Estimating Macroeconomic Models of Financial Crises: An Endogenous Regime Switching Approach*

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August 31, 2017

Abstract

We develop a novel approach to specifying, solving and estimating Dynamic Structural General Equilibrium (DSGE) models of financial crises. We first propose a new specification of the standard Kiyotaki-Moore type collateral constraint where the movement from the unconstrained state of the world to constrained state is a stochastic function of the endogenous leverage ratio in the model. This specification results in an endogenous regime switching model. Next, we develop perturbation methods to solve this model. Using the second order solution of the model, we then design an algorithm to estimate the parameters of the model with full-information Bayesian methods. Applying the framework to quarterly Mexican data since 1981, we find that the model's estimated crisis regime probabilities correspond closely with narrative dates for Sudden Stops in Mexico. Our results also shows that fluctuations in the non-crisis regime of the model are driven primarily by real shocks, while leverage shocks are the prime driver of the crisis regime. The paper provides the first set of structural estimates of financial shocks stressed in the normative literature and consistent with available reduced form evidence finding that financial/credit shocks only matter in crisis periods.

Keywords: Financial Crises, Regime Switching, Bayesian Estimation, Leverage Shocks.
JEL Codes: G01, E3, F41, C11.

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1 Introduction

In response to the Financial Crisis a large literature has emerged to model the impact of financial frictions. Much of this literature has highlighted the importance of collateral constraints in amplifying shocks and providing a theoretical justification for policy interventions. However, due to computational complexity this literature largely eschews formal econometric analysis of these models and the shocks that historically have driven crisis episodes. That is, a Smets and Wouters (2007) style evaluation of this class of models, an evaluation that is needed for implementation of policy recommendations, has not been done. In this paper, we bridge the econometric evaluation of DSGE models in the spirit of Smets and Wouters (2007) with the collateral constraint models emphasized in the recent normative literature on financial frictions.

This paper makes contributions to four areas of the literature. First, we propose a new specification of the standard Kiyotaki and Moore (1997) type collateral constraint where the movement from the unconstrained state of the world to constrained state is a stochastic function of the endogenous leverage ratio in the model. Our model is such that, as leverage rises, the probability of the constraint binding increases, but there is no specific leverage ratio where the constraint must bind. This specification results in an endogenous regime switching model. Our second contribution is to develop perturbation methods to solve endogenous regime switching models rapidly and to higher orders. Third, using the second order solution of the model, we design an algorithm to estimate the parameters of the model with full-information Bayesian methods, which has previously only been done for first order solutions of exogenous switching models. Our fourth contribution is to apply the framework to Mexican quarterly data since 1980 and provide the first formal econometric analysis of this class of models.

The model is estimated from 1981 to 2016 using data for Mexico. Our results reveal three novel empirical findings. First, we find that the probability of a crisis is an increasing function of leverage, but also that there is range of leverage ratios where a crisis is likely to occur. Second, the model provides estimated crisis regime probabilities which correspond closely with narrative dates for Sudden Stops in Mexico. Third, our results shows that fluctuations in the non-crisis regime of the model are driven primarily by the usual real shocks (TFP, world interest rate, and terms of trade). In the crisis regime, we find that leverage shocks are the prime driver of economic fluctuations. Our results then, provide the first structural estimates of financial shocks consistent with the reduced form literature which finds that financial/credit shocks only matter in periods of high financial stress.

The core of the new methodology is an endogenous regime switching approach to modeling financial crises. In the model there are two regimes, one a crisis regime, the second a regime for normal economic times. A crisis regime is a regime where an occasionally binding borrowing constraint binds (e.g. Mendoza, 2010) determined by economic variables in the economy. Likewise, the switch back to normal times is based on economic fundamentals. In our model the probability of moving to the crisis regime where the borrowing constraint binds is a logistic function of the debt to output ratio. This ratio in turn, is a function of endogenous state variables, exogenous shocks and control variables. Agents in the economy know of this probability and how debt, output and other choices map into the probability of moving in or out of the crisis state. That is, it is a rational expectations solution of the model. Our solution then ensures that decisions made in the normal state fully incorporate how those decision affect the probability of moving into the crisis state as well how the economy will operate in a crisis (i.e the decision rules in this crisis).

