph plots the first order impulse response functions (IRFs) for the levels of aggregate consumption, aggregate capital, and aggregate output for both discrete- and continuous-time models. The economic system is assumed to be in its deterministic steady state before the shock hits. The dash-dotted line corresponds to the deterministic steady state, and the dashed line to the approximated risky steady state.

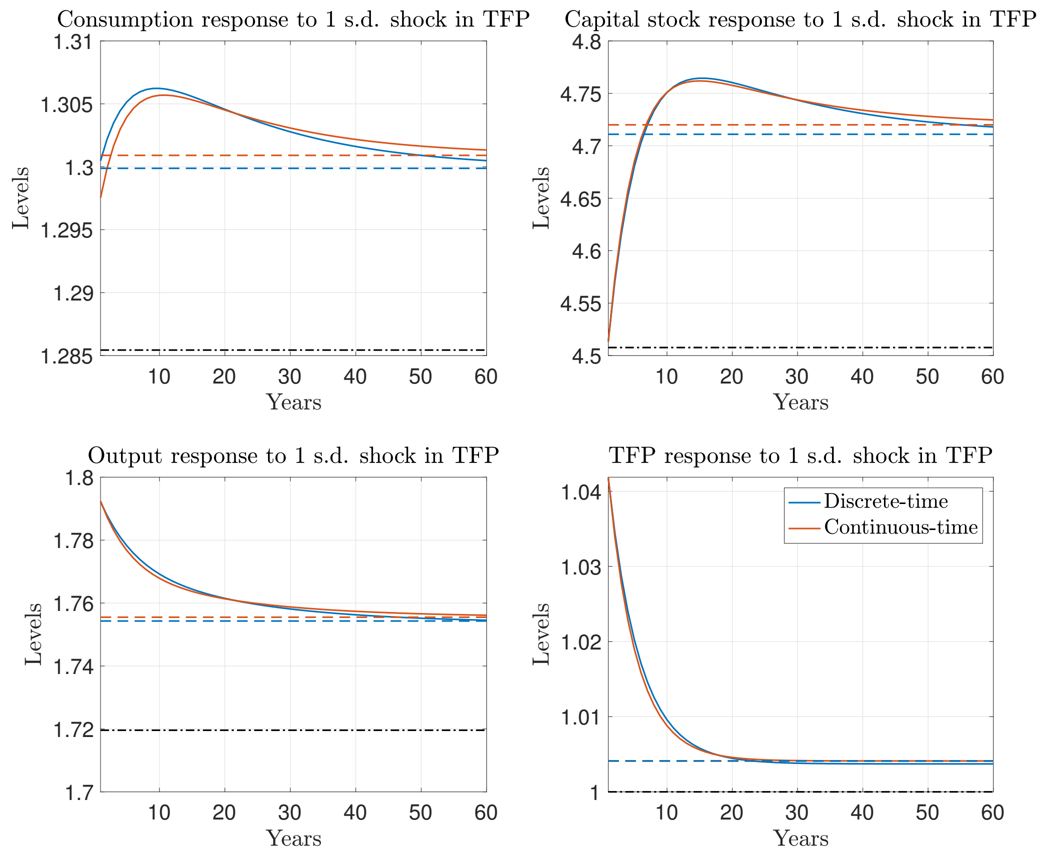

Figure 7. Second order Impulse-Response function to a TFP shock: Discrete-time vs. Continuous-time. The graph plots the second order impulse response functions (IRFs) for the levels of aggregate consumption, aggregate capital, and aggregate output for both discrete- and continuous-time models. The economic system is assumed to be in its deterministic steady state before the shock hits.The dash-dotted line corresponds to the deterministic steady state, and the dashed lines to the approximated risky steady states.

6 Conclusions

[TO BE COMPLETED]

Appendix

A Euler equation RBC in continuous-time

Using the first order condition (7), the maximized (concentrated) HJB equation reads:

$$\rho V(K_t, A_t) = \frac{C(K_t, A_t)^{1-\gamma}}{1-\gamma} + (A_t K_t^{\alpha} - C(K_t, A_t) - \delta K_t) V_K(K_t, A_t) \\ - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2) A_t V_A(K_t, A_t) + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AA}(K_t, A_t) \quad (33)$$

From (33), we obtain for the costate variable (using the envelope theorem)

$$\rho V_K(K_t, A_t) = (A_t K_t^{\alpha} - C(K_t, A_t) - \delta K_t) V_{KK}(K_t, A_t) + (\alpha A_t K_t^{\alpha-1} - \delta) V_K(K_t, A_t) \\ - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2) A_t V_{AK}(K_t, A_t) + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAK}(K_t, A_t)$$

such that

$$(\rho - \alpha A_t K_t^{\alpha-1} + \delta) V_K(K_t, A_t) = (A_t K_t^{\alpha} - C(K_t, A_t) - \delta K_t) V_{KK}(K_t, A_t) \\ - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2) A_t V_{AK}(K_t, A_t) + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAK}(K_t, A_t)$$

Using Itô's Lemma, the evolution of the costate is given by:

$$\begin{aligned} \mathrm{d}V_K(K_t, A_t) &= V_{KK}(K_t, A_t)\mathrm{d}K_t + V_{KA}(K_t, A_t)\mathrm{d}A_t + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{KAA}(K_t, A_t)\mathrm{d}t \\ &= (\rho - \alpha A_t K_t^{\alpha-1} + \delta) V_K(K_t, A_t)\mathrm{d}t + V_{KA}(K_t, A_t)\sigma_A A_t \mathrm{d}B_{A,t} \end{aligned} \quad (34)$$

Using once again the first-order condition, we may alternatively write:

$$\mathrm{d}C_t^{-\gamma} = (\rho - \alpha A_t K_t^{\alpha-1} + \delta) C_t^{-\gamma}\mathrm{d}t - \gamma C_t^{-\gamma-1} C_A \sigma_A A_t \mathrm{d}B_{A,t}$$

or

$$\frac{\mathrm{d}C_t}{C_t} = \left[\frac{1}{\gamma}\left(\alpha A_t K_t^{\alpha-1} - \delta - \rho\right) + \tfrac{1}{2}(1+\gamma)\left(\frac{C_A A_t}{C_t}\right)^2 \sigma_A^2\right]\mathrm{d}t + \left(\frac{C_A A_t}{C_t}\right)\sigma_A \mathrm{d}B_{A,t}$$

which is Equation (8).

B Conditional deterministic system

From the PDE for the costate variable we find that for $V_{KK}(K_t, A_t) \neq 0$:

$$C(K_t, A_t) = A_t K_t^{\alpha} - \delta K_t - (\rho - \alpha A_t K_t^{\alpha-1} + \delta)\frac{V_K(K_t, A_t)}{V_{KK}(K_t, A_t)} - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t \frac{V_{AK}(K_t, A_t)}{V_{KK}(K_t, A_t)} + \tfrac{1}{2}\sigma_A^2 A_t^2 \frac{V_{AAK}(K_t, A_t)}{V_{KK}(K_t, A_t)} \tag{35}$$

To obtain the necessary condition (35) from a (conditional) deterministic system, we start with the general consumption function, $C_t = C(K_t, A_t)$ which obeys:

$$\mathrm{d}C_t = C_K \mathrm{d}K_t + C_A \mathrm{d}A_t + \tfrac{1}{2}C_{AA}A_t^2\sigma_A^2 \mathrm{d}t \tag{36}$$

Inserting $\mathrm{d}A_t$ and $\mathrm{d}K_t$ from Equations (2) and (4) yields:

$$\mathrm{d}C_t = C_K(A_t K_t^{\alpha} - C_t - \delta K_t)\mathrm{d}t - C_A(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t \mathrm{d}t + C_A \sigma_A A_t \mathrm{d}B_t + \tfrac{1}{2}C_{AA}A_t^2\sigma_A^2 \mathrm{d}t$$

so

$$\mathrm{d}C_t - \tfrac{1}{2}C_{AA}A_t^2\sigma_A^2 \mathrm{d}t = C_K(A_t K_t^{\alpha} - C_t - \delta K_t)\mathrm{d}t - C_A(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t \mathrm{d}t + C_A \sigma_A A_t \mathrm{d}B_t$$

Inserting $\mathrm{d}C_t$ from Equation (8) we may eliminate time (and stochastic shocks) and arrive at:

$$\frac{1}{\gamma}\left(\alpha A_t K_t^{\alpha-1} - \delta - \rho\right) C_t + \tfrac{1}{2}(1+\gamma)C_t \left(\frac{C_A A_t}{C_t}\right)^2 \sigma_A^2 = C_K(A_t K_t^{\alpha} - C_t - \delta K_t) - C_A(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t + \tfrac{1}{2}C_{AA}A_t^2\sigma_A^2$$

which coincides with Equation (35) when using the condition $C_t = V_K^{-1/\gamma}$. A system of partial differential equations (PDEs) that implies the same policy function as in Equation (9) in the absence of shocks can be constructed from:

$$\begin{aligned}
\mathrm{d}C_t &= \left[\frac{1}{\gamma}\left(\alpha A_t K_t^{\alpha-1} - \delta - \rho\right) C_t + \tfrac{1}{2}(1+\gamma)C_t \left(\frac{C_A A_t}{C_t}\right)^2 \sigma_A^2 - \tfrac{1}{2}C_{AA}A_t^2\sigma_A^2\right]\mathrm{d}t \\
\mathrm{d}K_t &= (A_t K_t^{\alpha} - C_t - \delta K_t)\mathrm{d}t \\
\mathrm{d}A_t &= -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t \mathrm{d}t
\end{aligned}$$

together with

$$C_A = -\frac{1}{\gamma}V_K^{-\frac{1+\gamma}{\gamma}}V_{KA}, \quad C_{AA} = \frac{1+\gamma}{\gamma^2}V_K^{-\frac{1+\gamma}{\gamma}-1}V_{KA}^2 - \frac{1}{\gamma}V_K^{-\frac{1+\gamma}{\gamma}}V_{KAA}$$

such that $\mathrm{d}C_t = C_A\mathrm{d}A_t + C_K\mathrm{d}K_t$ with $\mathrm{d}C_t$, $\mathrm{d}K_t$, and $\mathrm{d}A_t$ from (10), (11), and (12), respectively, also solves the HJB equation.

Alternatively, we start with the general costate variable, $V_K = V_K(K_t, A_t)$, which obeys

$$\mathrm{d}V_K(K_t, A_t) = V_{KK}\mathrm{d}K_t + V_{AK}\mathrm{d}A_t + \tfrac{1}{2}V_{KAA}A_t^2\sigma_A^2\mathrm{d}t \tag{37}$$

Inserting $\mathrm{d}A_t$ and $\mathrm{d}K_t$ from Equations (2) and (4) yields:

$$\mathrm{d}V_K(K_t, A_t) = V_{KK}(A_tK_t^\alpha - C_t - \delta K_t)\mathrm{d}t - V_{AK}(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t + V_{AK}\sigma_A A_t \mathrm{d}B_t + \tfrac{1}{2}V_{KAA}A_t^2\sigma_A^2\mathrm{d}t$$

so

$$\mathrm{d}V_K(K_t, A_t) - \tfrac{1}{2}V_{KAA}A_t^2\sigma_A^2\mathrm{d}t = V_{KK}(A_tK_t^\alpha - C_t - \delta K_t)\mathrm{d}t - V_{AK}(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t + V_{AK}\sigma_A A_t \mathrm{d}B_t$$

Inserting $\mathrm{d}V_K$ from Equation (34) we may eliminate time (and stochastic shocks) and arrive at:

$$\begin{aligned}(\rho - \alpha A_tK_t^{\alpha-1} + \delta)V_K(K_t, A_t) = (A_tK_t^\alpha - C(K_t, A_t) - \delta K_t)V_{KK}(K_t, A_t)\\ - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_tV_{AK}(K_t, A_t) + \tfrac{1}{2}\sigma_A^2A_t^2V_{AAK}(K_t, A_t)\end{aligned}$$

which coincides with Equation (35).

