Solving Markov-Switching Models with Learning*

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Abstract

Markov-switching dynamic stochastic general equilibrium models allow for discrete, recurring shifts in the economic or policy environment. Typical solution methods for this class of models assume that agents perfectly observe the current regime. In this paper, we relax the perfect information assumption and instead allow agents to perform Bayesian learning about the current regime. Using this framework, we develop a general perturbation solution method to handle the learning framework. Our methodology relies on joint approximations to both the learning process and the decision rules, and highlights the necessity of second- and higher-order rather than simple linear approximations. We illustrate the method with three simple examples.

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1 Introduction

What happens when there is a sudden discrete change in the economic environment? Examples of such changes are plentiful in macroeconomics: Changes in fiscal or monetary policy (Davig and Leeper (2007) and Lubik and Schorfheide (2004), for example) and changes in the variances of exogenous shocks (Sims and Zha (2006)) are just two examples. Markov-switching full information rational expectations models (see Bianchi (2013), for example) offer one route to incorporate these sudden changes into economic models by modeling them as random variables that follow a discrete state Markov chain. Underlying these models is the assumption that all agents in the model are immediately aware both that a shift in the environment has occurred and that agents know the exact specification of the post-change environment.[1]

In this paper we show how to efficiently compute models where we endow agents with a different information set that does *not* include the current economic environment (i.e. the current state of the discrete Markov chain). Instead, the agents in our model use Bayes' law to infer the current economic environment (or, to be more precise, the probability associated with each possible environment). Is this assumption more or less reasonable than the assumption of full information that has been used by the bulk of the literature? The answer certainly depends on the specific application, but the full information assumption gives agents in the model a substantial informational advantage relative to any econometrician that analyzes the available data ex-post, whereas our approach makes the agents in the model face a filtering problem akin to the problem faced by econometricians. It might be worthwhile to emphasize that our agents are still fully rational in that they do not leave any information unexploited.

It is well known that solving full information Markov-switching rational expectations models that are linear apart from the discrete changes in environment is substantially more involved than solving corresponding models with Markov-switching. In this paper we face further difficulties because we introduce learning via Bayes' rule, which naturally introduces a nonlinearity in the equilibrium conditions. Furthermore, we are interested in the role of nonlinearities more generally, so a linear approximation will not suffice for our purposes. We build on Foerster et al. (2016) to show how to construct higher order perturbation solutions in Markov-switching models where agents have to infer the current economic environment. Using a perturbation-based method allows us to solve the model much faster than other global solution methods would allow. The key insight to our approach is that we *jointly* approximate the decision rules

[1]Another feature of these models that we will keep in our analysis is that agents are aware that sudden changes can possibly occur again in the future.

and the learning process (i.e. Bayes' rule).[2] The joint approximation is exactly allows us to rely on the methods developed by Foerster et al. (2016).

Previous papers that have introduced partial information into rational expectations Markov-switching models either had to resort to rather laborious global solution methods (Davig (2004)) or have used a less general information structure such as agents observing the current economic environment, but not knowing how persistent it is (Bianchi and Melosi (2017)), which simplifies the inference problem that agents in the model face. Our setup allows for information structures like those used in Bianchi and Melosi (2017), but can also accommodate situations where agents do not observe the current values of the parameters that are governed by the discrete state Markov chain. Markov switching environments when agents are not fully rational, but instead learn adaptively, have been studied by Branch et al. (2013). In terms of model solution, the assumption of adaptive learning makes solving the model easier because the way expectations are formed is predetermined (it is part of the model description in adaptive learning models) rather than jointly determined with the rest of the decision rules as in our approach.[3]

The remainder of the paper is as follows. In Section 2, we lay out a framework of the class of models we study, including the Bayesian learning process. Section 3 contains the main part of our methodology, which shows how to construct approximations to the solution of the framework considered. Sections 4, 5, and 6 show three examples that build intuition about the methodology and highlight its features: first a simple filtering example to demonstrate the learning process, a real business cycle example that layers on economic-decision making, and lastly a taxation example that contains feedback between the learning process and the economic decision-making. Finally, Section 7 concludes.

2 The General Framework

This section lays out the general framework that we consider. We show how a general class of models is combined with a Bayesian learning process, and then characterize the set of solutions to the problem. As an illustrative guide, we show how a real business cycle model with unobserved total factor productivity–which we study in detail in Section 5, fits into the framework.

[2] By studying examples that can be analytically solved, we show that with higher order approximations, the approximation error in the agents' model probabilities is small.

[3] To be very clear, we are not making any statement on whether adaptive learning or fully rational learning as in our approach is preferable. Which of these is preferred by the data most likely depends on the application. We are interested in developing a fast and reliable algorithm to solve Markov-switching rational expectations models with partial information, which would allow such a comparison to be undertaken.

2.1 The Economic Model

We consider a general class of dynamic, stochastic general equilibrium models where some of the parameters follow a discrete Markov process that is indexed by $s_t \in \{1, \ldots, n_s\}$. The regime variable s_t is not directly observed, but has known transition matrix $P = [p_{i,j}]$, where $p_{i,j} = \Pr(s_{t+1} = j | s_t = i)$. The set of parameters that follow a Markov process is given by $\boldsymbol{\theta}_t = \boldsymbol{\theta}(s_t)$.

The equilibrium conditions for the class of models that we study can be written as

$$\tilde{\mathbb{E}}_t \mathbf{f}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_t, \mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_{t+1}, \boldsymbol{\varepsilon}_t, \boldsymbol{\theta}_{t+1}, \boldsymbol{\theta}_t) = 0, \tag{1}$$

where $\mathbf{y}_t$ denotes non-predetermined variables at time t, $\mathbf{x}_t$ denotes predetermined variables, and $\boldsymbol{\varepsilon}_t$ denote the innovations which are serially uncorrelated and jointly distributed according with density function ϕ^{ε}.

In this case, the expectations operator $\tilde{\mathbb{E}}_t$ denotes expectations based on an information set given by $\mathcal{I}_t = \{\mathbf{y}^t, \mathbf{x}^t\}$. The history of innovations $\boldsymbol{\varepsilon}^t$, parameters $\boldsymbol{\theta}^t$, and regimes s^t is not part of the information set. The information set produces subjective probabilities of being in each regime $\{1, \ldots, n_s\}$, denoted by a vector $\boldsymbol{\psi}_t$, where $\psi_{i,t} = \Pr(s_t = i | \mathcal{I}_t)$. The subjective probabilities are updated via Bayesian learning.

As an example, consider a prototypical real business cycle economy where total factor productivity is a subject to both regime changes and idiosyncratic shocks, but the composition of these factors is not observed. In this example, a planner maximizes preferences

$$\tilde{\mathbb{E}}_0 \sum_{t=0}^{\infty} \beta^t \log c_t \tag{2}$$

subject to a budget constraint

$$c_t + k_t = \exp(z_t) k_{t-1}^{\alpha} + (1 - \delta) k_{t-1} \tag{3}$$

and a driving process for technology

$$z_t = \mu_t + \sigma_t \varepsilon_t. \tag{4}$$

where $\varepsilon_t \sim^{iid} N(0, 1)$. The process for z_t is governed in part by a Markov process with $\mu_t = \mu(s_t)$, and $\sigma_t = \sigma(s_t)$, where $s_t \in \{1, 2\}$. The information set at time t is the complete histories of consumption c^t, capital k^t, and technology z^t, but not the components s^t or ε^t. As a result, conditional on an observed value of z_t, the planner does not know if the observation was generated by regime 1 or regime 2. If regimes recur with some high probability, then knowing whether a realization of TFP is likely to be persistent due to the regime versus transitory due to the shock would impact consumption and savings decisions.

To solve for the equilibrium, standard optimization implies an Euler equation given by

$$1 = \beta \tilde{\mathbb{E}}_t \left[\frac{c_t}{c_{t+1}} \left(\alpha \exp\left(z_{t+1}\right) k_t^{\alpha-1} + 1 - \delta \right) \right], \tag{5}$$

which, combined with (3) and (4), produce a set of equilibrium conditions. The model is therefore in the form of equation (1), with $\mathbf{y}_t = [c_t, z_t]$, $\mathbf{x}_t = [k_t]$, and $\boldsymbol{\theta}_t = [\mu_t, \sigma_t]$, and

$$f\left(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_t, \mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_{t+1}, \boldsymbol{\varepsilon}_t, \boldsymbol{\theta}_{t+1}, \boldsymbol{\theta}_t\right) = \begin{bmatrix} \beta \frac{c_t}{c_{t+1}} \left(\alpha \exp\left(z_{t+1}\right) k_t^{\alpha-1} + 1 - \delta \right) - 1 \\ \exp\left(z_t\right) k_{t-1}^{\alpha} + (1-\delta) k_{t-1} - c_t - k_t \\ \mu_t + \sigma_t \varepsilon_t - z_t \end{bmatrix}. \tag{6}$$

2.2 The Learning Process

Since the regime s_t is not observed directly, the equilibrium dynamics depend on the subjective probabilities of being in each regime at time t, conditional on all past observables, which is given by $\psi_{i,t} = \Pr\left(s_t = i | \mathcal{I}_t\right)$. The subjective probabilities are updated via Bayesian learning, which involves combining rior beliefs in the form of last period's subjective probabilities $\psi_{i,t-1}$, with information about a known signal $\tilde{\mathbf{y}}_t \subseteq \mathbf{y}_t$.[4] The signal is generated by a combination of the predetermined variables $\mathbf{x}_{t-1}$ and the shocks $\boldsymbol{\epsilon}_t$ and depends on the regime s_t:

$$\tilde{\mathbf{y}}_t = \tilde{\lambda}_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t\right). \tag{7}$$

We assume that, given the regime s_t and the predetermined variables $\mathbf{x}_{t-1}$, there is a one-to-one mapping between shocks and signals, and hence can write

$$\boldsymbol{\varepsilon}_t = \lambda_{s_t}\left(\tilde{\mathbf{y}}_t, \mathbf{x}_{t-1}\right). \tag{8}$$

[4]The restriction that signals are part of the non-predetermined variables is without loss of generality, since auxiliary variables can be used to link elements of the set of predetermined and nonpredetermined variables.