The approach we develop allows us to capture all of the salient features one would want in an empirical model of financial crises. First, it captures the non-linear nature of a crisis: the crisis state can have very different properties/parameters from the normal state. Second, we solve the regime switching model using perturbation methods and a second order solution. This means that we can capture the change in decision rules as risk changes in a crisis. Third, since our solution method is perturbation-based we can handle multiple state variables and many shocks. That is, we are less constrained than current non-linear methods in terms of the size of the model. Fourth, the speed of the solution method means that we can use non-linear filters to calculate the likelihood function of the model for a full Bayesian estimation of the relevant shocks and frictions that characterize models of financial crises.

In the literature on Markov-switching DSGE models this paper is most closely related to Foerster et al. (2016), who develop perturbation methods to solve exogenous regime switching models working directly with the non-linear model. This differs from the Markov-Switching linear rational expectations (MSLRE) literature which starts with a system of linear rational expectations equations and imposes Markov Switching after linearizing the model (e.g. Leeper and Zha, 2003; Davig and Leeper, 2007; Farmer et al., 2011). Since our structural model has a regime switching at its core, this is the natural approach. It has the added benefit that the MSDGE model can be solved to higher orders, where the MSLRE model of course is restricted to first order solutions. Indeed, we find that the second order solution is critical for endogenous switching models to differ from exogenous switching models in the decision rules.

There is an emerging literature that focuses on solving endogenous regime switching models. Davig and Leeper (2008), Davig et al. (2010), and Alpanda and Ueberfeldt (2016) all consider endogenous regime switching, but employ computationally costly global solution methods that eliminate the possibility for likelihood-based estimation. Lind (2014) develops a regime-switching perturbation approach for approximating non-linear models, but the approach requires repeatedly refining the points of approximation and hence is not suitable for estimation purposes. Most closely related to our approach is the method developed by Barthlemy and Marx (2017), but who consider a class of models with regime-dependent steady states that our framework does not satisfy. In contrast, our extension of the Foerster et al. (2016) perturbation approach is well suited for solving a model of crises where regime-dependent steady states may not be relevant given the relatively short-lived nature of crises, and is fast enough to allow for likelihood-based estimation.

The application of the methodology that we propose is most closely related to the literature on emerging market business cycles, including among others Mendoza (1991 and 2010), Neumayer and Perri (2005), Aguiar and Gopinath (2007), Gacia-Chicco, Pancrazi and Uribe (2010). Like these contributions, our framework encompasses multiple sources of shocks. However, in this paper financial shocks are modeled explicitly as changes in the collateral requirements of lenders and allowed to differ during financial crises.

Following the seminal contribution of Mendoza (2010), models with occasionally binding collateral constraints have become the workhorse environment for normative analysis of macro-prudential policies and capital controls. Examples include Bianchi (2011), Benigno et al. (2013), Benigno et al. (2016), Bianchi (2011), Jeanne and Korinek (2010) and Bianchi and Mendoza (2010). Korinek and Mendoza (2013) review this literature and conclude by stating that an important future step is the "development of numerical methods that combine the strengths of global solution methods in describing non-linear dynamics with the power of perturbation methods in dealing with a large number of variables so as to analyze sudden stops in even richer macroeconomic models". Our paper develop such an approach allowing us to empirically evaluate this class of models with the potential to return to these normative questions in future work.

There are many possible applications of our approach to other classes of models. For example, Bocola (2015) builds and estimates a model of sovereign default. His estimation procedure is to first estimate the model outside of the crisis period, using a solution technique that assumes a crisis will not occur. Conditional on those parameter estimates a crisis probability that is exogenous is appended to the model. Our approach allows one to estimate model parameters fully incorporating the possibility of a crisis outside of the crisis period, and allowing for that crisis to be a function of the economic decisions. The methods here also apply to the literature on the zero-lower bound on interest rates. [1]

[1]Guerrieri and Iacoviello (2015) develop a set of procedures called OccBin to solve models with occasion-

The rest of the paper is organized as follows. Section 2 describes the model and introduces the new formulation for the collateral constraint. Section 3 develops the perturbation solution methodology for endogenous regime switching models. Section 4 describes our procedure for estimating the regime switching models using full information Bayesian Methods. Section 5 contains the empirical results and Section 6 concludes.

2 The Model

The model is a small, open, production economy with an occasionally binding collateral constraint subject to productivity, preference, income, interest rate, terms of trade, and financial shocks. The restriction on access to international credit markets that we specify depends on key endogenous variables of the model, including borrowing, capital, and its price. Capital and debt choices respond to exogenous shocks in the model and affect leverage. Leverage in turn affects the probability of a binding constraint. Because of the occasionally binding collateral constraint, this framework can potentially account for both normal business cycles as well as key aspects of financial crises in emerging market economies (Mendoza, 2010).