A system of partial differential equations (PDEs) that implies the same policy function as in Equation (9) in the absence of shocks can be constructed from:

$$\begin{aligned}\mathrm{d}V_K &= (\rho - \alpha A_tK_t^{\alpha-1} + \delta)V_K\mathrm{d}t - \tfrac{1}{2}\sigma_A^2A_t^2V_{KAA}\mathrm{d}t\\ \mathrm{d}K_t &= (A_tK_t^\alpha - C_t - \delta K_t)\mathrm{d}t\\ \mathrm{d}A_t &= -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t\end{aligned}$$

such that $\mathrm{d}V_K = V_{KA}\mathrm{d}A_t + V_{KK}\mathrm{d}K_t$ with $\mathrm{d}V_K$, $\mathrm{d}K_t$, and $\mathrm{d}A_t$ from (34), (11), and (12), respectively, also solves the HJB equation.

C Euler equation RBC in discrete-time

The maximized Bellman equation is given by:

$$V(K_t, A_t) = u(C_t(K_t, A_t)) + \beta \mathbb{E}_t V(K_{t+1}, A_{t+1}). \tag{38}$$

From (38), we obtain for the costate variable (using the envelope theorem)

$$\frac{\partial V(K_t, A_t)}{\partial K_t} = \beta \mathbb{E}_t \left[\frac{\partial V(K_{t+1}, A_{t+1})}{\partial K_{t+1}} \frac{\partial K_{t+1}}{\partial K_t} \right].$$

Inserting the first order condition in Equation (19) and the fact that from Equation (14) $\frac{\partial K_{t+1}}{\partial K_t} = \alpha A_t K_t^{\alpha-1} + 1 - \delta$, the Euler equation is obtained by iterating forward:

$$C_t^{-\gamma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\gamma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta \right) \right].$$

D First order perturbation for RBC in continuous-time

As an example, consider a first order approximation to the consumption function of the RBC model in continuous-time of Section 2. In other words, we are only interested in computing $\overline{C_K} \equiv C_K\left(\bar{K}, \bar{A}; 0\right)$, $\overline{C_A} \equiv C_A\left(\bar{K}, \bar{A}; 0\right)$ and $\overline{C_\eta} \equiv C_\eta\left(\bar{K}, \bar{A}; 0\right)$. Let f_i denote the partial derivative of the functional $f(\cdot)$ with respect to its i-th component. Then:

$$
\begin{aligned}
F_K\left(K_t, A_t; \eta\right) &= f_1 + f_3 C_K + f_4 C_{KK} + f_5 C_{AK} + f_6 C_{AAK} = 0 \\
F_A\left(K_t, A_t; \eta\right) &= f_2 + f_3 C_A + f_4 C_{KA} + f_5 C_{AA} + f_6 C_{AAA} = 0 \\
F_\eta\left(K_t, A_t; \eta\right) &= f_3 C_\eta + f_4 C_{K\eta} + f_5 C_{A\eta} + f_6 C_{AA\eta} + f_7 = 0
\end{aligned}
$$

which evaluated at the deterministic steady state reduces to:

$$
\begin{aligned}
F_K\left(\bar{K}, \bar{A}; 0\right) &= f_1 + f_3 \overline{C_K} = 0 \\
F_A\left(\bar{K}, \bar{A}; 0\right) &= f_2 + f_3 \overline{C_A} = 0 \\
F_\eta\left(K_t, A_t; \eta\right) &= f_7 + f_3 \overline{C_\eta} = 0
\end{aligned}
$$

which is a system of three non-linear equations in three unknowns, $\overline{C_K}$, $\overline{C_A}$, $\overline{C_\eta}$, where the constants f_1, f_2, f_3, f_7 are also functions of these unknowns.

To find the first of these constants, $\overline{C_K}$, we differentiate the extended conditional deterministic system with respect to the capital stock, $F_K\left(K_t, A_t; \eta\right)$, which evaluated at the deterministic steady state yields:

$$
F_K\left(\bar{K}, \bar{A}; 0\right) = \left(\alpha \bar{A} \bar{K}^{\alpha-1} - \delta - \overline{C_K}\right) \frac{\overline{C_K}}{\overline{C}} - \frac{1}{\gamma} \alpha (\alpha - 1) \bar{A} \bar{K}^{\alpha-2}.
$$

Since this derivative must be zero, we obtain the quadratic equation:

$$
\overline{C_K}^2 - \left(\alpha \bar{A} \bar{K}^{\alpha-1} - \delta\right) \overline{C_K} + \frac{1}{\gamma} \alpha (\alpha - 1) \bar{A} \bar{K}^{\alpha-2} \overline{C} = 0
$$

with roots:

$$
\overline{C_K} = \frac{\left(\alpha \bar{A} \bar{K}^{\alpha-1} - \delta\right)}{2} \pm \sqrt{\frac{\left(\alpha \bar{A} \bar{K}^{\alpha-1} - \delta\right)^2 - 4 \frac{1}{\gamma} \alpha (\alpha - 1) \bar{A} \bar{K}^{\alpha-2} \overline{C}}{4}}
$$

We pick the positive root since it is the only one that is consistent with a concave value function $V\left(K_t, A_t\right)$ in the capital stock dimension. To see why, recall that the first order condition:

$$
U_C\left(C\left(K_t, A_t\right)\right) = V_K\left(K_t, A_t\right)
$$

together with the assumptions on the utility function $U(C)$ imposes a necessary condition for concavity of the value function. A sufficient condition for concavity is given by the derivative of the first order condition:

$$
U_{CC}\left(C\left(K_t, A_t\right)\right) C_K\left(K_t, A_t\right) = V_{KK}\left(K_t, A_t\right)
$$

which suggests that $V_{KK}(K_t, A_t) < 0$ if and only if $C_K(K_t, A_t) > 0$ given that $U_{CC}(C(K_t, A_t)) < 0$.

To compute $\overline{C_A}$, we differentiate the extended conditional deterministic system with respect to the TFP, $F_A(K_t, A_t; \eta)$, which evaluated at the deterministic steady state yields:

$$F_A\left(\bar{K}, \bar{A}; 0\right) = \frac{\alpha K_t^{\alpha-1}}{\gamma}\overline{C} - \overline{C_K}\left(K_t^{\alpha} - \overline{C_A}\right) + \overline{C_A}\rho_A.$$

Since this derivative must be zero, we arrive to the following linear equation:

$$\overline{C_A} = \frac{1}{\left(\overline{C_K} + \rho_A\right)}\left[\overline{C_K}K_t^{\alpha} - \frac{\alpha K_t^{\alpha-1}}{\gamma}\overline{C}\right]$$

which can be readily computed once the value for $C_K\left(\bar{K}, \bar{A}; 0\right)$ is obtained from the first step.

To complete the first order perturbation we still need to compute the loading $\overline{C_\eta}$. To do so, differentiate the conditional deterministic system with respect to the perturbation parameter, $F_\eta(K_t, A_t; \eta)$, which evaluated at the deterministic steady state yields:

$$F_\eta\left(\bar{K}, \bar{A}; 0\right) = \overline{C_\eta C_K} + \frac{1}{2}(1+\gamma)\overline{C}\left(\frac{\overline{C_A}\bar{A}}{\overline{C}}\right)\sigma_A^2 - \frac{1}{2}\overline{C_A}\sigma_A^2\bar{A} - \frac{1}{2}\overline{C_{AA}}\bar{A}^2\sigma_A^2$$

Since $F_\eta(K_t, A_t; \eta) = 0$, we arrive to the linear equation:

$$\overline{C_\eta} = -\left(\overline{C_K}\right)^{-1}\left[\frac{1}{2}(1+\gamma)\overline{C}\left(\frac{\overline{C_A}\bar{A}}{\overline{C}}\right)^2\sigma_A^2 - \frac{1}{2}\overline{C_A}\sigma_A^2\bar{A} - \frac{1}{2}\overline{C_{AA}}\bar{A}^2\sigma_A^2\right].$$

Note however that in order to compute the loading $\overline{C_\eta}$ we require information on $\overline{C_{AA}}$ which can be only computed by solving the system of equations defined by $F_{KK}\left(\bar{K}, \bar{A}; 0\right) = 0$, $F_{KA}\left(\bar{K}, \bar{A}; 0\right) = 0$ and $F_{AA}\left(\bar{K}, \bar{A}; 0\right) = 0$. Hence, as pointed out in Judd and Guu (1993) it follows that for the computation of perturbations in continuous-time stochastic models, an $(n+2)$-th order deterministic approximation to the policy function is required in order to be able to compute an n-th order stochastic approximation. This restricts the number of derivatives that can be computed with respect to the perturbation parameter η.

Therefore, a second order approximation to the policy function is defined by a fourth deterministic approximation and a second order stochastic approximation. In between some cross derivatives will be required. It is important to note that after the first order approximation has been computed, the higher order approximations correspond to linear systems of equations.

E Risky steady states in discrete-time models

Consider the case of TFP. By defining the difference between the risky steady state and deterministic steady state as $A_\Delta = A_{rss} - \bar{A}$, the second order approximation of TFP evaluated at the risky steady state:

$$A_{rss} = \bar{A} + h_{2A}(A_{rss} - \bar{A}) + \frac{1}{2}\left[h_{2AA}(A_{rss} - \bar{A})^2 + h_{2\eta\eta}\eta^2\right].$$

can be written as:

$$\begin{aligned} -A_\Delta + h_{2A}A_\Delta + \frac{1}{2}h_{2AA}(A_\Delta)^2 + \frac{1}{2}h_{2\eta\eta}\eta^2 &= 0 \\ \Rightarrow (A_\Delta)^2 + \frac{2\left(h_{2A} - 1\right)}{h_{2AA}}A_\Delta + \frac{h_{2\eta\eta}\eta^2}{h_{2AA}} &= 0 \end{aligned}$$

which has the two solutions:

$$A_\Delta = -\frac{h_{2A} - 1}{h_{2AA}} \pm \sqrt{\left(\frac{h_{2A} - 1}{h_{2AA}}\right)^2 - \frac{h_{2\eta\eta}\eta^2}{h_{2AA}}}, \tag{39}$$

where one of the roots can be discarded since it leads to a negative TFP level. The risky steady state follows from $A_{rss} = \bar{A} + A_\Delta$.

In a similar fashion, let us define K_Δ to be equal to the difference between the risky and the deterministic steady state value of aggregate capital. Then, the second order approximation of TFP evaluated at the risky steady state:

$$\begin{aligned} K_{rss} &= \overline{K} + \overline{h_{1K}}(K_{rss} - \bar{K}) + \overline{h_{1A}}(A_{rss} - \bar{A}) \\ &+ \frac{1}{2}\left[\overline{h_{1KK}}(K_{rss} - \bar{K})^2 + 2\overline{h_{1KA}}(K_{rss} - \bar{K})(A_{rss} - \bar{A}) + \overline{h_{1AA}}(A_{rss} - \bar{A})^2 + \overline{h_{1\eta\eta}}\eta^2\right]. \end{aligned}$$

can be written as:

$$-K_\Delta + \overline{h_{1K}}K_\Delta + \overline{h_{1A}}A_\Delta + \frac{1}{2}\left[\overline{h_{1KK}}(K_\Delta)^2 + 2\overline{h_{1KA}}K_\Delta A_\Delta + \overline{h_{1AA}}(A_\Delta)^2 + \overline{h_{1\eta\eta}}\eta^2\right] = 0.$$

Inserting the solution for A_Δ from (39) and rearranging constant, linear, and quadratic terms in K_Δ leads to:

$$(K_\Delta)^2 + \frac{\left(\overline{h_{1K}} - 1 + \overline{h_{1KA}}A_\Delta\right)}{\overline{h_{1KK}}}2K_\Delta + \frac{2\overline{h_{1A}}A_\Delta + \overline{h_{1AA}}(A_\Delta)^2 + \overline{h_{1\eta\eta}}\eta^2}{\overline{h_{1KK}}} = 0$$

with solutions:

$$K_\Delta = -\frac{\overline{h_{1K}} - 1 + \overline{h_{1KA}}A_\Delta}{\overline{h_{1KK}}} \pm \sqrt{\left(\frac{\overline{h_{1K}} - 1 + \overline{h_{1KA}}A_\Delta}{\overline{h_{1KK}}}\right)^2 - \frac{2\overline{h_{1A}}A_\Delta + \overline{h_{1AA}}(A_\Delta)^2 + \overline{h_{1\eta\eta}}\eta^2}{\overline{h_{1KK}}}},$$

where one of the roots can be discarded as it results in a negative capital stock. Finally, K_{rss} follows from $K_{rss} = \bar{K} + K_\Delta$.