The Jacobian of this mapping is given by

$$J_{s_t}\left(\tilde{\mathbf{y}}_t, \mathbf{x}_{t-1}\right) = \left|\frac{\partial \lambda_{s_t}\left(\tilde{\mathbf{y}}_t, \mathbf{x}_{t-1}\right)}{\partial \tilde{\mathbf{y}}_t}\right|$$

where $|\cdot|$ denotes the determinant, and we

Given known signals $\tilde{\mathbf{y}}_t \subseteq \mathbf{y}_t$, predetermined variables $\mathbf{x}_{t-1}$, prior probabilities $\boldsymbol{\psi}_{t-1}$, transiton probabilities $P = [p_{i,j}]$, and a joint density function of the errors ϕ^{ε}, Bayesian updating implies

$$\psi_{i,t} = \frac{J_{s_t=i}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right) \phi^{\varepsilon}\left(\lambda_{s_t=i}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right)\right) \sum_{s=1}^{n_s} p_{s,i} \psi_{s,t-1}}{\sum_{j=1}^{n_s} J_{s_t=j}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right) \phi^{\varepsilon}\left(\lambda_{s_t=j}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right)\right) \sum_{s=1}^{n_s} p_{s,j} \psi_{s,t-1}}. \tag{9}$$

For ease in constructing approximations that appropriately bound the probabilities between zero and one, define the logit of the probabilities $\eta_{i,t} = \log\left(\frac{\psi_{i,t}}{1-\psi_{i,t}}\right)$, which in turn implies $\psi_{i,t} = \frac{1}{1+\exp\left(-\eta_{i,t}\right)}$. These logits are expressed as

$$\exp \eta_{i,t} = \frac{J_{s_t=i}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right) \phi^{\varepsilon}\left(\lambda_{s_t=i}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right)\right) \sum_{s=1}^{n_s} p_{s,i} \frac{1}{1+\exp\left(-\eta_{s,t-1}\right)}}{\sum_{j=1, j\neq i}^{n_s} J_{s_t=j}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right) \phi^{\varepsilon}\left(\lambda_{s_t=j}\left(\mathbf{y}_t, \mathbf{x}_{t-1}\right)\right) \sum_{s=1}^{n_s} p_{s,j} \frac{1}{1+\exp\left(-\eta_{s,t-1}\right)}}. \tag{10}$$

We denote the vector of probabilities as $\boldsymbol{\psi}_t$, and the vector of logits of the probabilities as $\boldsymbol{\eta}_t$.[5]

As a result, we write the equations characterizing the learning process as

$$\Phi\left(\mathbf{y}_t, \mathbf{x}_{t-1}, \boldsymbol{\eta}_t, \boldsymbol{\eta}_{t-1}, \boldsymbol{\Theta}\right) = 0 \tag{11}$$

where the i-th equation of Φ is given by rearranging (10), and $\boldsymbol{\Theta} = \left[\boldsymbol{\theta}(1), \cdots, \boldsymbol{\theta}(n_s)\right]'$ denotes the complete set of regime-switching parameters. The full vector of regime-switching parameters, and not just the current regime's values, matter for the learning process because the Bayesian updating weighs the relative likelihood of the observables being generated by each possible regime.

Turning back to the RBC model, the signal is generated by equation (4), and so conditional on the regime and the capital stock (which is irrelevant in this case), the shock could inferred by inverting the signal equation:

$$\varepsilon_t = \frac{z_t - \mu\left(s_t\right)}{\sigma\left(s_t\right)}$$

[5]In practice, we can use the fact that $\sum_{i=1}^{n_s} \psi_{i,t} = 1$ and only generate approximations to $n_s - 1$ elements of the vector $\boldsymbol{\eta}_t$.

Hence, via Bayes rule, the subjective probability of regime i given an observed level of technology z_t is given by

$$\psi_{i,t} = \frac{\frac{1}{\sigma(i)}\phi\left(\frac{z_t-\mu(i)}{\sigma(i)}\right)\left(p_{1,i}\psi_{1,t-1}+p_{2,i}\psi_{2,t-1}\right)}{\sum_{j=1}^{2}\frac{1}{\sigma(j)}\phi\left(\frac{z_t-\mu(j)}{\sigma(j)}\right)\left[p_{1,j}\psi_{1,t-1}+p_{2,j}\psi_{2,t-1}\right]} \tag{12}$$

where $\phi(\cdot)$ denotes the PDF of the standard normal distribution. As a result, the logit of the probabilities are

$$\Phi\left(\mathbf{y}_t,\mathbf{x}_{t-1},\boldsymbol{\eta}_t,\boldsymbol{\eta}_{t-1},\boldsymbol{\Theta}\right) = \left[\begin{array}{c} \dfrac{\frac{1}{\sigma(1)}\phi\left(\frac{z_t-\mu(1)}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,1}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}{\frac{1}{\sigma(2)}\phi\left(\frac{z_t-\mu(2)}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,2}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)} - \exp\eta_{1,t} \\ \dfrac{\frac{1}{\sigma(2)}\phi\left(\frac{z_t-\mu(2)}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,2}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}{\frac{1}{\sigma(1)}\phi\left(\frac{z_t-\mu(1)}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,1}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)} - \exp\eta_{2,t} \end{array}\right]. \tag{13}$$

Note that, since the learning process is independent of future variables, we trivially have

$$\tilde{\mathbb{E}}_t\Phi\left(\mathbf{y}_t,\mathbf{x}_{t-1},\boldsymbol{\eta}_t,\boldsymbol{\eta}_{t-1},\boldsymbol{\Theta}\right) = \Phi\left(\mathbf{y}_t,\mathbf{x}_{t-1},\boldsymbol{\eta}_t,\boldsymbol{\eta}_{t-1},\boldsymbol{\Theta}\right). \tag{14}$$

2.3 Equilibrium Conditions with Learning

To characterize the full equilibrium conditions with the Bayesian updating of subjective probabilities, we can simply append the equations (9) to the original equilibrium conditions in equation (1). This produces a system of the form

$$\begin{aligned} \tilde{\mathbb{E}}_t\widetilde{\mathbf{f}}\left(\mathbf{y}_{t+1},\mathbf{y}_t,\mathbf{x}_t,\mathbf{x}_{t-1},\boldsymbol{\eta}_t,\boldsymbol{\eta}_{t-1},\boldsymbol{\varepsilon}_{t+1},\boldsymbol{\varepsilon}_t,\boldsymbol{\theta}_{t+1},\boldsymbol{\theta}_t,\boldsymbol{\Theta}\right) &= \\ \tilde{\mathbb{E}}_t\left[\begin{array}{c}\mathbf{f}\left(\mathbf{y}_{t+1},\mathbf{y}_t,\mathbf{x}_t,\mathbf{x}_{t-1},\boldsymbol{\varepsilon}_{t+1},\boldsymbol{\varepsilon}_t,\boldsymbol{\theta}_{t+1},\boldsymbol{\theta}_t\right)\\ \Phi\left(\mathbf{y}_t,\mathbf{x}_t,\mathbf{x}_{t-1},\boldsymbol{\eta}_t,\boldsymbol{\eta}_{t-1},\boldsymbol{\Theta}\right)\end{array}\right] &= 0. \end{aligned} \tag{15}$$

Further, the expectation can be decomposed into the subjective probabilities, transitions between regimes, and expectations over future shocks:

$$\sum_{s=1}^{n_s}\sum_{s'=1}^{n_s}\frac{p_{s,s'}}{1+\exp\left(-\eta_{s,t}\right)}\int\widetilde{\mathbf{f}}\left(\mathbf{y}_{t+1},\mathbf{y}_t,\mathbf{x}_t,\mathbf{x}_{t-1},\boldsymbol{\eta}_t,\boldsymbol{\eta}_{t-1},\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}_t,\boldsymbol{\theta}\left(s'\right),\boldsymbol{\theta}\left(s_t\right),\boldsymbol{\Theta}\right)\phi_{\varepsilon}\left(\varepsilon'\right) = 0. \tag{16}$$