2.1 The Borrowing Constraint

The collateral constraint limits total debt to a fraction of the market value of physical capital (i.e. it is a limit on leverage). As in Mendoza (2010), Kiyotaki and Moore (1997), Aiyagari and Gertler (1999), and Kocherlakota (2000), among others, the collateral constraint is not derived from an optimal credit contract, but imposed directly on the economy. However, the borrowing constraint may result from limited enforcement problems preventing lenders from collecting more than a fraction of the value of the collateral. When the constraint binds, the model produces endogenous risk premia over the world interest rate at which borrowers would agree to contract while satisfying it. Like the specifications in the literature above, when the constraint binds, debt is limited to a fraction of the market value of the capital stock. Here we follow Mendoza (2010) and include also working capital in the borrowing limit to pin down a well-behaved supply response of the economy during financial crises.

We model the occasionally binding nature of the constraint as an endogenous regime switching process. Thus, there is one regime in which the constraint binds (a crisis regime),

ally binding constraints. OccBin is a certainty equivalent solution method which requires agents to know precisely how long a regime (the one you are not currently in) will apply if there are no shocks, making it functionally quite similar to perfect foresight methods. These methods rule out precautionary effects, which are important for the model in this paper.

and one in which it does not (a normal regime). The probabilities of switching from one regime to the other are assumed to depend on key endogenous variables in the model. The probability to switch from the normal regime to the crisis regime is assumed to be a logistic function of the distance between actual borrowing and the borrowing limit. Therefore it is affected by all endogenous variables that enter the credit constraint. The probability to switch from a crisis regime to the normal one, instead, is assumed to be a function of the borrowing multiplier.

While our constraint is the same as in the quantitative financial friction literature above, we propose a new specification of its occasionally binding nature that is more tractable and has appealing empirical properties. The main difference between our specification and the formulation in the literature is that we transform a deterministic relationship between leverage and a binding borrowing constraint into a stochastic relationship. In the deterministic specification there is one specific leverage ratio that leads to a binding constraint, in our specification increased leverage raises the probability of a binding constraint but does not necessarily force the constraint to bind.

Our model captures a key finding of the empirical literature on financial crises, which documents that the likelihood of a financial crisis increases with leverage, but high leverage does not require a crisis to occur. From an empirical perspective, not having a given leverage ratio that triggers a crisis event (i.e., the collateral constraint binding), but rather leverage affecting the likelihood of the constraint binding in a smooth manner adds an element of realism to the model. Borrowing constraints don't bind at any particular leverage ratio in the real world, they are stochastic functions of leverage ratios.[2]

Most importantly, agents in our regime switching model, know that higher leverage and borrowing levels (and hence lower collateral) increase the probability of switching to a constrained regime (and vice versa). This preserves the interaction in agents' behavior between the two regimes that gives rise to precautionary behaviors and distinguishes this class of models from those in which financial frictions are always binding or are approximated with solution methods that eliminate these interactions across regimes.

[2]The empirical literature finds that when a borrower hits the leverage limit, expenditure is adjusted gradually because other source of financing such as cash, precautionary credit lines, asset sales, etc. can be tapped into. See Capello, Graham, and Harvey (2010) for survey information on behavior of financially constrained firms and Ivashina and Scharfstein (2010) for loan level data showing that credit origination dropped during the crisis because firms drew down from pre-existing credit lines in order to satisfy their liquidity needs.

2.2 Representative Household-Firm

There is a representative household that maximizes the following utility function

$$U \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ d_t \beta^t \frac{1}{1-\rho} \left(C_t - \frac{H_t^{\omega}}{\omega} \right)^{1-\rho} \right\}, \tag{1}$$

with C_t denoting the individual consumption and H_t the individual supply of labor. The elasticity of labor supply is ω, while ρ is the coefficient of relative risk aversion. The variable d_t represents an exogenous and stochastic preference shock. Households choose consumption, labor, capital, intermediate inputs, and holdings of real, one-period international bonds maximizing utility subject to the budget constraint

$$C_t + I_t = A_t K_{t-1}^{\eta} H_t^{\alpha} V_t^{1-\alpha-\eta} - P_t V_t - \phi r_t \left(W_t H_t + P_t V_t \right) + E_t - \frac{1}{(1+r_t)} B_t + B_{t-1}. \tag{2}$$

The first term of the right hand side of equation is the production function. Goods are produced with capital (K_{t-1}), labor (H_t) and imported intermediate goods (V_t). P_t is the relative price of intermediate imports, which follows a stochastic process specified below. The shock to this process interpreted as a terms of trade shock.