F Euler equation errors

Figure 8 compares the Euler equation errors between discrete and continuous-time models for each of the orders of approximation along the aggregate capital lattice, while Figure 9 compares the Euler equation errors for the first and second order approximations to the policy functions for consumption for each of the timing assumptions.

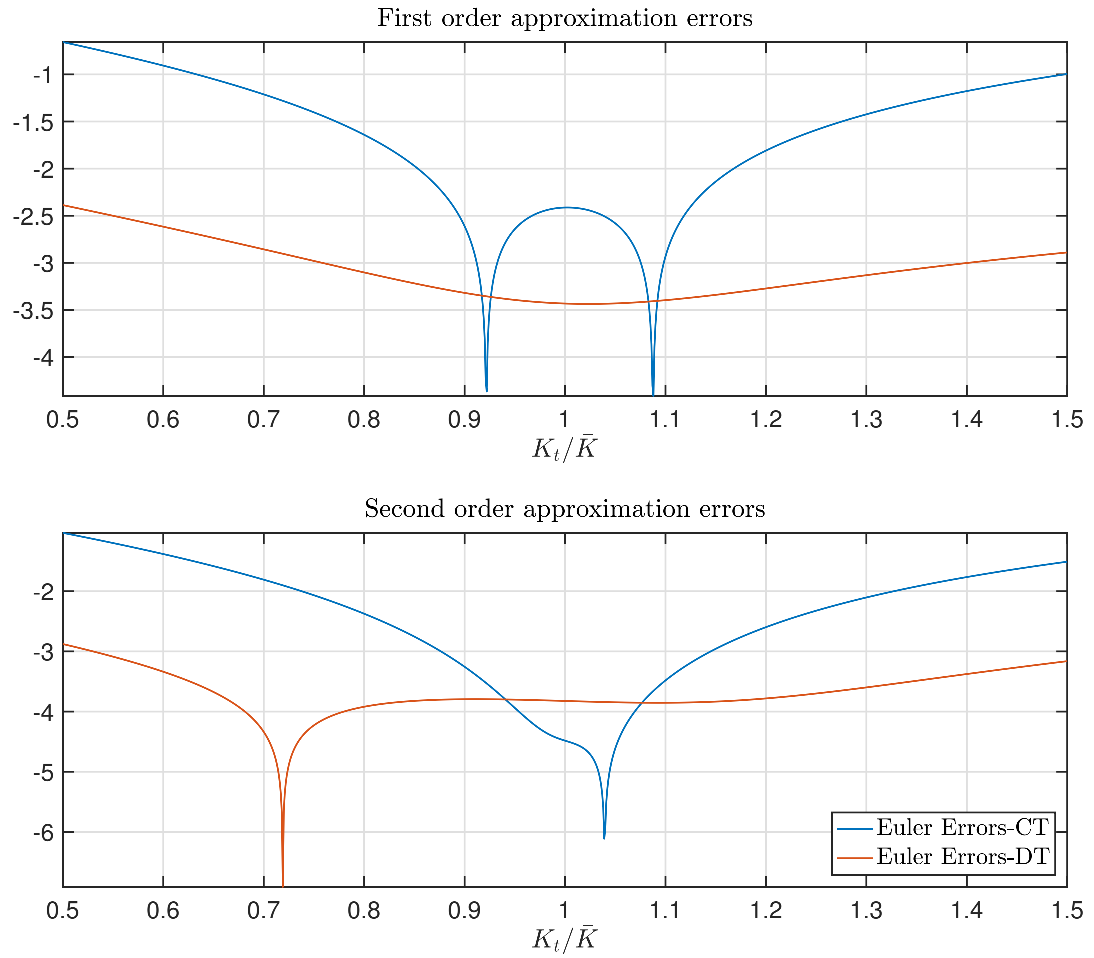

Figure 8. Euler Equation Errors. The graph plots the log10 of the absolute value of the Euler equation errors for the first order approximation (top panel) and the second order approximation (bottom panel) to the policy function of aggregate consumption along the capital lattice while keeping productivity at its determinisitic steady state, $EE\left(K, \bar{A}\right)$.

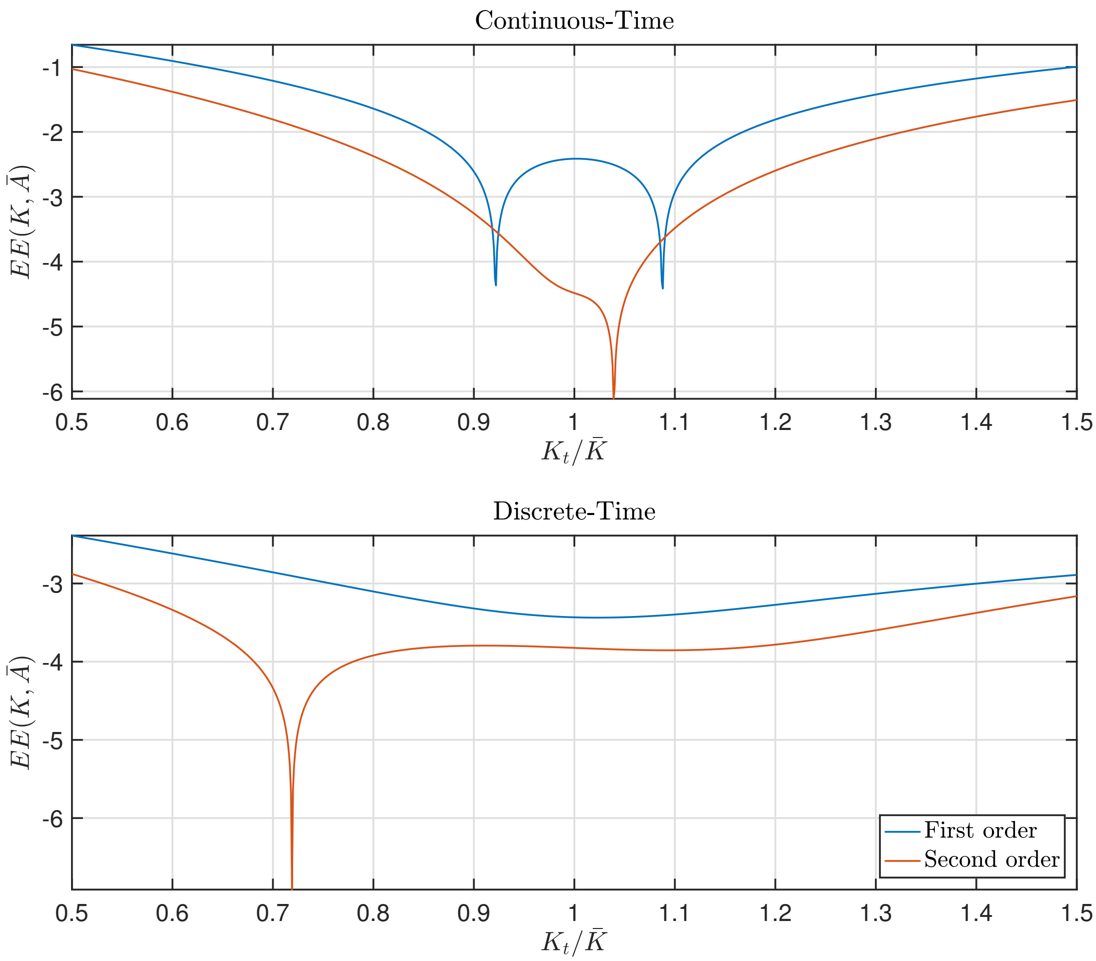

Figure 9. Euler Equation Errors. The graph plots the log10 of the absolute value of the first and second order approximation Euler equation errors along the capital lattice while keeping productivity at its determinisitic steady state, $EE\left(K,\bar{A}\right)$ for the RBC model in continuous-time (top panel) and the model in discrete-time (bottom panel).

G A model with capital adjustment cost and habit formation

G.1 Continuous-time

G.1.1 Technology

Consider the problem faced by a benevolent planner with a production function:

$$Y_t = A_t K_t^{\alpha}, \tag{40}$$

where A_t is the total factor productivity (TFP) and K_t is the aggregate capital stock.

The capital stock increases if the costly gross investment exceeds depreciation,

$$\mathrm{d}K_t = (\Phi(I_t/K_t) - \delta)K_t \mathrm{d}t \tag{41}$$

where the capital adjustment cost function is given by the cost specification in Jermann (1998):

$$\Phi(I_t/K_t) = \frac{a_1}{1-1/\xi}\left(\frac{I_t}{K_t}\right)^{1-1/\xi} + a_2$$

and where $\xi > 0$ denotes the elasticity of the investment-to-capital ratio with respect to Tobin's q and $a_1 \geq 0$ and $a_2 \geq 0$ are parameters. Following Boldrin et al. (2001), we set $a_1 = \delta^{1/\xi}$ and $a_2 = \frac{\delta}{1-\xi}$ such that the steady state is invariant to ξ, and hence the steady state investment-to-capital ratio equals the deprecation rate, $\bar{I}/\bar{K} = \delta$ [7]. By inserting the market clearing condition:

$$Y_t = C_t + I_t,$$

the dynamics for the capital stock can be rewritten as:

$$\mathrm{d}K_t = \left(\Phi\left(\frac{A_t K_t^{\alpha} - C_t}{K_t}\right) - \delta\right) K_t \mathrm{d}t \quad K_0 > 0 \tag{42}$$

Finally, the logarithm of the TFP follows an Ornstein-Uhlenbeck process with mean reversion $\rho_A > 0$ of the form:

$$\begin{aligned} \mathrm{d}\log A_t &= -\rho_A \log A_t \mathrm{d}t + \sigma_A \mathrm{d}B_{A,t} \\ \Leftrightarrow\ \mathrm{d}A_t &= -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t \mathrm{d}t + \sigma_A A_t \mathrm{d}B_{A,t} \quad A_0 > 0 \end{aligned} \tag{43}$$

where $B_{A,t}$ is a standard Brownian motions with volatility σ_A.

[7] Given this parameterization it can be shown that in the steady $\Phi(\bar{I}/\bar{K}) = \Phi(\delta) = \delta$, $\Phi'(\bar{I}/\bar{K}) = \Phi'(\delta) = 1$, and $\Phi''(\bar{I}/\bar{K}) = \Phi''(\delta) = -\frac{1}{\xi\delta}$, i.e. the slope of Φ' depends negatively on ξ and δ.