In our simple RBC example, the new set of equilibrium conditions is written as

$$\widetilde{\mathbf{f}}\left(\mathbf{y}_{t+1},\mathbf{y}_t,\mathbf{x}_t,\mathbf{x}_{t-1},\boldsymbol{\eta}_{t,}\boldsymbol{\eta}_{t-1},\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}_t,\boldsymbol{\theta}\left(s'\right),\boldsymbol{\theta}\left(s_t\right),\boldsymbol{\Theta}\right) \tag{17}$$

$$= \begin{bmatrix} \beta\frac{c_t}{c_{t+1}}\left(\alpha\exp\left(z_{t+1}\right)k_t^{\alpha-1}+1-\delta\right)-1 \\ \exp\left(z_t\right)k_{t-1}^{\alpha}+\left(1-\delta\right)k_{t-1}-c_t-k_t \\ \mu\left(s_t\right)+\sigma\left(s_t\right)\varepsilon_t-z_t \\ \dfrac{\frac{1}{\sigma(1)}\phi\left(\frac{z_t-\mu(1)}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,1}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}{\frac{1}{\sigma(2)}\phi\left(\frac{z_t-\mu(2)}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,2}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}-\exp\eta_{1,t} \\ \dfrac{\frac{1}{\sigma(2)}\phi\left(\frac{z_t-\mu(2)}{\sigma(2)}\right)\left(\frac{p_{1,2}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,2}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}{\frac{1}{\sigma(1)}\phi\left(\frac{z_t-\mu(1)}{\sigma(1)}\right)\left(\frac{p_{1,1}}{1+\exp\left(-\eta_{1,t-1}\right)}+\frac{p_{2,1}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}-\exp\eta_{2,t} \end{bmatrix}.$$

3 Generating Approximations

3.1 Solutions

Extending Foerster et al. (2016) to the case where subjective probabilities are now a state variable, minimum state variable solutions to the model in equation (16) have the form

$$\mathbf{y}_t = g_{s_t}\left(\mathbf{x}_{t-1},\boldsymbol{\eta}_{t-1},\boldsymbol{\varepsilon}_t,\chi\right), \tag{18}$$

$$\mathbf{x}_t = h^x_{s_t}\left(\mathbf{x}_{t-1},\boldsymbol{\eta}_{t-1},\boldsymbol{\varepsilon}_t,\chi\right), \tag{19}$$

and

$$\boldsymbol{\eta}_t = h^\eta_{s_t}\left(\mathbf{x}_{t-1},\boldsymbol{\eta}_{t-1},\boldsymbol{\varepsilon}_t,\chi\right), \tag{20}$$

The form of these solutions show that the evolution of non-predetermined variables $\mathbf{y}_t$, predetermined variables $\mathbf{x}_t$, and beliefs $\boldsymbol{\eta}_t$, depends upon the actual realized regime s_t, the previous values of $\mathbf{x}_{t-1}$ and $\boldsymbol{\eta}_{t-1}$, the realization of shocks $\boldsymbol{\varepsilon}_t$, and a perturbation parameter χ.

Perturbation seeks to generate Taylor series expansions to the functions g_{s_t}, $h^x_{s_t}$, and $h^\eta_{s_t}$, around a given point. The following section turns to how to define this point and construct the approximations.

3.2 The Refined Partition Principle

The Partition Principle in Foerster et al. (2016) dictates separating the switching parameters $\boldsymbol{\theta}(k)$ into blocks denoted $\boldsymbol{\theta}_1(k)$ and $\boldsymbol{\theta}_2(k)$, for $k \in \{1, \ldots, n_s\}$, where the first set are perturbed and the second set are not. This partition of the parameters allows for finding a steady state and preserving the maximum information at lower orders of approximation. In particular, the perturbation function is

$$\boldsymbol{\theta}(k, \chi) = \chi \begin{bmatrix} \boldsymbol{\theta}_1(k) \\ \boldsymbol{\theta}_2(k) \end{bmatrix} + (1 - \chi) \begin{bmatrix} \bar{\boldsymbol{\theta}}_1 \\ \boldsymbol{\theta}_2(k) \end{bmatrix} \tag{21}$$

for for $k \in \{1, \ldots, n_s\}$. The set of parameters included in $\boldsymbol{\theta}_2(k)$ is chosen to be the maximal set such that a steady state is defined.

The definition of a steady state is when $\boldsymbol{\varepsilon}_t = 0$, $\chi = 0$, and hence for all s_t

$$\mathbf{y}_{ss} = g_{s_t}(\mathbf{x}_{ss}, \boldsymbol{\eta}_{ss}, \mathbf{0}, 0), \tag{22}$$

$$\mathbf{x}_{ss} = h^x_{s_t}(\mathbf{x}_{ss}, \boldsymbol{\eta}_{ss}, \mathbf{0}, 0), \tag{23}$$

and

$$\boldsymbol{\eta}_{ss} = h^\eta_{s_t}(\mathbf{x}_{ss}, \boldsymbol{\eta}_{ss}, \mathbf{0}, 0). \tag{24}$$

In the case with learning as shown in the set of equilibrium equations (16), the presence of all the regime-switching parameters $\boldsymbol{\Theta}$ in the learning process poses a challenge. A natural extension of the Partition Principle would suggest perturbing the same sets of parameters in $\boldsymbol{\Theta}$ that are perturbed in equation (21). That is, we could write

$$\boldsymbol{\Theta}(\chi) = \chi \begin{bmatrix} \boldsymbol{\theta}_1(1) \\ \boldsymbol{\theta}_2(1) \\ \vdots \\ \boldsymbol{\theta}_1(n_s) \\ \boldsymbol{\theta}_2(n_s) \end{bmatrix} + (1 - \chi) \begin{bmatrix} \bar{\boldsymbol{\theta}}_1 \\ \boldsymbol{\theta}_2(1) \\ \vdots \\ \bar{\boldsymbol{\theta}}_1 \\ \boldsymbol{\theta}_2(n_s) \end{bmatrix}, \tag{25}$$

and the steady state would be defined by

$$\widetilde{\mathbf{f}}\left(\mathbf{y}_{ss}, \mathbf{y}_{ss}, \mathbf{x}_{ss}, \mathbf{x}_{ss}, \boldsymbol{\eta}_{ss}, \boldsymbol{\eta}_{ss}, \mathbf{0}, \mathbf{0}, \bar{\boldsymbol{\theta}}_1, \bar{\boldsymbol{\theta}}_2(s'), \bar{\boldsymbol{\theta}}_1, \bar{\boldsymbol{\theta}}_2(s), \boldsymbol{\Theta}(0)\right) = 0$$

for all s', s.

However, this assumption would lead to a loss of information in the steady state and at low orders of approximation. To see this point, let us return to the RBC example, where in the context of a full information regime-switching model, the Partition Principle would require perturbing μ_t and not σ_t. In the case with learning, though, the steady state would be defined by

$$\begin{bmatrix} \beta\left(\alpha\exp\left(z_{ss}\right)k_{ss}^{\alpha-1}+1-\delta\right)-1 \\ \exp\left(z_{ss}\right)k_{ss}^{\alpha}+\left(1-\delta\right)k_{ss}-c_{ss}-k_{ss} \\ \bar{\mu}-z_{ss} \\ \frac{\frac{1}{\sigma(1)}\phi\left(\frac{z_{ss}-\bar{\mu}}{\sigma(1)}\right)\left(\frac{P_{1,1}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{P_{2,1}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}{\frac{1}{\sigma(2)}\phi\left(\frac{z_{ss}-\bar{\mu}}{\sigma(2)}\right)\left(\frac{P_{1,2}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{P_{2,2}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}-\exp\eta_{1,ss} \\ \frac{\frac{1}{\sigma(2)}\phi\left(\frac{z_{ss}-\bar{\mu}}{\sigma(2)}\right)\left(\frac{P_{1,2}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{P_{2,2}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}{\frac{1}{\sigma(1)}\phi\left(\frac{z_{ss}-\bar{\mu}}{\sigma(1)}\right)\left(\frac{P_{1,1}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{P_{2,1}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}-\exp\eta_{2,ss} \end{bmatrix}=0$$

Note that, while the first three equations are absent $\sigma\left(s_t\right)$, both the learning equuations are still functions of $\sigma\left(1\right)$ and $\sigma\left(2\right)$. So, one could simply perturb the parameter σ_t as well, in which case the steady state is defined by

$$\begin{bmatrix} \beta\left(\alpha\exp\left(z_{ss}\right)k_{ss}^{\alpha-1}+1-\delta\right)-1 \\ \exp\left(z_{ss}\right)k_{ss}^{\alpha}+\left(1-\delta\right)k_{ss}-c_{ss}-k_{ss} \\ \bar{\mu}-z_{ss} \\ \frac{\frac{1}{\bar{\sigma}}\phi\left(\frac{z_{ss}-\bar{\mu}}{\bar{\sigma}}\right)\left(\frac{P_{1,1}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{P_{2,1}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}{\frac{1}{\bar{\sigma}}\phi\left(\frac{z_{ss}-\bar{\mu}}{\bar{\sigma}}\right)\left(\frac{P_{1,2}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{P_{2,2}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}-\exp\eta_{1,ss} \\ \frac{\frac{1}{\bar{\sigma}}\phi\left(\frac{z_{ss}-\bar{\mu}}{\bar{\sigma}}\right)\left(\frac{P_{1,2}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{P_{2,2}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}{\frac{1}{\bar{\sigma}}\phi\left(\frac{z_{ss}-\bar{\mu}}{\bar{\sigma}}\right)\left(\frac{P_{1,1}}{1+\exp\left(-\eta_{1,ss}\right)}+\frac{P_{2,1}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}-\exp\eta_{2,ss} \end{bmatrix}=0$$

which can be solved in a straightforward manner to get

$$z_{ss}=\bar{\mu},\ k_{ss}=\left(\frac{1}{\alpha\exp\left(z_{ss}\right)}\left(\frac{1}{\beta}-1+\delta\right)\right)^{\frac{1}{\alpha-1}},$$

$$c_{ss}=\exp\left(z_{ss}\right)k_{ss}^{\alpha}-\delta k_{ss},$$

$$\eta_{1,ss} = \log\left(\frac{1-P_{2,2}}{1-P_{1,1}}\right), \text{ and } \eta_{2,ss} = \log\left(\frac{1-P_{1,1}}{1-P_{2,2}}\right).$$

This ability to find a steady state that is independent of Markov-switching parameters comes at a cost, though, in that perturbing σ_t leads to a loss of information at low levels of approximation.