G.1.2 Households

The economy is assumed to be inhabited by a sufficiently large number of identical individuals, which maximize their discounted life-time utility

$$U_0 \equiv \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} \mathrm{d}t \right], \tag{44}$$

where ρ is the household's subjective discount rate and γ is related to the coefficient of relative risk aversion. Equation (44) exhibits adjacent complementarity in consumption as defined in Ryder and Heal (1973). Hence, an increase in consumption increases the marginal utility of consumption at adjacent dates relative to the marginal utility of consumption at distant ones. The consumption choice made by households is assumed to be non-negative, $C_t \geq 0$, and not to fall below a subsistence level of consumption, $C_t \geq X_t$. Habit in consumption is defined endogenously within the model according to[8]:

$$X_t = e^{-at} X_0 + b \int_0^t e^{a(s-t)} C_s \mathrm{d}s, \quad X_0 > 0$$

or equivalently,

$$\mathrm{d}X_t = (bC_t - aX_t)\mathrm{d}t. \tag{45}$$

Hence, X_t is a weighted sum of past consumption, with weights declining exponentially into the past. The larger is b, the less weight is given to past consumption in determining X_t and viceversa. The special case $b = X_0 = 0$ correspond to the case of time-separable utility with constant relative risk aversion (see also Constantinides (1990)).

Households maximize Equation (44) subject to the dynamics in (42), (45) and (43).

G.1.3 The HJB equation and the first-order conditions

The benevolent planner chooses a path for consumption in order to maximize expected life-time utility of a representative household. Define the value of the optimal program

$$V(K_0, X_0, A_0) = \max_{\{C_t \geq X_t \in \mathbb{R}^+\}_{t=0}^{\infty}} U_0 \quad s.t. \quad (42), (43) \quad \text{and} \quad (45) \tag{46}$$

in which $C_t \geq X_t \in \mathbb{R}^+$ denotes the control at instant $t \in \mathbb{R}^+$.

As the first step, the *Hamilton-Jacobi-Bellman equation* (HJB) reads for any $t \in [0, \infty)$

$$\rho V(K_t, X_t, A_t) = \max_{C_t \geq X_t \in \mathbb{R}^+} \left\{ \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} + \frac{1}{\mathrm{d}t} \mathbb{E}_t \mathrm{d}V(K_t, X_t, A_t) \right\} \tag{47}$$

where

$$\mathrm{d}V(K_t, X_t, A_t) \quad = \quad V_K \mathrm{d}K_t + V_X \mathrm{d}X_t + V_A \mathrm{d}A_t + \tfrac{1}{2} V_{AA} A_t^2 \sigma_A^2 \mathrm{d}t$$

[8]This is in contrast with the relative consumption model (catching up with the Joneses), or external habit model, where the habit is aggregate consumption and thus exogenous to the households.

Using the martingale difference properties of stochastic integrals, we arrive at

$$
\begin{aligned}
0 = \max_{C_t \geq X_t \in \mathbb{R}^+} \Bigg\{ & \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} + (\Phi((A_t K_t^\alpha - C_t)/K_t)K_t - \delta K_t)V_K(K_t, X_t, A_t) \\
& + (bC_t - aX_t)V_X(K_t, X_t, A_t) - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t V_A(K_t, X_t, A_t) \\
& + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AA}(K_t, X_t, A_t) - \rho V(K_t, X_t, A_t) \Bigg\}
\end{aligned}
$$

The first-order condition for any interior solution reads:

$$
(C_t - X_t)^{-\gamma} + bV_X(K_t, X_t, A_t) = \Phi'(I_t/K_t)V_K(K_t, X_t, A_t), \tag{48}
$$

making optimal consumption a function of the state variables, $C_t = C(K_t, X_t, A_t)$, where $\Phi'(I_t/K_t) = \mathrm{d}\Phi(I_t/K_t)/\mathrm{d}(I_t/K_t) = a_1 (I_t/K_t)^{-1/\xi}$.

The maximized (concentrated) HJB equation is then:

$$
\begin{aligned}
\rho V(K_t, X_t, A_t) \quad = \quad & \frac{(C(K_t, X_t, A_t) - X_t)^{1-\gamma}}{1-\gamma} + (\Phi((A_t K_t^\alpha - C(K_t, X_t, A_t))/K_t)K_t \\
& -\delta K_t)\, V_K(K_t, X_t, A_t) + (bC(K_t, X_t, A_t) - aX_t)V_X(K_t, X_t, A_t) \\
& -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t V_A(K_t, X_t, A_t) + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AA}(K_t, X_t, A_t)
\end{aligned} \tag{49}
$$

From (49), we obtain for the costate variable with respect to capital (using the envelope theorem):

$$
\begin{aligned}
\rho V_K(K_t, X_t, A_t) \quad = \quad & (\Phi((A_t K_t^\alpha - C_t)/K_t)K_t - \delta K_t)V_{KK}(K_t, X_t, A_t) \\
& +(\Phi((A_t K_t^\alpha - C_t)/K_t) + \Phi'((A_t K_t^\alpha - C_t)/K_t)((\alpha - 1)A_t K_t^{\alpha-1} \\
& +C_t/K_t) - \delta)V_K(K_t, X_t, A_t) + (bC_t - aX_t)V_{XK}(K_t, X_t, A_t) \\
& -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t V_{AK}(K_t, X_t, A_t) + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAK}(K_t, X_t, A_t)
\end{aligned}
$$

such that:

$$
\begin{aligned}
&(\rho - \Phi((A_t K_t^\alpha - C_t)/K_t) - \Phi'((A_t K_t^\alpha - C_t)/K_t)((\alpha - 1)A_t K_t^{\alpha-1} + C_t/K_t) + \delta)V_K(K_t, X_t, A_t) = \\
&\qquad (\Phi((A_t K_t^\alpha - C_t)/K_t)K_t - \delta K_t)V_{KK}(K_t, X_t, A_t) + (bC_t - aX_t)V_{XK}(K_t, X_t, A_t) \\
&\qquad -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t V_{AK}(K_t, X_t, A_t) + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAK}(K_t, X_t, A_t)
\end{aligned}
$$

Using Itô's Lemma, the evolution of the costate with respect to capital is:

$$
\begin{aligned}
\mathrm{d}V_K(K_t,X_t,A_t) &= V_{KK}(K_t,X_t,A_t)\mathrm{d}K_t + V_{KX}(K_t,X_t,A_t)\mathrm{d}X_t \\
&\quad + V_{KA}(K_t,X_t,A_t))\mathrm{d}A_t + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{KAA}(K_t,X_t,A_t)\mathrm{d}t \\
&= V_{KK}(K_t,X_t,A_t)(\Phi(I_t/K_t)K_t - \delta K_t)\mathrm{d}t \\
&\quad + V_{KX}(K_t,X_t,A_t)(bC_t - aX_t)\mathrm{d}t - V_{KA}(K_t,X_t,A_t)(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t \\
&\quad + V_{KA}(K_t,X_t,A_t)\sigma_A A_t \mathrm{d}B_{A,t} + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{KAA}(K_t,X_t,A_t)\mathrm{d}t \\
&= (\rho - \Phi((A_tK_t^\alpha - C_t)/K_t) - \Phi'((A_tK_t^\alpha - C_t)/K_t)((\alpha-1)A_tK_t^{\alpha-1} + C_t/K_t) \\
&\quad + \delta)V_K(K_t,X_t,A_t)\mathrm{d}t + V_{KA}(K_t,X_t,A_t)\sigma_A A_t \mathrm{d}B_{A,t} \qquad (50)
\end{aligned}
$$

From (49), the costate variable with respect to the habit level (using the envelope theorem) reads:

$$
\begin{aligned}
\rho V_X(K_t,X_t,A_t) &= -(C_t - X_t)^{-\gamma} + (\Phi((A_tK_t^\alpha - C_t))K_t - \delta K_t)V_{KX}(K_t,X_t,A_t) \\
&\quad + (bC_t - aX_t)V_{XX}(K_t,X_t,A_t) - aV_X(K_t,X_t,A_t) \\
&\quad - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t V_{AX}(K_t,X_t,A_t) + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAX}(K_t,X_t,A_t)
\end{aligned}
$$

such that:

$$
\begin{aligned}
(\rho + a)V_X(K_t,X_t,A_t) + (C_t - X_t)^{-\gamma} &= (\Phi((A_tK_t^\alpha - C_t)/K_t)K_t - \delta K_t)V_{KX}(K_t,X_t,A_t) \\
&\quad + (bC_t - aX_t)V_{XX}(K_t,X_t,A_t) \\
&\quad - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t V_{AX}(K_t,X_t,A_t) \\
&\quad + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAX}(K_t,X_t,A_t)
\end{aligned}
$$

Using Itô's Lemma, the evolution of the costate with respect to the habit level is:

$$
\begin{aligned}
\mathrm{d}V_X(K_t,X_t,A_t) &= V_{XK}(K_t,X_t,A_t)\mathrm{d}K_t + V_{XX}(K_t,X_t,A_t)\mathrm{d}X_t \\
&\quad + V_{XA}(K_t,X_t,A_t)\mathrm{d}A_t + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{XAA}(K_t,X_t,A_t)\mathrm{d}t \\
&= V_{XK}(K_t,X_t,A_t)(\Phi(I_t/K_t)K_t - \delta K_t)\mathrm{d}t + V_{XX}(K_t,X_t,A_t)(bC_t - aX_t)\mathrm{d}t \\
&\quad - V_{XA}(K_t,X_t,A_t)(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t + V_{XA}(K_t,X_t,A_t)\sigma_A A_t \mathrm{d}B_{A,t} \\
&\quad + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{XAA}(K_t,X_t,A_t)\mathrm{d}t \\
&= ((\rho + a)V_X(K_t,X_t,A_t) + (C_t - X_t)^{-\gamma})\mathrm{d}t + V_{XA}(K_t,X_t,A_t)\sigma_A A_t \mathrm{d}B_{A,t} \qquad (51)
\end{aligned}
$$

G.1.4 Equilibrium

The equilibrium in this economy in the time-domain is given by the sequence $\{V_{K,t}, V_{X,t}, K_t, X_t, A_t\}_{t=0}^{\infty}$ that solves the following system of differential equations:

$$
\begin{aligned}
\mathrm{d}V_{K,t} &= (\rho - \Phi((A_t K_t^{\alpha} - C_t)/K_t) - \Phi'((A_t K_t^{\alpha} - C_t)/K_t)((\alpha-1)A_t K_t^{\alpha-1} + C_t/K_t) \\
&\quad +\delta)V_{K,t}\mathrm{d}t + V_{KA,t}\sigma_A A_t \mathrm{d}B_{A,t} \\
\mathrm{d}V_{X,t} &= ((\rho + a)V_{X,t} + (C_t - X_t)^{-\gamma})\mathrm{d}t + V_{XA,t}\sigma_A A_t \mathrm{d}B_{A,t} \\
\mathrm{d}K_t &= (\Phi(I_t/K_t)K_t - \delta K_t)\mathrm{d}t \\
\mathrm{d}X_t &= (bC_t - aX_t)\mathrm{d}t \\
\mathrm{d}A_t &= -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t + \sigma_A A_t \mathrm{d}B_{A,t}
\end{aligned}
$$

together with initial conditions $K(0) = K_0$, $X(0) = X_0$, and $A(0) = A_0$ and where C_t solves the non-linear algebraic equation:

$$
(C_t - X_t)^{-\gamma} + bV_{X,t} = \Phi'((A_t K_t^{\alpha} - C_t)/K_t)V_{K,t}. \tag{52}
$$

Alternatively, the equilibrium of the economy in the state-space domain is given by the set of policy functions $\{V_K(K_t, X_t, A_t), V_X(K_t, X_t, A_t), C(K_t, X_t, A_t)\}$ that satisfy the following system of functional equations:

$$
\begin{aligned}
0 &= (\rho - \Phi((A_t K_t^{\alpha} - C(K_t, X_t, A_t))/K_t) \\
&\quad -\Phi'((A_t K_t^{\alpha} - C(K_t, X_t, A_t))/K_t)((\alpha-1)A_t K_t^{\alpha-1} + C(K_t, X_t, A_t)/K_t) \\
&\quad +\delta)V_{K,t}(K_t, X_t, A_t) - (\Phi((A_t K_t^{\alpha} - C(K_t, X_t, A_t))/K_t)K_t - \delta K_t)V_{KK}(K_t, X_t, A_t) \\
&\quad -(bC(K_t, X_t, A_t) - aX_t)V_{XK}(K_t, X_t, A_t) + (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t V_{AK}(K_t, X_t, A_t) \\
&\quad -\tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAK}(K_t, X_t, A_t) \\
0 &= (\rho + a)V_X(K_t, X_t, A_t) + (C(K_t, X_t, A_t) - X_t)^{-\gamma} \\
&\quad -(\Phi((A_t K_t^{\alpha} - C(K_t, X_t, A_t))/K_t)K_t - \delta K_t)V_{KX}(K_t, X_t, A_t) \\
&\quad -(bC(K_t, X_t, A_t) - aX_t)V_{XX}(K_t, X_t, A_t) + (\rho_A \log A_t + \tfrac{1}{2}\sigma_A^2)A_t V_{AX}(K_t, X_t, A_t) \\
&\quad -\tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAX}(K_t, X_t, A_t) \\
0 &= (C(K_t, X_t, A_t) - X_t)^{-\gamma} + bV_X(K_t, X_t, A_t) - \Phi'((A_t K_t^{\alpha} - C(K_t, X_t, A_t))/K_t)V_K(K_t, X_t, A_t)
\end{aligned}
$$

together with the dynamic equations for the state variables:

$$
\mathrm{d}K_t = (\Phi((A_t K_t^{\alpha} - C(K_t, X_t, A_t))/K_t)K_t - \delta K_t)\mathrm{d}t \tag{53}
$$

$$
\mathrm{d}X_t = (bC(K_t, X_t, A_t) - aX_t)\mathrm{d}t \tag{54}
$$

$$
\mathrm{d}A_t = -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t + \sigma_A A_t \mathrm{d}B_{A,t} \tag{55}
$$

subject to initial conditions $K(0) = K_0$, $X(0) = X_0$ and $A(0) = A_0$.

G.1.5 Deterministic steady state

A deterministic steady state is defined as the limiting behavior of the economy under the assumption that all variables in the economy do not grow and agents do not anticipate the effects of future shocks. The steady state of the economy is given by the values $\{\bar{C}, \bar{I}, \bar{V}_K, \bar{V}_X, \bar{K}, \bar{X}, \bar{A}\}$ that solve the following system of equations:

$$\rho - \Phi(\bar{I}/\bar{K}) - \Phi'(\bar{I}/\bar{K})((\alpha-1)\bar{A}\bar{K}^{\alpha-1} + \bar{C}/\bar{K}) + \delta = 0 \tag{56}$$

$$(\rho + a)\bar{V}_X + (\bar{C} - \bar{X})^{-\gamma} = 0 \tag{57}$$

$$\Phi\left(\bar{I}/\bar{K}\right) - \delta = 0 \tag{58}$$

$$b\bar{C} - a\bar{X} = 0 \tag{59}$$

$$\left(\bar{C} - \bar{X}\right)^{-\gamma} + b\bar{V}_X - \Phi'\left(\bar{I}/\bar{K}\right)\bar{V}_K = 0 \tag{60}$$

$$\bar{I}/\bar{K} - \frac{\bar{A}\bar{K}^{\alpha} - \bar{C}}{\bar{K}} = 0 \tag{61}$$

$$\bar{A} - 1 = 0. \tag{62}$$

The solution to this system of equation is entirely determined by the steady state value of the investment-capital ratio $(\bar{I}/\bar{K})$. Given the values of a_1 and a_2, it is possible to show that for any value of ξ:

$$\bar{I}/\bar{K} = \delta.$$

Note that for the steady-state value of the investment-capital ratio, $\Phi(\delta) = \delta$, $\Phi'(\delta) = 1$, and $\Phi''(\bar{I}/\bar{K}) = \Phi''(\delta) = -\frac{1}{\xi\delta}$. Now, from Equations (56) and (61) we find the steady state value of the capital stock as:

$$\bar{K} = \left[\frac{\alpha\bar{A}}{(\rho+\delta)}\right]^{\frac{1}{1-\alpha}}. \tag{63}$$

Using Equation (61) we find the steady state value of consumption:

$$\bar{C} = \bar{A}\bar{K}^{\alpha} - \delta\bar{K}. \tag{64}$$

From Equation (59) we pin down the steady state value of the habit as:

$$\bar{X} = \frac{b}{a}\bar{C}. \tag{65}$$

Finally using Equations (57) and (60) we find the long-run values of the costate variables:

$$\bar{V}_X = -\frac{1}{\rho+a}\left(\bar{C} - \bar{X}\right)^{-\gamma} \tag{66}$$

$$\bar{V}_K = \left(1 - \frac{b}{\rho+a}\right)\left(\bar{C} - \bar{X}\right)^{-\gamma}. \tag{67}$$

G.1.6 Conditional deterministic system

We start with the general costate variable, $V_K = V_K(K_t, X_t, A_t)$, which obeys:

$$\mathrm{d}V_K = V_{KK}\mathrm{d}K_t + V_{KX}\mathrm{d}X_t + V_{KA}\mathrm{d}A_t + \tfrac{1}{2}V_{KAA}\sigma_A^2 A_t^2\mathrm{d}t \tag{68}$$

Inserting $\mathrm{d}K_t$, $\mathrm{d}A_t$, and $\mathrm{d}X_t$ yields:

$$\begin{aligned}\mathrm{d}V_K &= V_{KK}(\Phi((A_tK_t^\alpha - C_t)/K_t)K_t - \delta K_t)\mathrm{d}t + V_{KX}(bC_t - aX_t)\mathrm{d}t \\ &\quad - V_{KA}(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t + \tfrac{1}{2}V_{KAA}\sigma_A^2A_t^2\mathrm{d}t + V_{KA}\sigma_A A_t\mathrm{d}B_{A,t}\end{aligned}$$

so

$$\begin{aligned}\mathrm{d}V_K - \tfrac{1}{2}V_{KAA}\sigma_A^2A_t^2\mathrm{d}t &= V_{KK}(\Phi((A_tK_t^\alpha - C_t)/K_t)K_t - \delta K_t)\mathrm{d}t + V_{KX}(bC_t - aX_t)\mathrm{d}t \\ &\quad - V_{KA}(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t + V_{KA}\sigma_A A_t\mathrm{d}B_{A,t}\end{aligned}$$

Inserting $\mathrm{d}V_K$ from Equation (50) we may eliminate time (and stochastic shocks) and arrive at the the costate obtained from the maximized HJB equation with respect to the capital stock:

$$\begin{aligned}(\rho - \Phi((A_tK_t^\alpha - C_t)/K_t) - \Phi'((A_tK_t^\alpha - C_t)/K_t)((\alpha-1)A_tK_t^{\alpha-1} + C_t/K_t) + \delta)V_K - \tfrac{1}{2}V_{KAA}\sigma_A^2A_t^2 = \\ V_{KK}(\Phi((A_tK_t^\alpha - C_t)/K_t)K_t - \delta K_t) + V_{KX}(bC_t - aX_t) - V_{KA}(\rho_A\log A_t - \tfrac{1}{2}\sigma_A^2)A_t\end{aligned}$$

Similarly, $V_X = V_X(K_t, X_t, A_t)$ obeys:

$$\mathrm{d}V_X = V_{XK}\mathrm{d}K_t + V_{XX}\mathrm{d}X_t + V_{XA}\mathrm{d}A_t + \tfrac{1}{2}V_{XAA}A_t^2\sigma_A^2\mathrm{d}t \tag{69}$$

Inserting $\mathrm{d}K_t$, $\mathrm{d}X_t$, and $\mathrm{d}A_t$ yields:

$$\begin{aligned}\mathrm{d}V_X &= V_{XK}(\Phi((A_tK_t^\alpha - C_t)/K_t)K_t - \delta K_t)\mathrm{d}t + V_{XX}(bC_t - aX_t)\mathrm{d}t \\ &\quad - V_{XA}(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t + V_{XA}\sigma_A A_t\mathrm{d}B_{A,t} + \tfrac{1}{2}V_{XAA}\sigma_A^2A_t^2\mathrm{d}t\end{aligned}$$

so

$$\begin{aligned}\mathrm{d}V_X - \tfrac{1}{2}V_{XAA}\sigma_A^2A_t^2\mathrm{d}t &= V_{XK}(\Phi((A_tK_t^\alpha - C_t)/K_t)K_t - \delta K_t)\mathrm{d}t + V_{XX}(bC_t - aX_t)\mathrm{d}t \\ &\quad - V_{XA}(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\mathrm{d}t + V_{XA}\sigma_A A_t\mathrm{d}B_{A,t}\end{aligned}$$

Inserting $\mathrm{d}V_X$ from Equation (51) we may eliminate time (and stochastic shocks) and arrive at the the costate obtained from the maximized HJB equation with respect to the habit:

$$\begin{aligned}(\rho + a)V_X + (C_t - X_t)^{-\gamma} - \tfrac{1}{2}V_{XAA}\sigma_A^2A_t^2 &= V_{XK}(\Phi((A_tK_t^\alpha - C_t)/K_t)K_t - \delta K_t) \\ &\quad + V_{XX}(bC_t - aX_t) - V_{XA}(\rho_A\log A_t - \tfrac{1}{2}\sigma_A^2)A_t.\end{aligned}$$

A system of partial differential equations (PDEs) that implies the same policy function in the absence of shocks can be constructed from:

$$
\begin{aligned}
\mathrm{d}V_K &= [(\rho - \Phi((A_t K_t^{\alpha} - C_t)/K_t) \\
&\quad -\Phi'((A_t K_t^{\alpha} - C_t)/K_t)((\alpha-1)A_t K_t^{\alpha-1} + C_t/K_t) + \delta)V_K - \tfrac{1}{2}V_{KAA}\sigma_A^2 A_t^2]\,\mathrm{d}t \\
\mathrm{d}V_X &= \left[(\rho + a)V_X + (C_t - X_t)^{-\gamma} - \tfrac{1}{2}\sigma_A^2 A_t^2 V_{XAA}\right]\mathrm{d}t \\
\mathrm{d}K_t &= [\Phi((A_t K_t^{\alpha} - C_t)/K_t)K_t - \delta K_t]\,\mathrm{d}t \\
\mathrm{d}X_t &= [bC_t - aX_t]\,\mathrm{d}t \\
\mathrm{d}A_t &= \left[-(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\right]\mathrm{d}t
\end{aligned}
$$

where C_t is implicitly defined by:

$$(C_t - X_t)^{-\gamma} + bV_X(K_t, X_t, A_t) = \Phi'((A_t K_t^{\alpha} - C_t)/K_t)V_K(K_t, X_t, A_t).$$

such that $\mathrm{d}V_K = V_{KK}\mathrm{d}K_t + V_{KX}\mathrm{d}X_t + V_{KA}\mathrm{d}A_t$ with $\mathrm{d}V_K$, $\mathrm{d}K_t$, $\mathrm{d}X_t$, and $\mathrm{d}A_t$ from (34), (11), and (12), respectively, also solves the HJB equation.

G.2 Discrete-time

G.2.1 Firms

A representative firm produces its output according to the production function:

$$Y_t = A_t K_t^{\alpha}, \tag{70}$$

where A_t is total factor productivity and K_t is the aggregate capital stock.

The log of total factor productivity (TFP) follows an AR(1) process

$$
\begin{aligned}
\log A_{t+1} &= \tilde{\rho}_A \log A_t + \tilde{\sigma}_A \epsilon_{A,t+1} \\
\Leftrightarrow \quad A_{t+1} &= A_t^{\tilde{\rho}_A} \exp(\tilde{\sigma}_A \epsilon_{A,t+1})
\end{aligned} \tag{71}
$$

where $\tilde{\rho}_A$ denotes the AR coefficient of the TFP process in discrete time, $\tilde{\sigma}_A$ its standard deviation, and $\epsilon_{A,t} \sim \mathcal{N}(0,1)$ a shock on it.