To resolve this issue, one option is to not perturb any part of $\mathbf{\Theta}$, treating it differently than $\boldsymbol{\theta}_t$ entirely. In this *Partition Principle Refinement*, we would write leave $\mathbf{\Theta}$ unchanged, and hence the steady state would be defined by

$$\widetilde{\mathbf{f}}\left(\mathbf{y}_{ss}, \mathbf{y}_{ss}, \mathbf{x}_{ss}, \mathbf{x}_{ss}, \boldsymbol{\eta}_{ss,} \boldsymbol{\eta}_{ss}, \mathbf{0}, \mathbf{0}, \bar{\boldsymbol{\theta}}_1, \bar{\boldsymbol{\theta}}_2\left(s'\right), \bar{\boldsymbol{\theta}}_1, \bar{\boldsymbol{\theta}}_2\left(s\right), \mathbf{\Theta}\right) = 0 \tag{26}$$

for all s and s'.

Returning again to the RBC example, the steady state would thus satisfy

$$\begin{bmatrix} \beta\left(\alpha \exp\left(z_{ss}\right) k_{ss}^{\alpha-1} + 1 - \delta\right) - 1 \\ \exp\left(z_{ss}\right) k_{ss}^{\alpha} + (1-\delta) k_{ss} - c_{ss} - k_{ss} \\ \bar{\mu} - z_{ss} \\ \dfrac{\frac{1}{\sigma(1)}\phi\left(\frac{z_{ss}-\mu(1)}{\sigma(1)}\right)\left(\frac{P_{1,1}}{1+\exp\left(-\eta_{1,ss}\right)} + \frac{P_{2,1}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}{\frac{1}{\sigma(2)}\phi\left(\frac{z_{ss}-\mu(2)}{\sigma(2)}\right)\left(\frac{P_{1,2}}{1+\exp\left(-\eta_{1,ss}\right)} + \frac{P_{2,2}}{1+\exp\left(-\eta_{2,ss}\right)}\right)} - \exp \eta_{1,ss} \\ \dfrac{\frac{1}{\sigma(2)}\phi\left(\frac{z_{ss}-\mu(2)}{\sigma(2)}\right)\left(\frac{P_{1,2}}{1+\exp\left(-\eta_{1,ss}\right)} + \frac{P_{2,2}}{1+\exp\left(-\eta_{2,ss}\right)}\right)}{\frac{1}{\sigma(1)}\phi\left(\frac{z_{ss}-\mu(1)}{\sigma(1)}\right)\left(\frac{P_{1,1}}{1+\exp\left(-\eta_{1,ss}\right)} + \frac{P_{2,1}}{1+\exp\left(-\eta_{2,ss}\right)}\right)} - \exp \eta_{2,ss} \end{bmatrix} = 0$$

where z_{ss}, k_{ss}, and c_{ss} are unchanged from above, and $\eta_{1,ss}$ and $\eta_{2,ss}$ can be solved nonlinearly.

3.3 Perturbation Setup

Having discussed a Refined Partition Principle, we return to a full definition of the equilibrium, which is

$$\sum_{s=1}^{n_s}\sum_{s'=1}^{n_s} \frac{p_{s,s'}}{1+\exp\left(-\eta_{s,t}\right)} \int \widetilde{\mathbf{f}}\left(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_t, \mathbf{x}_{t-1}, \boldsymbol{\eta}_{t,} \boldsymbol{\eta}_{t-1}, \boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}_t, \boldsymbol{\theta}_1\left(s'\right), \boldsymbol{\theta}_2\left(s'\right), \boldsymbol{\theta}_1\left(s_t\right), \boldsymbol{\theta}_1\left(s_t\right), \mathbf{\Theta}\right) \phi_{\varepsilon}\left(\varepsilon'\right) = 0. \tag{27}$$

Using the functional forms (18), (19), (20), and

$$\begin{bmatrix} \boldsymbol{\theta}_1(k) \\ \boldsymbol{\theta}_2(k) \end{bmatrix} = \chi \begin{bmatrix} \boldsymbol{\theta}_1(k) \\ \boldsymbol{\theta}_2(k) \end{bmatrix} + (1-\chi) \begin{bmatrix} \bar{\boldsymbol{\theta}}_1 \\ \boldsymbol{\theta}_2(k) \end{bmatrix} \tag{28}$$

for $k = s_t, s'$ produces an equation of the form

$$\mathbf{F}_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\eta}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right) = 0.$$

We will take derivatives of this function, evaluated at steady state, to find approximations to the policy functions (18), (19), (20).

3.4 Steady State

The steady state is given by the set of equations

$$\mathbf{F}_{s_t}\left(\mathbf{x}_{ss}, \boldsymbol{\eta}_{ss}, \mathbf{0}, 0\right) = 0,$$

for all s_t. The definitions of the functions at steady state imply

$$\begin{aligned} \widetilde{\mathbf{f}}\left(\mathbf{y}_{ss}, \mathbf{y}_{ss}, \mathbf{x}_{ss}, \mathbf{x}_{ss}, \boldsymbol{\eta}_{ss}, \boldsymbol{\eta}_{ss}, \mathbf{0}, \mathbf{0}, \bar{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_2\left(s'\right), \bar{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_2\left(s_t\right), \boldsymbol{\Theta}\right) &= \\ \left[\begin{array}{c} \mathbf{f}\left(\mathbf{y}_{ss}, \mathbf{y}_{ss}, \mathbf{x}_{ss}, \mathbf{x}_{ss}, \mathbf{0}, \mathbf{0}, \bar{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_2\left(s'\right), \bar{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_2\left(s_t\right)\right) \\ \Phi\left(\mathbf{y}_{ss}, \mathbf{x}_{ss}, \mathbf{x}_{ss}, \boldsymbol{\eta}_{ss}, \boldsymbol{\eta}_{ss}, \boldsymbol{\Theta}\right) \end{array}\right] &= 0. \end{aligned} \tag{29}$$

Since the first n equations of $\tilde{\mathbf{f}}$ are the original equilibrium conditions in equation (1), then the steady state satisfies

$$f\left(\mathbf{y}_{ss}, \mathbf{y}_{ss}, \mathbf{x}_{ss}, \mathbf{x}_{ss}, \mathbf{0}, \mathbf{0}, \bar{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_2\left(s_{t+1}\right), \bar{\boldsymbol{\theta}}_1, \boldsymbol{\theta}_2\left(s_t\right)\right) = 0, \tag{30}$$

or in other words, is identical to the steady state to a version of the model with full information and can be used to solve for the n unknowns $\{\mathbf{y}_{ss}, \mathbf{x}_{ss}\}$. The second set of n_s equations of $\tilde{f}$ are the Bayesian updating equations, and so

$$\Phi\left(\mathbf{y}_{ss}, \mathbf{x}_{ss}, \mathbf{x}_{ss}, \boldsymbol{\eta}_{ss}, \boldsymbol{\eta}_{ss}, \boldsymbol{\Theta}\right) = 0$$

pins down the n_s unknowns $\boldsymbol{\eta}_{ss}$.

After solving for the steady state, the next sections discuss generating first- and higher-order approximations.

3.5 Deriving Approximations

We can take derivatives of $\mathbf{F}_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\eta}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right)$ with respect to its arguments to get equations that allow us to solve for the coefficients in the expansions of (18), (19), and (20).