Capital is assumed to be owned by households, which lend it out to firms in each period. Hence, the firm maximizes profits given by

$$\Pi_t = A_t K_t^{\alpha} - r_t K_t$$

where r_t is the net rental rate of capital which is given by the first order condition of the firm as

$$r_t = \alpha A_t K_t^{\alpha-1}$$

leading to profits of

$$\Pi_t = (1-\alpha)A_t K_t^{\alpha},$$

which are transferred to households, as well.

G.2.2 Households

The economy is assumed to be inhabited by a sufficiently large number of identical individuals, which maximize their discounted life-time utility

$$U_0 \equiv \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} \right], \tag{72}$$

where β denotes the household's subjective discount factor and X_t is the subsistence level of consumption. Our definition of the subsistence level in continuous time, given by (45), can be written in discrete-time as (see Grishchenko, 2010):

$$X_t \;=\; \tilde{b} \sum_{s=0}^{\infty} (1-\tilde{a})^s C_{t-1-s},$$

which can be written in recursive form as

$$X_t \;=\; \tilde{b} C_{t-1} + (1-\tilde{a}) X_{t-1} \tag{73}$$

implying a steady state habit-to-consumption ratio of $\frac{\bar{X}}{\bar{C}} = \frac{\tilde{b}}{\tilde{a}}$[9].

The capital stock, which is owned by households and lent out to firms, increases if gross investment exceeds capital adjustment costs and depreciation

$$K_{t+1} = \left(\Phi\left(\frac{I_t}{K_t}\right) + 1 - \delta \right) K_t, \tag{74}$$

where the capital adjustment cost function is given by the cost specification in Jermann (1998):

$$\Phi(I_t/K_t) = \frac{a_1}{1-1/\xi} \left(\frac{I_t}{K_t}\right)^{1-1/\xi} + a_2$$

and where $\xi > 0$ denotes the elasticity of the investment-to-capital ratio with respect to Tobin's q and $a_1 \geq 0$ and $a_2 \geq 0$ are parameters.

Households own the capital stock, invest in physical capital facing the capital adjustment costs and get rents $r_t K_t = \alpha A_t K_t^{\alpha-1} K_t$ as well as profits $\Pi_t = (1-\alpha)A_t K_t^{\alpha}$ from firms summing up to $\alpha A_t K_t^{\alpha} + (1-\alpha)A_t K_t^{\alpha} = A_t K_t^{\alpha} = Y_t$. Hence, the budget constraint of a representative household

[9] Since both timing assumptions lead to the same long-run habit-to-consumption ratio, $\left(\frac{X}{C}\right)^{CT} = \frac{b}{a}$ (see (65)) and $\left(\frac{X}{C}\right)^{DT} = \frac{\tilde{b}}{\tilde{a}}$, in the following simulations we set $\tilde{b} = b$ and $\tilde{a} = a$.

reads:

$$K_{t+1} = \left(\Phi\left(\frac{A_t K_t^{\alpha} - C_t}{K_t}\right) + 1 - \delta\right) K_t \tag{75}$$

G.2.3 The HJB Equation and the first-order conditions

The representative household chooses the paths of consumption and investment that maximize its expected life-time utility. Define the value of the optimal program

$$V(K_0, A_0, X_0) = \max_{\{C_t\}_{t=0}^{\infty}} U_0 \quad s.t. \quad (71), \qquad \text{and} \quad (75) \tag{76}$$

in which $C_t \in \mathbb{R}^+$ and $K_{t+1} \in \mathbb{R}^+$ define the control variables at time $t \in \mathbb{Z}$.

The *Bellman* equation of the household reads for any $t \in \{0, 1, 2, \dots\}$

$$V(K_t, A_t, X_t) = \max_{C_t \in \mathbb{R}^+} \left\{ \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t V(K_{t+1}, A_{t+1}, X_{t+1}) \right\} \tag{77}$$

with associated first order condition

$$\begin{aligned}
(C_t - X_t)^{-\gamma} &= -\beta \mathbb{E}_t \left[\frac{\partial V(K_{t+1}, A_{t+1}, X_{t+1})}{\partial C_t} \right] \\
&= -\beta \mathbb{E}_t \left[\frac{\partial V(K_{t+1}, A_{t+1}, X_{t+1})}{\partial K_{t+1}} \frac{\partial K_{t+1}}{\partial C_t} + \frac{\partial V(K_{t+1}, A_{t+1}, X_{t+1})}{\partial X_{t+1}} \frac{\partial X_{t+1}}{\partial C_t} \right] \\
&= \beta \mathbb{E}_t \left[\frac{\partial V(K_{t+1}, A_{t+1}, X_{t+1})}{\partial K_{t+1}} \left(\Phi'\left(\frac{I_t}{K_t}\right)\right) + (C_{t+1} - X_{t+1})^{-\gamma} \tilde{b} \right] \\
\Rightarrow \beta \mathbb{E}_t \left[\frac{\partial V(K_{t+1}, A_{t+1}, X_{t+1})}{\partial K_{t+1}} \right] &= \left((C_t - X_t)^{-\gamma} - \tilde{b} \beta \mathbb{E}_t \left[(C_{t+1} - X_{t+1})^{-\gamma} \right] \right) \left(\Phi'\left(\frac{I_t}{K_t}\right) \right)^{-1} \qquad (78)
\end{aligned}$$

making optimal consumption a function of the state variables, $C_t = C(K_t, A_t, X_t)$.

The maximized Bellman equation is

$$V(K_t, A_t, X_t) = u(C_t(K_t, A_t, X_t)) + \beta \mathbb{E}_t V(K_{t+1}, A_{t+1}, X_{t+1}). \tag{79}$$

From (79), we obtain for the costate variable (using the envelope theorem)

$$\frac{\partial V(K_t, A_t, X_t)}{\partial K_t} = \beta \mathbb{E}_t \left[\frac{\partial V(K_{t+1}, A_{t+1}, X_{t+1})}{\partial K_{t+1}} \frac{\partial K_{t+1}}{\partial K_t} \right]$$

Inserting the first order condition in Equation (78) and the fact that from Equation (74) it easy to see that

$$\frac{\partial K_{t+1}}{\partial K_t} = \Phi\left(\frac{I_t}{K_t}\right) + 1 - \delta + \Phi'\left(\frac{I_t}{K_t}\right)\left(r_t - \frac{I_t}{K_t}\right)$$

we arrive at

$$\frac{\partial V(K_t, A_t, X_t)}{\partial K_t} = \frac{(C_t - X_t)^{-\gamma} - \tilde{b}\beta \mathbb{E}_t\left[(C_{t+1} - X_{t+1})^{-\gamma}\right]}{\Phi'\left(\frac{I_t}{K_t}\right)} \left(\Phi\left(\frac{I_t}{K_t}\right) + 1 - \delta + \Phi'\left(\frac{I_t}{K_t}\right)\left(r_t - \frac{I_t}{K_t}\right)\right).$$

The Euler equation is obtained by iterating forward and inserting into (78)

$$\begin{aligned} & \frac{(C_t - X_t)^{-\gamma} - \tilde{b}\beta \mathbb{E}_t\left[(C_{t+1} - X_{t+1})^{-\gamma}\right]}{\Phi'\left(\frac{I_t}{K_t}\right)} \\ = \quad & \beta \mathbb{E}_t\left[\frac{(C_{t+1} - X_{t+1})^{-\gamma} - \tilde{b}\beta (C_{t+2} - X_{t+2})^{-\gamma}}{\Phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} \left(\Phi\left(\frac{I_t}{K_t}\right) + 1 - \delta + \Phi'\left(\frac{I_t}{K_t}\right)\left(r_t - \frac{I_t}{K_t}\right)\right)\right] \end{aligned} \tag{80}$$

G.2.4 Equilibrium

The equilibrium in the economy is given by the sequence $\{C_t, X_t, I_t, K_t, A_t\}_{t=0}^{\infty}$ that solves the following system of equations:

$$\begin{aligned} & \frac{(C_t - X_t)^{-\gamma} - \tilde{b}\beta \mathbb{E}_t\left[(C_{t+1} - X_{t+1})^{-\gamma}\right]}{\Phi'\left(\frac{I_t}{K_t}\right)} \\ = \quad & \beta \mathbb{E}_t\left[\frac{(C_{t+1} - X_{t+1})^{-\gamma} - \tilde{b}\beta (C_{t+2} - X_{t+2})^{-\gamma}}{\Phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} \left(\Phi\left(\frac{I_t}{K_t}\right) + 1 - \delta + \Phi'\left(\frac{I_t}{K_t}\right)\left(r_t - \frac{I_t}{K_t}\right)\right)\right] \\ X_t \quad = \quad & \tilde{b} C_{t-1} + (1 - \tilde{a}) X_{t-1} \\ K_{t+1} \quad = \quad & \left(\Phi\left(\frac{I_t}{K_t}\right) + 1 - \delta\right) K_t \\ I_t \quad = \quad & A_t K_t^{\alpha} - C_t \\ \log A_{t+1} \quad = \quad & \tilde{\rho}_A \log A_t + \tilde{\sigma}_A \epsilon_{A,t+1} \end{aligned}$$

The equilibrium of the economy is characterized by a system of 5 stochastic difference equations in 5 variables whose solution delivers the optimal paths of the exogenous variable A_t and the endogenous variables, K_t, I_t, C_t, and X_t.

G.2.5 Steady State

The deterministic steady state can be derived as follows. The equation for capital accumulation reads in the deterministic steady state

$$\begin{aligned} 1 \quad &= \quad \Phi\left(\frac{\bar{I}}{\bar{K}}\right) + 1 - \delta \\ &= \quad \frac{a_1}{1 - 1/\xi}\left(\frac{\bar{I}}{\bar{K}}\right)^{1-1/\xi} + a_2 + 1 - \delta. \end{aligned}$$

Inserting $a_1 = \delta^{1/\xi}$ and $a_2 = \frac{\delta}{1-\xi}$, we arrive at

$$1 = \frac{\delta^{1/\xi}}{1-1/\xi}\left(\frac{\bar{I}}{\bar{K}}\right)^{1-1/\xi} + \frac{\delta}{1-\xi} + 1 - \delta.$$

implying an investment-to-capital ratio of

$$\left(\frac{\bar{I}}{\bar{K}}\right) = \left(\frac{\delta - \frac{\delta}{1-\xi}}{\frac{\delta^{1/\xi}}{1-1/\xi}}\right)^{\frac{1}{1-1/\xi}} = \delta.$$

Using the Euler equation, we get the capital stock

$$\bar{K} = \left[\frac{\frac{1}{\beta} - 1 + \delta}{\alpha A}\right]^{\frac{1}{\alpha-1}}.$$

From the capital-to-investment ratio, we can compute the corresponding deterministic steady state value for investment and from the aggregate resource constraint the one for consumption. Further, we have $\bar{A} = 1$ and $\frac{\bar{X}}{\bar{C}} = \frac{\tilde{b}}{\tilde{a}}$.

G.3 Calibration

Table 5 summarizes the prototype models described above. The model is calibrated annually and the parameters should be interpreted accordingly. The values of the parameters are the same as those used in Section 2. The calibration of the habit formation process is taken from Constantinides (1990)[10]. Table 6 summarizes the calibration used for the model with habit formation and capital adjustment costs.

G.4 Extended RBC model: Results

G.4.1 Comparison of steady states

Using the calibration above, Table 7 presents both the deterministic and the risky the steady state values for the variables of the model.

G.4.2 Approximated policy functions

Using the calibration in Table 6, Figure (10) plots the approximated policy function for consumption for the discrete- and continuous-time models along a discretized grid containing $n_K = 1001$ values for the capital stock around the interval $K \in \left[0.5\bar{K}, 1.5\bar{K}\right]$, while keeping the habit persistence and the TFP levels at their deterministic steady states. We plot the first and second order policy functions.

[10] The calibration of the parameters a and b in the dynamics of the habit formation match, in the steady state, the value of the weight of habit in the utility use in Jermann (1998)

Table 5. Summary of the two modeling approaches.

	Discrete-time	Continuous-time
Objective function (U_0)	$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\frac{(C_t-X_t)^{1-\gamma}}{1-\gamma}\right]$	$\mathbb{E}_0\left[\int_0^{\infty}e^{-\rho t}\frac{(C_t-X_t)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right]$
Market clearing	$A_tK_t^{\alpha}=C_t+I_t$	$A_tK_t^{\alpha}=C_t+I_t$
Capital dynamics	$K_{t+1}=$ $\left(\Phi\left(I_t/K_t\right)+(1-\delta)\right)K_t$	$\mathrm{d}K_t=\left(\Phi\left(I_t/K_t\right)-\delta\right)K_t\mathrm{d}t$
Habit dynamics	$X_t=bC_{t-1}+(1-a)\,X_{t-1}$	$\mathrm{d}X_t=\left(bC_t-aX_t\right)\mathrm{d}t$
TFP dynamics	$\log A_{t+1}=\tilde{\rho}_A\log A_t$ $+\tilde{\sigma}_A\epsilon_{A,t+1}$	$\mathrm{d}\log A_t=-\rho_A\log A_t\mathrm{d}t$ $+\sigma_A\mathrm{d}B_{A,t}$
Uncertainty	$\epsilon_{A,t}\sim N(0,1)$	$\left(B_{A,t+\Delta}-B_{A,t}\right)\sim N(0,\Delta)$

Table 6. Parameter values for the RBC model with habit persistence and capital adjustment costs.

Parameter	Discrete-time	Continuous-time
Discounting, β/ρ	0.9606	0.0410
Relative risk aversion, γ	5.0000	5.0000
Depreciation rate, δ	0.0963	0.0963
Capital share in output, α	0.3600	0.3600
Persistence of TFP, $\tilde{\rho}_A/\rho_A$	0.8145	0.2052
Volatility of TFP, $\tilde{\sigma}_A/\sigma_A$	0.0372	0.0410
Adjustment cost parameter, ξ		0.4350
Weight of current cons., b		0.3500
Weight of past cons., a		0.6000

Figure (11) plots compares the first and second order approximations to the policy functions for consumption for both the discrete- and the continuous-time model.

The first order approximation to the consumption and the shadow prices of capital and habit are:

$$C_t^{(1),CT} = 1.2854+0.0232\left(K_t-\bar{K}\right)+0.7850\left(X_t-\bar{X}\right)+0.2824\left(A_t-\bar{A}\right)-\mathbf{0.0087}$$

Table 7. Comparison of steady states values.

Variable	Determ.	Risky				
		Discrete-time		Continuous-time		
		First	Second	First	Second	Global
A	1.0000	1.0000	1.0037	1.0041	1.0041	1.0041
X	0.7498	0.7498	0.7623	0.7631	0.7645	0.7641
K	4.5077	4.5077	4.9077	4.9243	4.9932	4.9761
C	1.2854	1.2854	1.3069	1.3082	1.3105	1.3100

while the second order approximations are:

$$\begin{aligned} C_t^{(2),CT} = \; & 1.2854 + \left(0.0232 - \mathbf{1.8768 \times 10^{-4}}\right)\left(K_t - \bar{K}\right) + \left(0.7850 - \mathbf{0.0033}\right)\left(X_t - \bar{X}\right) \\ & + \left(0.2824 - \mathbf{0.0066}\right)\left(A_t - \bar{A}\right) - \mathbf{0.0087} + 0.0390\left(K_t - \bar{K}\right)\left(X_t - \bar{X}\right) \\ & -0.0423\left(K_t - \bar{K}\right)\left(A_t - \bar{A}\right) + 0.6533\left(X_t - \bar{X}\right)\left(A_t - \bar{A}\right) \\ & + \frac{1}{2}\left[-0.0081\left(K_t - \bar{K}\right)^2 - 0.4065\left(X_t - \bar{X}\right)^2 - 1.1480\left(A_t - \bar{A}\right)^2 - \mathbf{9.4246 \times 10^{-4}}\right] \end{aligned}$$

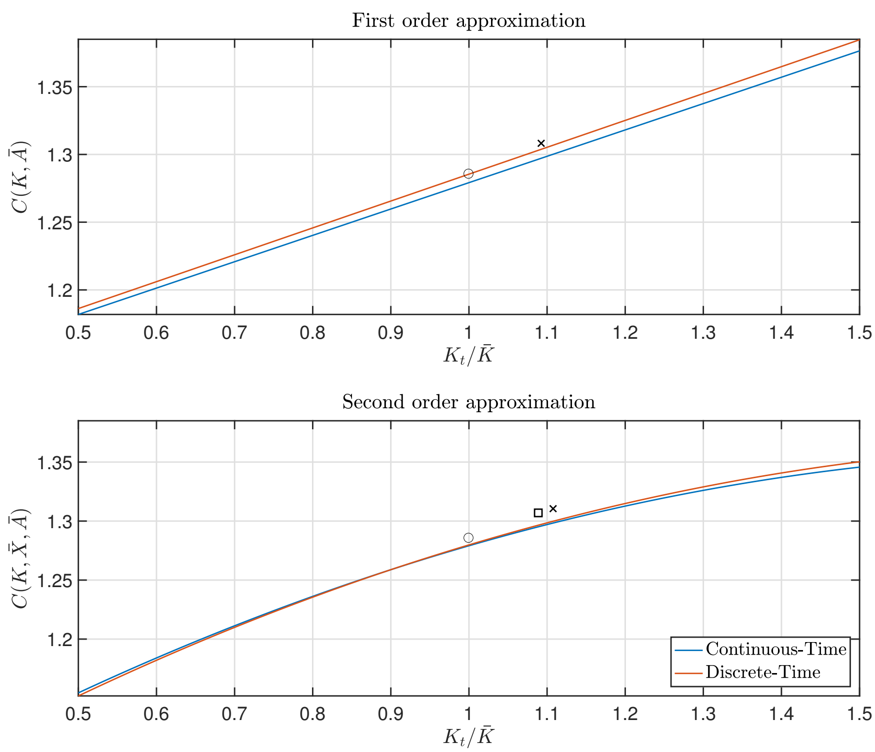

Figure 10. Policy functions for consumption. The graph plots the first order approximation (top panel) and the second order approximation (bottom panel) to the policy function for aggregate consumption along the capital lattice while keeping habit persistence and productivity at their deterministic steady states, $C\left(K, \bar{X}, \bar{A}\right)$. A circle denotes the deterministic steady state, a star denotes the risky steady state approximated from the continuous-time model, and a square the risky steady state approximated from the discrete-time model

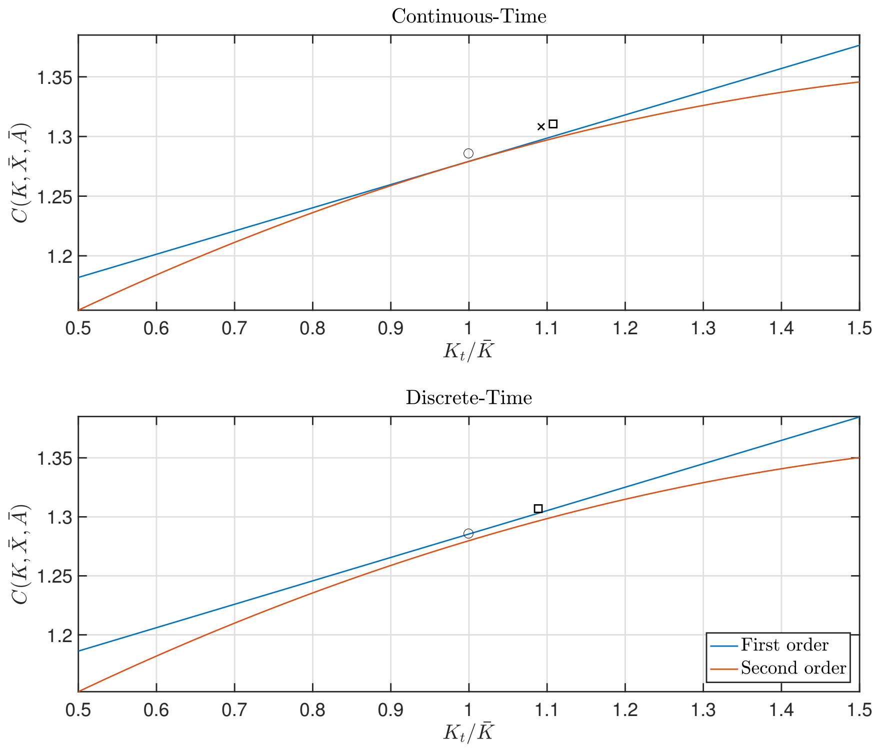

Figure 11. Policy functions for consumption. The graph plots the first and second order approximation to the policy function for aggregate consumption along the capital lattice while keeping habit persistence and productivity at their deterministic steady state, $C\left(K, \bar{X}, \bar{A}\right)$ for the model in continuous-time (top panel) and the model in discrete-time (bottom panel). A circle denotes the deterministic steady state, a star denotes a first order approximation to the risky steady state approximated, while a square denotes a second order approximation to the risky steady state.

G.4.3 Impulse Response Functions

This section compares the response of the endogenous variables of the discrete- and continuous-time models to a temporary shock on the level of total factor productivity.

Figure 12 plots the impulse response functions (IRFs) for the levels of aggregate consumption, aggregate capital, habit persistence, and future productivity when the discrete-time economy is subject to a one standard deviation shock in the TFP. The blue lines plot the IRFs based on the first order approximation, and the red lines those based on the second order approximation to the policy functions. For the sake of comparison, we assume that before the shock hits, the economy rests in its deterministic steady state. Up to a first order approximation, the model converges to the deterministic steady state as time passes. However, when the model is approximated to a second order, the economy converges instead to its (approximated) risky steady state.

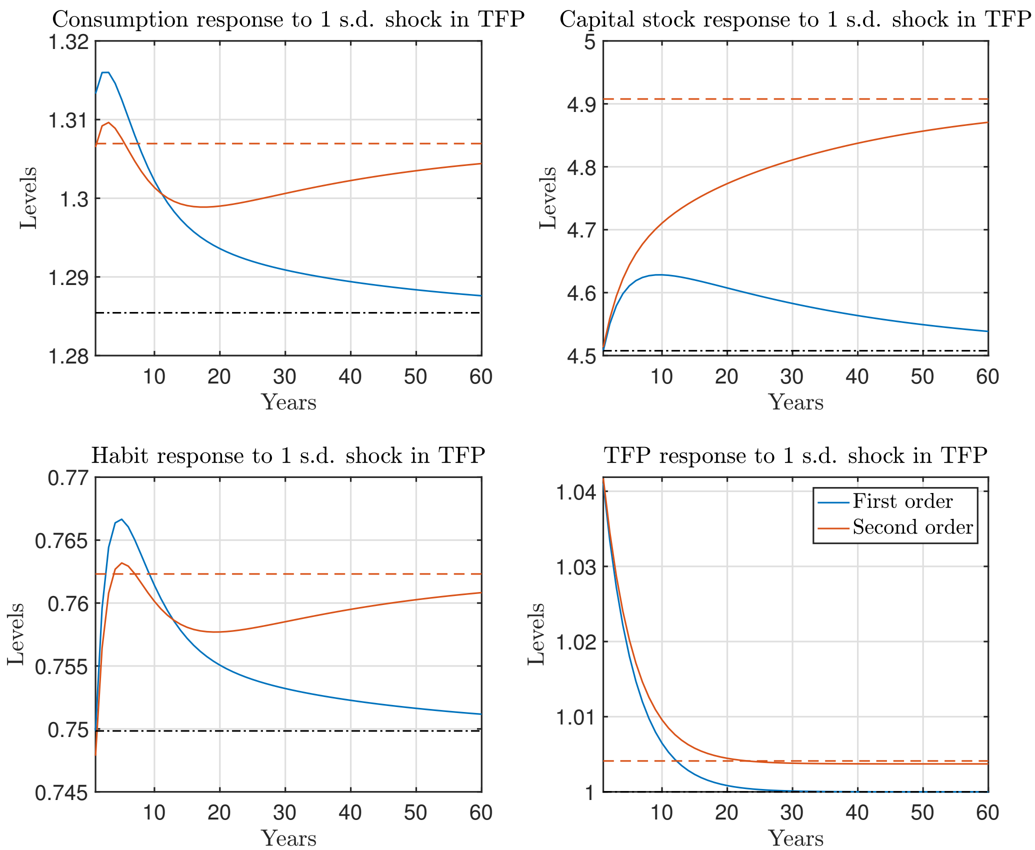

Figure 12. Impulse-Response function to a TFP shock: Discrete-time. The graph plots the impulse response functions (IRFs) for the levels of aggregate consumption, aggregate capital, habit persistence when time in the economy is assumed to be discrete. The economic system is assumed to be in its deterministic steady state before the shock hits. The dash-dotted line corresponds to the deterministic steady state, and the dashed line to the approximated risky steady state.