First, the derivative with respect to $\mathbf{x}_{t-1}$ evaluated at steady state is given by

$$\mathbf{F}_{\mathbf{x}_{t-1},s_t} = \sum_{s=1}^{n_s}\sum_{s'=1}^{n_s} \frac{p_{s,s'}}{1+\exp\left(-\eta_{s,ss}\right)} \left[\begin{array}{c} \tilde{\mathbf{f}}_{\mathbf{y}_{t+1}}\left(s', s_t\right)\left(g_{\mathbf{x}_{t-1},s'} h^x_{\mathbf{x}_{t-1},s_t} + g_{\boldsymbol{\eta}_{t-1},s'} h^\eta_{\mathbf{x}_{t-1},s_t}\right) \\ +\tilde{\mathbf{f}}_{\mathbf{y}_t}\left(s', s_t\right) g_{\mathbf{x}_{t-1},s_t} + \tilde{\mathbf{f}}_{\mathbf{x}_t}\left(s', s_t\right) h^x_{\mathbf{x}_{t-1},s_t} \\ +\tilde{\mathbf{f}}_{\mathbf{x}_{t-1}}\left(s', s_t\right) + \tilde{\mathbf{f}}_{\boldsymbol{\eta}_t}\left(s', s_t\right) h^\eta_{\mathbf{x}_{t-1},s_t} \end{array}\right] = 0 \quad (31)$$

and with respect to $\boldsymbol{\eta}_{t-1}$ is given by

$$\mathbf{F}_{\boldsymbol{\eta}_{t-1},s_t} = \sum_{s=1}^{n_s}\sum_{s'=1}^{n_s} \frac{p_{s,s'}}{1+\exp\left(-\eta_{s,ss}\right)} \left[\begin{array}{c} \tilde{\mathbf{f}}_{\mathbf{y}_{t+1}}\left(s', s_t\right)\left(g_{\mathbf{x}_{t-1},s'} h^x_{\mathbf{x}_{t-1},s_t} + g_{\boldsymbol{\eta}_{t-1},s'} h^\eta_{\boldsymbol{\eta}_{t-1},s_t}\right) \\ +\tilde{\mathbf{f}}_{\mathbf{y}_t}\left(s', s_t\right) g_{\boldsymbol{\eta}_{t-1},s_t} + \tilde{\mathbf{f}}_{\mathbf{x}_t}\left(s', s_t\right) h^x_{\boldsymbol{\eta}_{t-1},s_t} \\ +\tilde{\mathbf{f}}_{\boldsymbol{\eta}_t}\left(s', s_t\right) h^\eta_{\boldsymbol{\eta}_{t-1},s_t} + \tilde{\mathbf{f}}_{\boldsymbol{\eta}_{t-1}}\left(s', s_t\right) \end{array}\right] = 0 \quad (32)$$

These two expressions can be concatenated together and across regimes to produce a system of the form

$$\mathbf{F}_{\left[\mathbf{x}_{t-1}\boldsymbol{\eta}_{t-1}\right]} = \left[\begin{array}{cc} \mathbf{F}_{\mathbf{x}_{t-1},s_t=1} & \mathbf{F}_{\boldsymbol{\eta}_{t-1},s_t=1} \\ \vdots & \vdots \\ \mathbf{F}_{\mathbf{x}_{t-1},s_t=n_s} & \mathbf{F}_{\boldsymbol{\eta}_{t-1},s_t=n_s} \end{array}\right] = 0, \quad (33)$$

which is a quadratic form in the unknowns $\left\{g_{\mathbf{x}_{t-1},s_t}, g_{\boldsymbol{\eta}_{t-1},s_t}, h^x_{\mathbf{x}_{t-1},s_t}, h^x_{\boldsymbol{\eta}_{t-1},s_t}, h^\eta_{\mathbf{x}_{t-1},s_t}, h^\eta_{\boldsymbol{\eta}_{t-1},s_t}\right\}_{s_t=1}^{n_s}$. Further, the expression is of the form shown by Foerster et al. (2016) to be a general quadratic form that is solvable by using Gröbner bases rather than the standard Eigenvalue problem found in constant parameter models. Gröbner bases have the advantage that they will find all possible solutions to the quadratic form (33), each of which can be checked for stability. The concept of mean square stability (MSS), defined in Costa et al. (2005), and advocated by Farmer et al. (2009) and Foerster et al. (2016) allows for unbounded paths provided that first and second moments of the solutions are finite. In the context of learning, MSS has the advantage over an alternative concept of bounded stability–which requires all possible solutions to be bounded–since it allows for $\eta_{i,t} \to \pm\infty$ and hence the subjective probabilities to become arbitrarily close to either 0 or 1. In other words, it includes possible realizations of shocks and regimes such that the current regime is learned with near-perfect precision.

After solving equation (33), the other terms that make up the first order approximations can be solved via standard linear methods. First, the derivative with respect to $\boldsymbol{\varepsilon}_t$ is given by

$$\mathbf{F}_{\boldsymbol{\varepsilon}_t,s_t} = \sum_{s=1}^{n_s}\sum_{s'=1}^{n_s} \frac{p_{s,s'}}{1+\exp\left(-\eta_{s,ss}\right)} \left[\begin{array}{c} \tilde{\mathbf{f}}_{\mathbf{y}_{t+1}}\left(s',s_t\right)\left(g_{\mathbf{x}_{t-1},s'}h^{x}_{\boldsymbol{\varepsilon}_t,s_t} + g_{\boldsymbol{\eta}_{t-1},s'}h^{\eta}_{\boldsymbol{\varepsilon}_t,s_t}\right) \\ +\tilde{\mathbf{f}}_{\mathbf{y}_t}\left(s',s_t\right)g_{\boldsymbol{\varepsilon}_t,s_t} + \tilde{\mathbf{f}}_{\mathbf{x}_t}\left(s',s_t\right)h^{x}_{\boldsymbol{\varepsilon}_t,s_t} \\ +\tilde{\mathbf{f}}_{\boldsymbol{\eta}_t}\left(s',s_t\right)h^{\eta}_{\boldsymbol{\varepsilon}_t,s_t} + \tilde{\mathbf{f}}_{\boldsymbol{\varepsilon}_t}\left(s',s_t\right) \end{array} \right] \tag{34}$$

which can be concatenated to generate

$$\mathbf{F}_{\boldsymbol{\varepsilon}_t} = \left[\begin{array}{c} \mathbf{F}_{\boldsymbol{\varepsilon}_t,s_t=1} \\ \vdots \\ \mathbf{F}_{\boldsymbol{\varepsilon}_t,s_t=n_s} \end{array} \right] = 0, \tag{35}$$

which is a linear system in the unknowns $\left\{g_{\varepsilon_t,s_t}, h^{x}_{\boldsymbol{\varepsilon}_t,s_t}, h^{\eta}_{\boldsymbol{\varepsilon}_t,s_t}\right\}_{s_t=1}^{n_s}$.

Lastly the derivative with respect to χ is given by

$$\mathbf{F}_{\chi,s_t} = \sum_{s=1}^{n_s}\sum_{s'=1}^{n_s} \frac{p_{s,s'}}{1+\exp\left(-\eta_{s,ss}\right)} \left[\begin{array}{c} \tilde{\mathbf{f}}_{\mathbf{y}_{t+1}}\left(s',s_t\right)\left(g_{\mathbf{x}_{t-1},s'}h^{x}_{\chi,s_t} + g_{\boldsymbol{\eta}_{t-1},s'}h^{\eta}_{\chi,s_t} + g_{\chi,s'}\right) \\ +\tilde{\mathbf{f}}_{\mathbf{y}_t}\left(s',s_t\right)g_{\chi,s_t} + \tilde{\mathbf{f}}_{\mathbf{x}_t}\left(s',s_t\right)h^{x}_{\chi,s_t} + \tilde{\mathbf{f}}_{\boldsymbol{\eta}_t}\left(s',s_t\right)h^{\eta}_{\chi,s_t} \\ \tilde{\mathbf{f}}_{\boldsymbol{\theta}_{1,t+1}}\left(\boldsymbol{\theta}_1\left(s'\right)-\bar{\boldsymbol{\theta}}_1\right) + \tilde{\mathbf{f}}_{\boldsymbol{\theta}_{1,t}}\left(\boldsymbol{\theta}_1\left(s_t\right)-\bar{\boldsymbol{\theta}}_1\right) \end{array} \right] \tag{36}$$

which produces a system given by

$$\mathbf{F}_{\chi} = \left[\begin{array}{c} \mathbf{F}_{\chi,s_t=1} \\ \vdots \\ \mathbf{F}_{\chi,s_t=n_s} \end{array} \right] = 0, \tag{37}$$

which is linear in the unknowns $\left\{g_{\chi,s_t}, h^{x}_{\chi,s_t}, h^{\eta}_{\chi,s_t}\right\}_{s_t=1}^{n_s}$.

These expressions thus can be used to solve for the coefficients of the first-order expansion. We can take derivatives of $\left(\mathbf{x}_{t-1}, \boldsymbol{\eta}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right)$ multiple times to solve for second- or higher-order approximations, which are simply progressively larger linear systems. For expositional simplicity, we relegate these expressions to the appendix.

3.6 Properties of Approximations

Having provided the analytical expressions for the derivatives, we now characterize several important features of the solution and approximation.

- Subjective probabilities are constant at first order, show up at second order

- Important interaction that shows up in third order.

4 A Simple Filtering Example

To illustrate the impact of approximating the subjective probabilities, we now consider a simple filtering problem devoid of any economic decision making.

4.1 Filtering Model

In this example, the signal follows a process akin to that in our RBC example equation (4)

$$z_t = \mu(s_t) + \sigma(s_t)\varepsilon_t, \tag{38}$$

where z_t is observable but the individual components are not. In this example $\mathbf{y}_t = z_t$, $\mathbf{x}_t$ is empty, $\boldsymbol{\theta}_1(s_t) = \mu(s_t)$, and $\theta_2(s_t) = \sigma(s_t)$. Simple Bayesian updating implies the probabilities of being in each regime evolve according to

$$\psi_{i,t} = \frac{\phi\left(\frac{z_t-\mu(i)}{\sigma(i)}\right)\left(P_{1,i}\psi_{1,t-1} + P_{2,i}\psi_{2,t-1}\right)}{\sum_{j=1}^{2}\phi\left(\frac{z_t-\mu(j)}{\sigma(j)}\right)\left[P_{1,j}\psi_{1,t-1} + P_{2,j}\psi_{2,t-1}\right]} \tag{39}$$

for $i = 1, 2$. Then we can write the system (16) as

$$\begin{bmatrix} \mu(s_t) + \sigma(s_t)\varepsilon_t - z_t \\ \log\left(\frac{\phi\left(\frac{z_t-\mu(1)}{\sigma(1)}\right)\left(\frac{P_{1,1}}{1+\exp\left(-\eta_{1,t-1}\right)} + \frac{P_{2,1}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}{\phi\left(\frac{z_t-\mu(2)}{\sigma(2)}\right)\left(\frac{P_{1,2}}{1+\exp\left(-\eta_{1,t-1}\right)} + \frac{P_{2,2}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}\right) - \eta_{1,t} \\ \log\left(\frac{\phi\left(\frac{z_t-\mu(2)}{\sigma(2)}\right)\left(\frac{P_{1,2}}{1+\exp\left(-\eta_{1,t-1}\right)} + \frac{P_{2,2}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}{\phi\left(\frac{z_t-\mu(1)}{\sigma(1)}\right)\left(\frac{P_{1,1}}{1+\exp\left(-\eta_{1,t-1}\right)} + \frac{P_{2,1}}{1+\exp\left(-\eta_{2,t-1}\right)}\right)}\right) - \eta_{2,t} \end{bmatrix} = 0, \tag{40}$$

and we note that since this a simple filtering example, there is no forward looking component, so expectations are irrelevant.

We calibrate as shown in Table 1, with two separate parameterizations. In the switching volatility case, the first regime has a high average signal with low variance, and the second regime has a low average signal with high variance. In the constant volatility case, the average

Table 1: Calibration of Filtering Example

	$P_{1,1}$	$P_{2,2}$	$\mu(1)$	$\mu(2)$	$\sigma(1)$	$\sigma(2)$
Example 1	0.95	0.95	0.01	0.005	0.0031	0.0075
Example 2	0.95	0.95	0.01	0.005	0.0031	0.0031
Example 3	0.95	0.95	0.0075	0.0075	0.0031	0.0075

signal changes across regimes, but both regimes have low variance. These two examples will help illustrate our methodology when the regime is relatively hard to identify in the switching volatility case, versus relatively easy to identify in the constant volatility case.

4.2 Filtering Results

We first consider the implications of the switching volatility model for the evolution of subjective probabilities. Figure 1 shows the results of simulations and their implications for the distributions of the signal z_t and the probabilities $\psi_{1,t}$ and $\psi_{2,t}$. The top left panel shows the unconditional distribution of z_t, which is left skewed as the mixture of two normal distributions. The top right panel shows the conditional distributions based on the realization of each regime. In regime $s_t = 1$, the realizations of z_t tend to be relatively tightly centered around the conditional mean, while in regime $s_t = 2$ the distribution has larger variance around a lower mean. The implications for the learning process from these conditional distributions is that $s_t = 2$ is relatively easier to identify because of its tendency to produce outcomes that are very low probability events in the $s_t = 1$ regime. On the other hand, regime $s_t = 1$ is relatively hard to identify, as outcomes tend to be somewhat likely under both regimes.

The remaining panels of Figure 1 show the distribution of subjective probabilities, conditional on the realization of each regime, for different orders of approximation. The second row of panels show that, with a third-order approximation, the first regime is rarely identified correctly, as most of the time there is a low probability placed on being in the first regime, $\psi_{1,t}$, when the regime is actually $s_t = 1$. On the other hand, the second regime is identified correctly nearly all of the time, with most of the mass of $\psi_{2,t}$ being near 1 when the regime is actually $s_t = 1$. The third row of panels show these traits are true at a second-order approximation as well. Given the conditional distributions of z_t in each regime, we would in fact expect this behavior for the subjective probabilities; the inability to correctly identify regime 1 while identifying regime 2 with high accuracy is a feature of the environment that our approximation picks up well, rather than being an indication if inaccurate approximations. We highlight this momentarily with

Figure 1: Simulations of the Switching Volatility Model

our second calibration and some additional accuracy checks. However, it is important to note that the last row of panels shows the distribution of beliefs for a first-order approximation, and how they are invariant at the steady state of the probabilities, which is 0.5 for this numerical example. In other words, first-order approximations are not enough to pick up any variation in beliefs.

The second calibration, which has constant volatilities across regimes rather than a higher volatility in the second regime, demonstrates how inference about the regime can be more accurate given different signal processes. Figure 2 shows simulation results for this alternative calibration. The upper right panel highlights there is now symmetry of the conditional regimes across regime, suggesting that the learning process should be symmetric. The panels showing third- and second-order approximations verify this feature, with both regimes being correctly identified with high probability, but with some slight chance of placing low weight on the true regime. Again, however, the first-order approximations are completely uninformative, with all mass being placed on the steady state probability and being invariant to the signal.

Turning back to the issue of accuracy of our approximation, Figures 1 and 2 are insufficient for this purpose, as they compare beliefs with actual realizations of the regime, and without full information there is no reason to believe these should be fully accurate given the distribution of the signal. A more proper way of assessing accuracy of the approximation is to compare these simulations with the fully nonlinear updating of beliefs in equation (39). In Table 2 we compare mean square error of our approximations, which is given by

$$MSE^{order} = T^{-1} \sum_{t=1}^{T} \left(\hat{\psi}_{1,t}^{order} - \psi_{1,t} \right)^2 \tag{41}$$

where $\hat{\psi}_{1,t}^{order}$ is the approximated beliefs for $order \in \{1, 2, 3\}$, and $\psi_{1,t}$ is the true beliefs based upon the fully nonlinear updating. The table shows that for the switching volatility example, the second- and third-order approximations achieve a high degree of accuracy, while the first-order approximation performs relatively poorly. The constant volatility parameterization, which has a slightly different distribution of subjective probabilities, produces slightly less accurate approximations, but the degree of accuracy remains high for second- and third-order approximations, but low for first-order approximations.

The results in this simple filtering example therefore lead us to conclude that approximating

Figure 2: Simulations of the Constant Volatility Model

the Bayesian updating of subjective probabilities can be done with high accuracy using second- and third-orders, while first-order approximations are insufficient. However this example was by design without any economic decision-making, which limits our ability to characterize how the learning process interacts with other parts of the economy. In the following section, we return to the RBC example to supplement our results about the learning process with how it affects consumption and capital accumulation. In Section 6, we consider another example where learning and economic decision-making have feedback in both directions.

5 Application to the RBC Model

Having studied the learning mechanism in isolation, we now return to the RBC example introduced in Section 2. The set of parameters is given in Table 3. The parameters for preferences and production are standard values, while the processes for technology are exactly identical to those considered in the switching volatility parameterization in Section 4. In this way, we can supplement our analysis that focused solely on the learning to one in which learning impacts the consumption and capital accumulation decisions of a planner.

Figure 3 highlights the impact of learning and the importance of higher-order approximations in the RBC example by plotting simulated distributions of consumption and capital.[6] Given the parameterization in Table 3, the distribution of TFP and of the subjective probabilities under learning are identical to those shown in Figure 1. The top panels of Figure 3 show the distributions of capital and consumption under full information where the planner perfectly observes the regime. For first-, second-, and third-order of approximation, the distributions of capital and consumption look nearly identical, suggesting in this parameterization there are not major differences from higher-order approximations. Given regime-switching in the volatility of shocks, using second-order approximations may capture precautionary behavior, but given log preferences the effect appears to be modest.

The bottom panels in Figure 3 show that, with learning, a precautionary effect only shows up at third-order. When the regime is not perfectly revealed, recall from the discussion of Figure 1 that the first regime–with high average productivity and low variance–is relatively hard to identify, while the second regime–with low average productivity and high variance–is relatively

[6]For improved accuracy, and since log-linearization tends to be more typical in RBC models than simple linearization, we approximate the solution in logarithms rather than levels and present the corresponding results.

Table 2: Accuracy Check - MSEs

	order=3	order=2	order=1
Example 1	0.0273	0.0273	0.1544
Example 2	0.0862	0.0865	0.1789
Example 3	0.0081	0.0081	0.1848

easy to identify. As a result, there is a strong precautionary effect that induces the planner to accumulate additional capital and hence allow a higher level of consumption. However, this effect is absent in first- and second-order terms.

6 A Model with Feedback to Learning

One of the possible downsides of considering the RBC model as an application is that it lacks any feedback between economic decision-making and the learning mechanism. In other words, the learning is dependent only on an exogenous process and not endogenous variables. We now turn to an example where there is in fact feedback, as the speed of learning about a taxation regime depends on output, which in turn varies depending on the learning.