Figure 13 plots the responses of aggregate consumption, aggregate capital stock, habit persistence and future productivity to the same TFP shock when time in the economy is assumed to be continuous. The blue lines plot the IRFs based on the first order approximation, and the red lines those based on the second order approximation to the policy functions. The IRF's are computed by iterating forward, from the deterministic steady state, the system of equations consisting of Euler-discretized versions of the stochastic process for the state variables and the approximated consumption function.

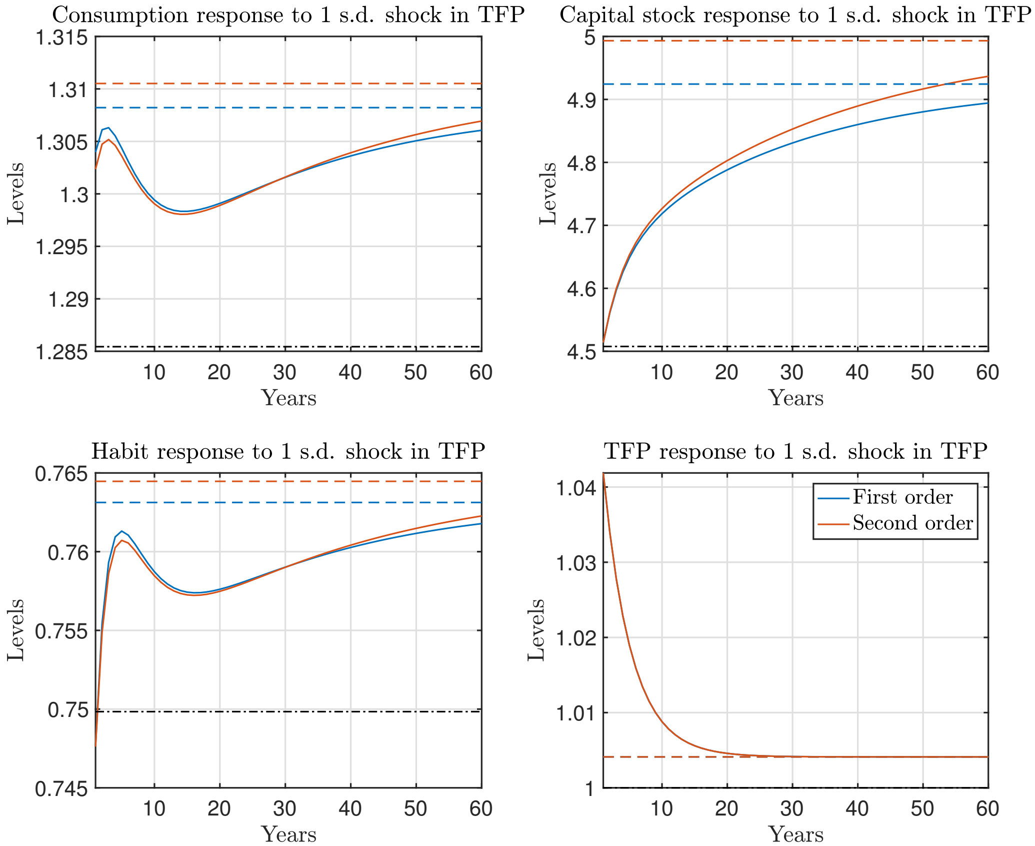

Figure 13. Impulse-Response function to a TFP shock: Continuous-time. The graph plots the impulse response functions (IRFs) for the levels of aggregate consumption, aggregate capital, and habit persistence when time in the economy is assumed to be continuous. The economic system is assumed to be in its deterministic steady state before the shock hits. The dash-dotted line corresponds to the deterministic steady state.

For completeness, Figures 14 and 15, compare the effect of the timing assumption on the IRFs for each type of approximation.

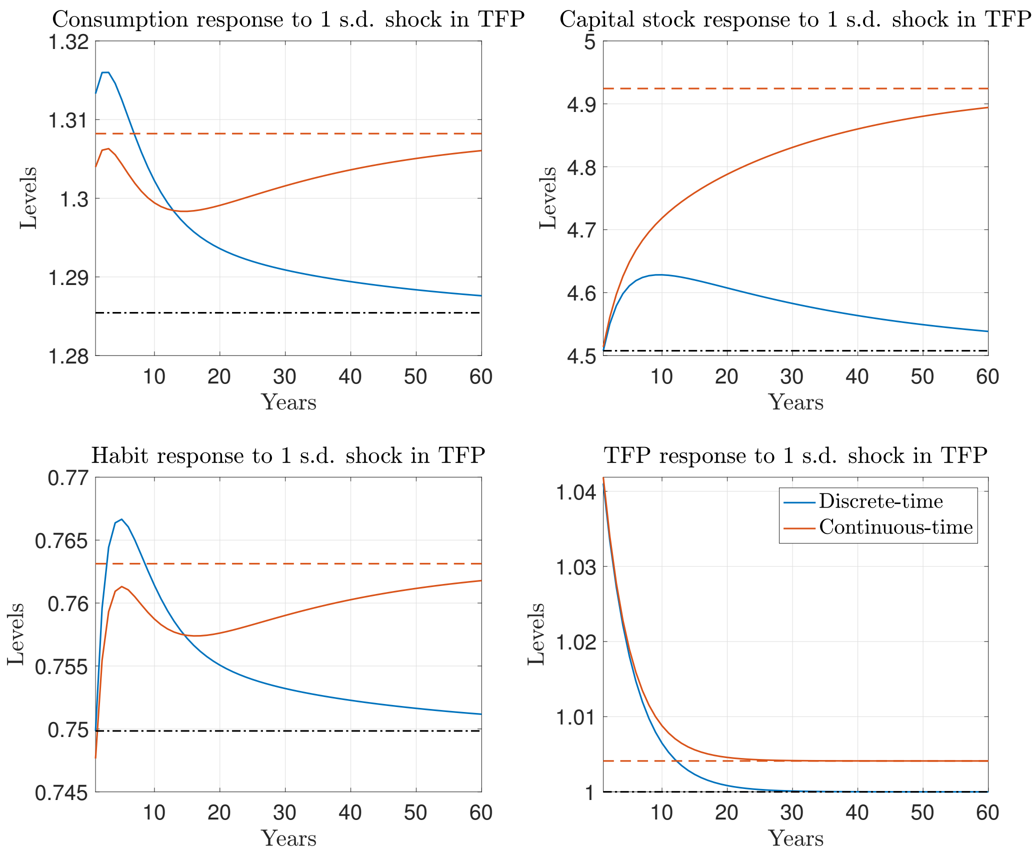

Figure 14. First order Impulse-Response function to a TFP shock: Discrete-time vs. Continuous-time. The graph plots the first order impulse response functions (IRFs) for the levels of aggregate consumption, aggregate capital, and habit persistence for both discrete- and continuous-time models. The economic system is assumed to be in its deterministic steady state before the shock hits.

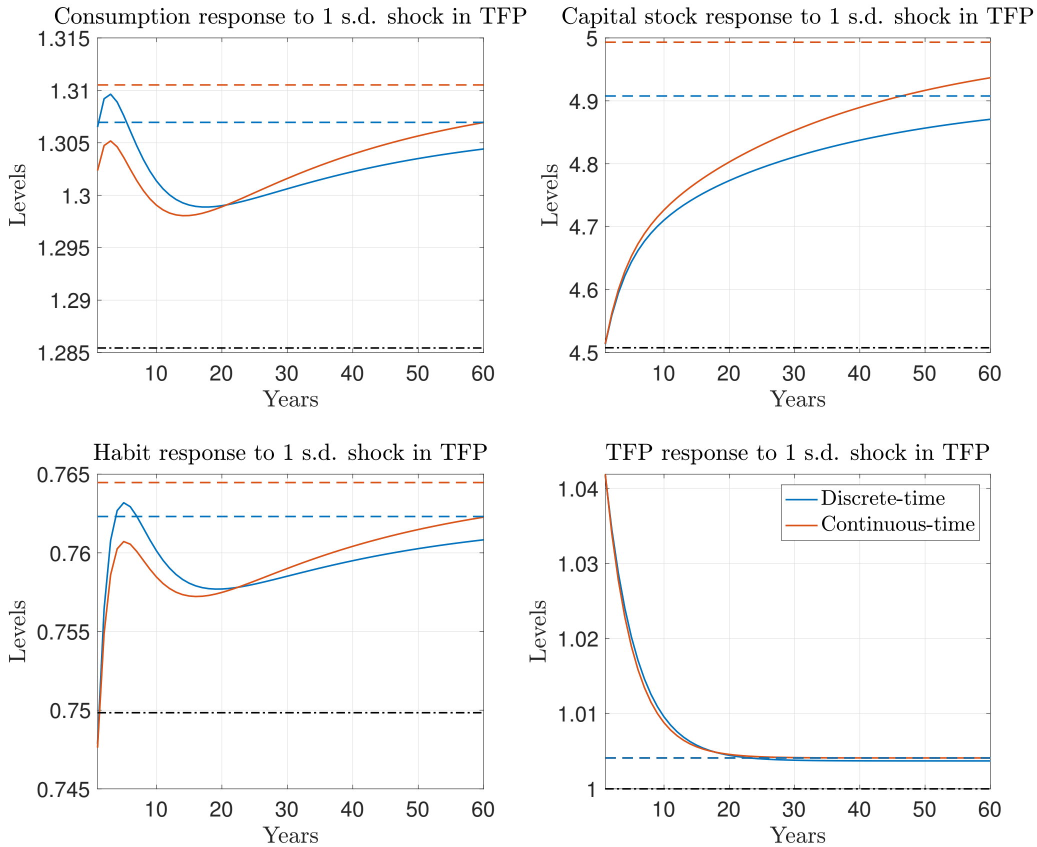

Figure 15. Second order Impulse-Response function to a TFP shock: Discrete-time vs. Continuous-time. The graph plots the second order impulse response functions (IRFs) for the levels of aggregate consumption, aggregate capital, and habit persistence for both discrete- and continuous-time models. The economic system is assumed to be in its deterministic steady state before the shock hits.The dash-dotted line corresponds to the deterministic steady state.

H A model with rare disasters

[WORK IN PROGRESS] Posch and Trimborn (2013) model, solution using first-order perturbation; shows how first-order perturbation performs relatively to the certainty equivalent solution

H.1 Continuous-time: Central planner

H.1.1 Technology

H.1.2 Households

H.1.3 The HJB equation and the first-order conditions

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