In this example, Households supply $n_t = 1$ units of labor inelastically and maximize

$$\hat{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t \tag{42}$$

subject to a sequence of budget constraints

$$c_t + k_t = w_t + (1 - \tau_t)\, r_t k_{t-1} + (1 - \delta)\, k_{t-1}, \tag{43}$$

where w_t denotes the real wage, r_t is the rental rate on capital, and τ_t is the rate of capital taxation. Standard optimality conditions give an Euler equation of the form

$$1 = \beta \hat{E}_t \frac{c_t}{c_{t+1}} \left((1 - \tau_{t+1})\, r_{t+1} + 1 - \delta \right). \tag{44}$$

Firms produce according to

$$y_t = \exp(z_t)\, k_{t-1}^{\alpha} n_t^{1-\alpha} \tag{45}$$

where productivity

$$z_t = \sigma_z \varepsilon_{z,t}, \tag{46}$$

Table 3: Calibration of RBC Example

α	β	δ	$P_{1,1}$	$P_{2,2}$	$\mu(1)$	$\mu(2)$	$\sigma(1)$	$\sigma(2)$
0.33	0.99	0.025	0.95	0.95	0.01	0.005	0.0031	0.0075

and $\varepsilon_{z,t} \sim N(0,1)$. The tax rate follows the feedback rule

$$\tau_t = \mu + \gamma(s_t)(\log y_{t-1} - \log y_{ss}) + \sigma_\tau \varepsilon_{\tau,t}, \tag{47}$$

and government purchases equalling $g_t = \tau_t r_t k_{t-1}$ are unproductive. The information set of households and firms includes $\mathcal{I}_t = \{c^t, k^t, r^t, w^t, y^t, \tau^t, z^t\}$, but importantly the components of the tax process are not observed.[7]

We can simplify to produce a set of equilibrium conditions of the form in equation (1),

$$f(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_t, \mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_{t+1}, \boldsymbol{\varepsilon}_t, \boldsymbol{\theta}_{t+1}, \boldsymbol{\theta}_t) = \tag{48}$$

$$\begin{bmatrix} \beta \frac{c_t}{c_{t+1}} \left((1-\tau_{t+1}) \alpha \exp(z_{t+1}) k_t^{\alpha-1} + 1 - \delta \right) - 1 \\ y_t + (1-\delta) k_{t-1} - c_t - k_t - \tau_t \alpha y_t \\ \exp(z_t) k_{t-1}^{\alpha} - y_t \\ \sigma_z \varepsilon_{z,t} - z_t \\ \mu + \gamma(s_t)(\log y_{t-1} - \log y_{ss}) + \sigma_\tau \varepsilon_{\tau,t} - \tau_t \end{bmatrix}. \tag{49}$$

where $\mathbf{y}_t = [c_t, z_t, \tau_t]$, $\mathbf{x}_t = [k_t, y_t]$, $\boldsymbol{\theta}_{1,t}$ is empty, and $\boldsymbol{\theta}_{2,t} = \gamma(s_t)$.

7 Conclusion

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[7] Technically, z_t is observed and $\varepsilon_{z,t}$ is not, but the shock is obviously exactly identified given the observables.

Figure 3: Simulations of the RBC Model

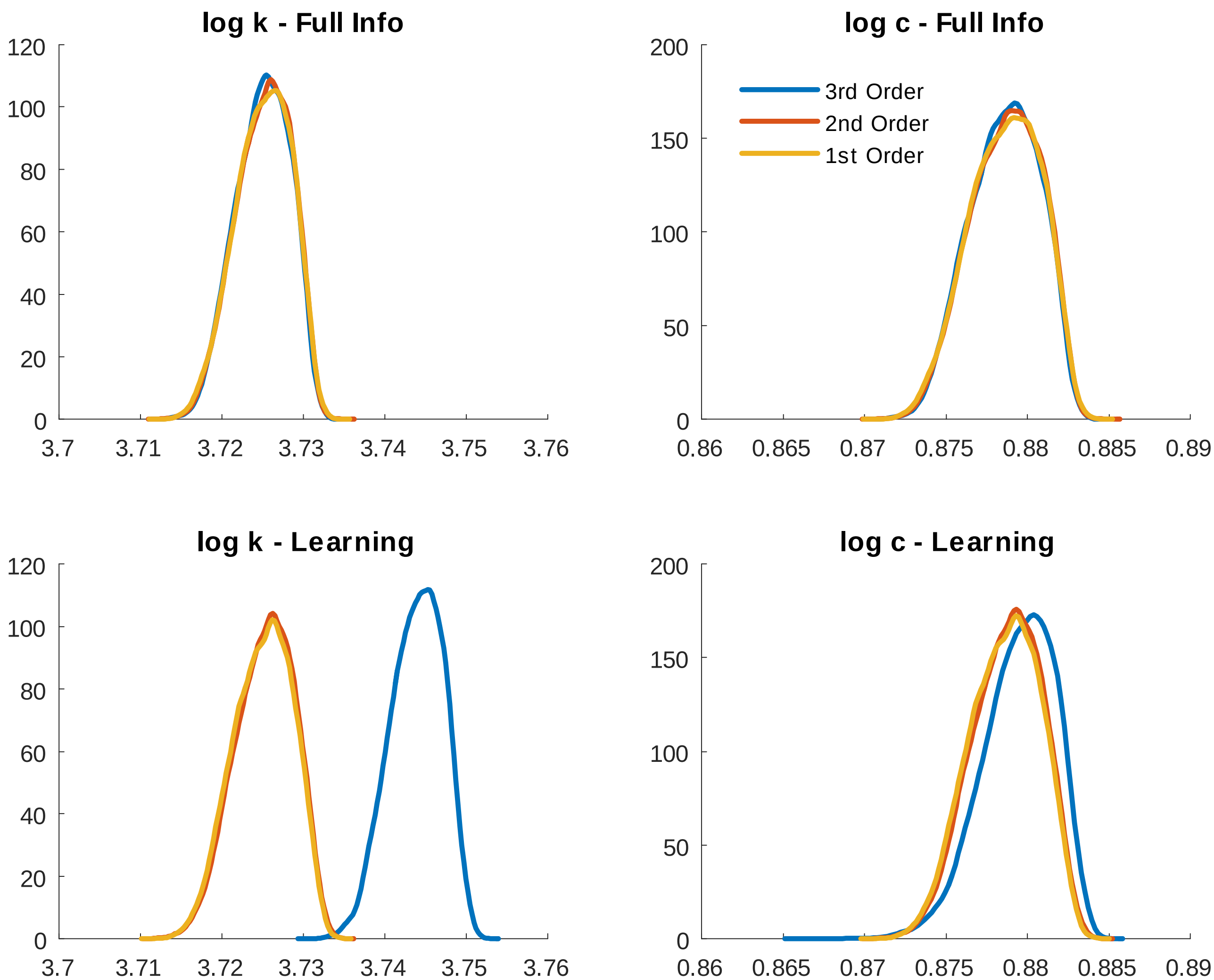

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Table 4: Calibration of Taxation Example

α	β	δ	$P_{1,1}$	$P_{2,2}$	μ	$\gamma(1)$	$\gamma(2)$	σ_τ	σ_z
0.33	0.99	0.025	0.95	0.95	0.20	0	5.0	0.01	0.01

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Figure 4: Conditional Simulations of the Taxation Model with Full Information

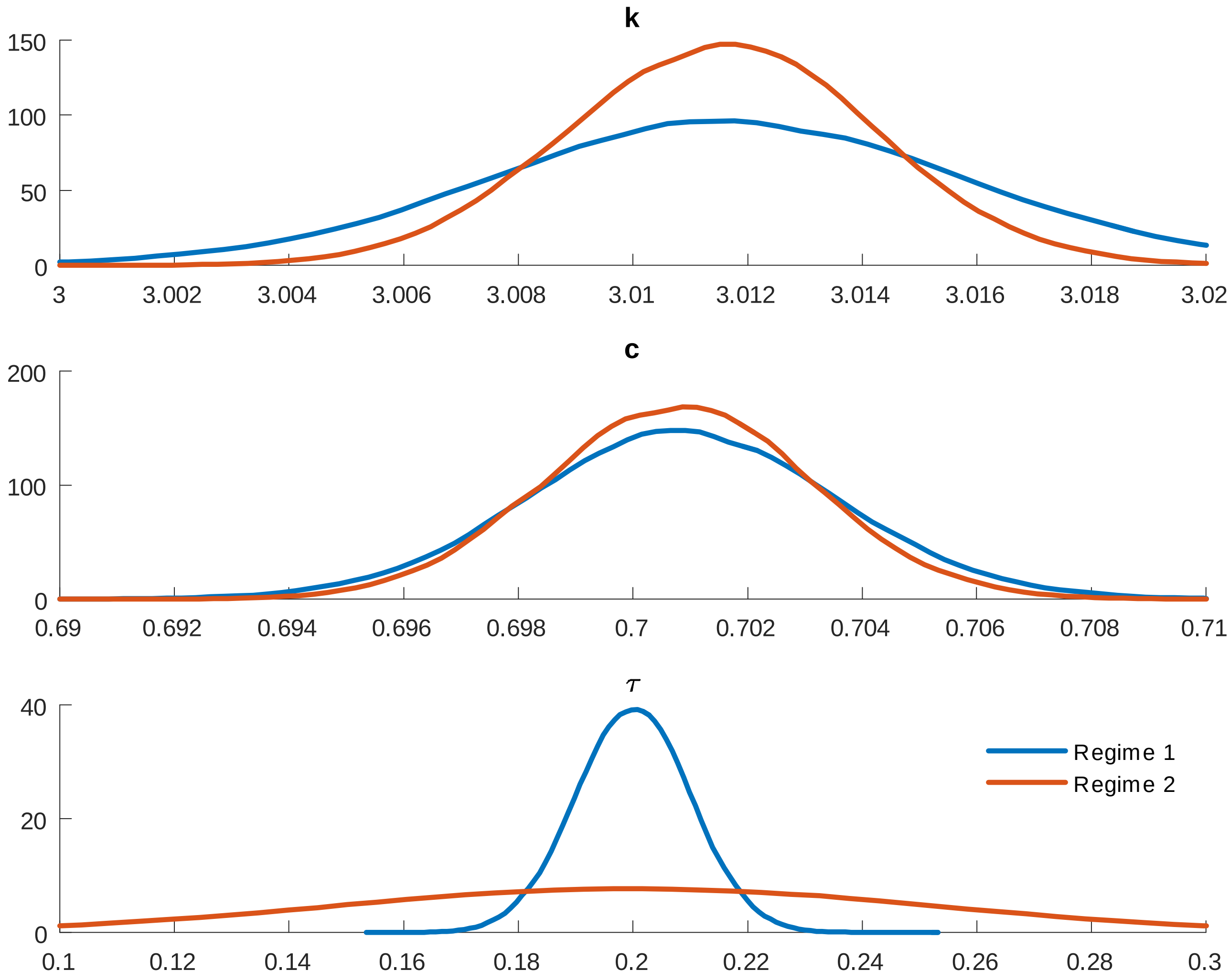


Figure 5: Comparison of Unconditional Simulations of the Taxation Model

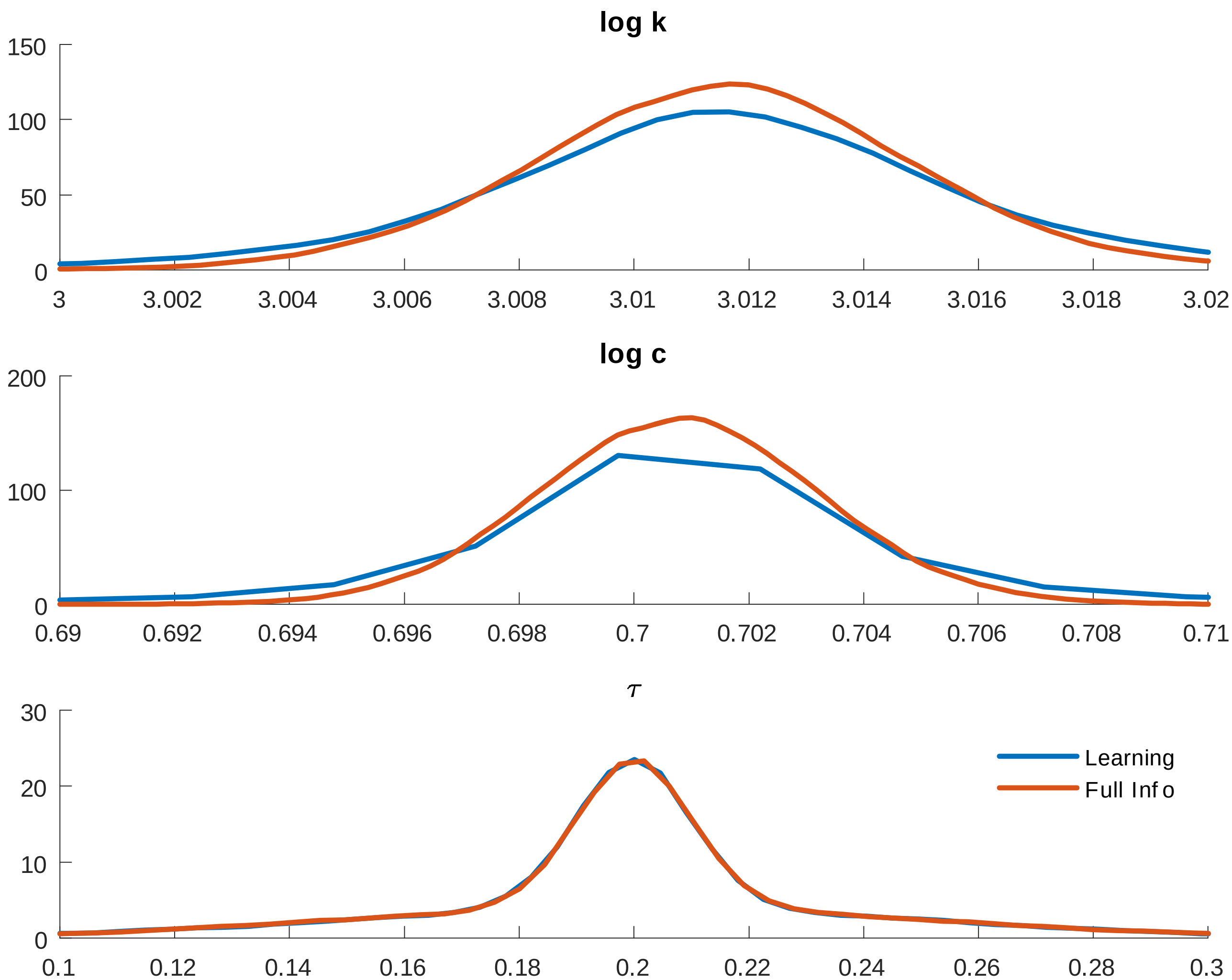
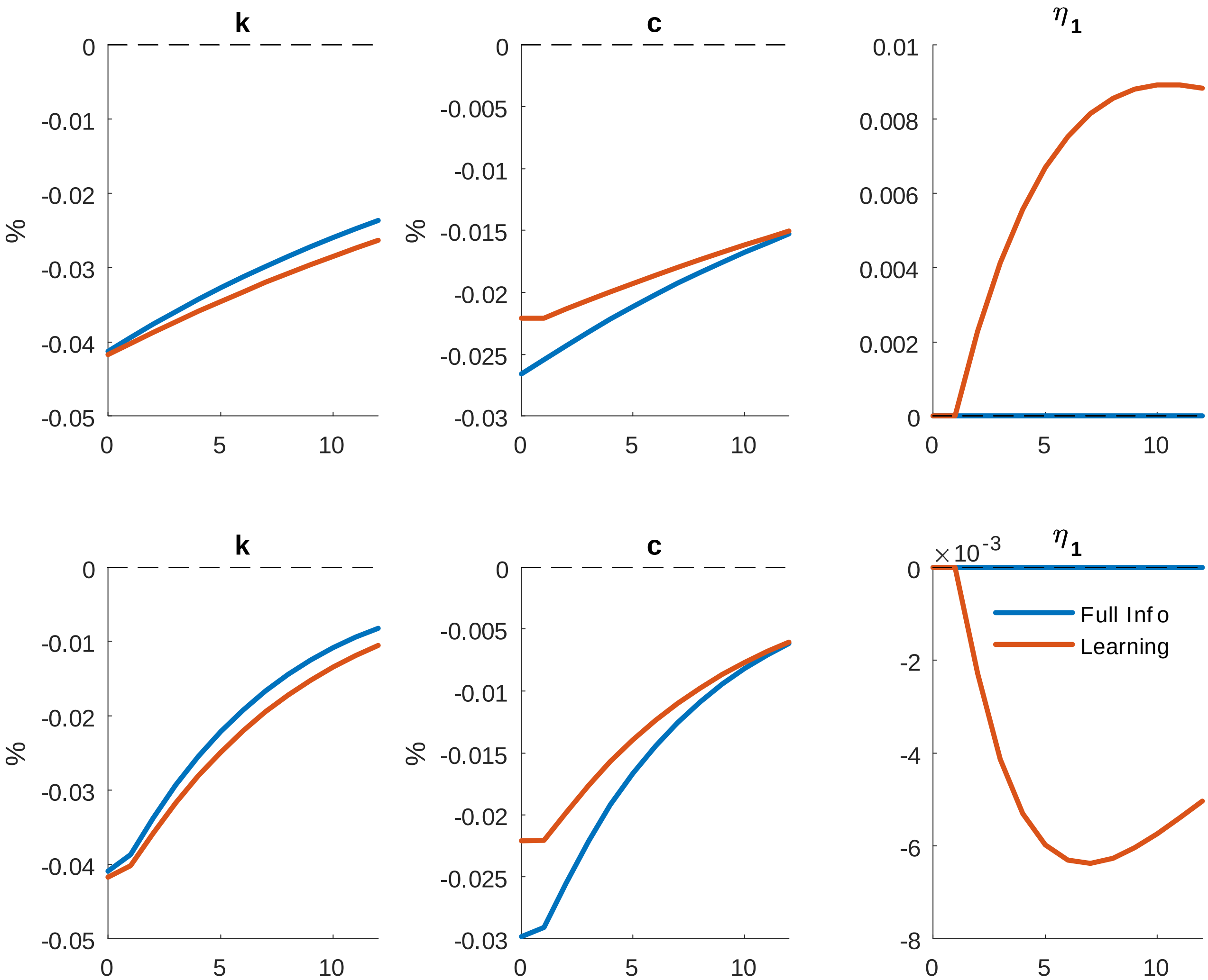


Figure 6: Response to a Positive Tax Shock in the Taxation Model



Note: Top graphs are conditional on regime $s_t = 1$, bottom graphs are conditional on regime $s_t = 2$