B_t is a one-period international bond with net interest rate r_t discussed below. The interest rate faced by borrowers, when the constraint is not binding, is given by

$$r_t = r_t^* + \psi \left(e^{\overline{B} - B_t} - 1 \right) \tag{3}$$

In normal times, therefore, the interest rate has an exogenous stochastic component equal to the world interest rate and a debt elastic component, which pins down a well defined steady state in the non-binding regime.

The term ϕr_t is the working capital constraint and says that a fraction of both wages and intermediate goods must be paid for in advance of production with borrowed funds. The price of labor and capital are given by w_t and q_t, both of which are endogenous variables, but taken as given by the household. We allow also for an exogenous spending shock represented by the variable E_t. Gross investment I_t is subject to adjustment costs as a function of net investment:

$$I_t = \delta K_{t-1} + (K_t - K_{t-1}) \left(1 + \frac{\iota}{2} \left(\frac{K_t - K_{t-1}}{K_{t-1}} \right) \right). \tag{4}$$

Households face a regime specific collateral constraint, where the regimes are denoted by $s_t \in \{0, 1\}$. When $s_t = 1$, the constraint binds strictly, and total borrowing is equal to a fraction of the value of collateral

$$\frac{1}{(1+r_t)}B_t - \phi(1+r_t)(W_tH_t + P_tV_t) = -\kappa_t q_t K_t \tag{5}$$

On the left hand side of this equation we have total debt and working capital loans. The presence of the binding constraint limits both international borrowing (hence consumption smoothing) as well as borrowing to pay for intermediate inputs. The latter limit constraints output, which may cause the constraint to bind even tighter. In this regime, as the quantity and value of capital fluctuates, the amount of borrowing will also fluctuate. When $s_t = 0$, the constraint is slack and the value of the collateral is enough for international lenders to finance all the desired borrowing levels. Thus,

$$\frac{1}{(1+r_t)}B_t - \phi(1+r_t)(W_tH_t + P_tV_t) \tag{6}$$

is unconstrained by $\kappa_t q_t K_t$ in the current period. The tightness of the borrowing constraint, κ_t, is time-varying and subject to shocks according to a process specified below.

In order to specify how the economy changes regimes it is useful to first define the notion of "borrowing cushion" as the distance of the borrowing from the debt limit in (6):

$$B_t^* = \frac{1}{(1+r_t)}B_t - \phi(1+r_t)(W_tH_t + P_tV_t) + \kappa_t q_t K_t. \tag{7}$$

When the borrowing cushion is small then the constraint is close to binding. In this case, the leverage ratio is high because borrowing relative to the value of the collateral is high. In regime $s_t = 0$, when the constraint is not binding, the probability that it binds the next period depends on the value of borrowing cushion in (7) in a logistic way. That is, the transition probability from regime 0 to regime 1 is a function of all endogenous variables in B_t^*:

$$\Pr(s_{t+1} = 1 | s_t = 0) = \frac{\exp(-\gamma_0 B_t^*)}{1 + \exp(-\gamma_0 B_t^*)}. \tag{8}$$

The parameter γ_0 controls how the likelihood of hitting the debt limit is linked to the borrowing cushion. For small values of this parameter, the cushion has little impact on the probability of a transition to the binding regime. For large values of this parameter, the probability of a crisis moves rapidly from 0 to 1 as B_t^* approaches 0.

In regime 1, when the constraint is binding, the Lagrange multiplier associated with the constraint is non-zero. Denoting the multiplier as λ_t, the transition probability from the binding regime to the non-binding regime is given by:

$$\Pr\left(s_{t+1}=0|s_t=1\right)=\frac{\exp\left(-\gamma_1\lambda_t\right)}{1+\exp\left(-\gamma_1\lambda_t\right)}. \tag{9}$$

This expression implies that as the multiplier approaches 0, the probability of transitioning back to the non-binding state rises.

This logistic function can be given a structural interpretation by adding a stochastic monitoring (or enforcement) shock ϵ_t^M to the standard borrowing constraint used in the literature:

$$\frac{1}{(1+r_t)}B_t-\phi\left(1+r_t\right)\left(W_tH_t+P_tV_t\right)=-\kappa q_tK_t+\epsilon_t^M$$

This shock has two interpretations, based on its sign. When the shocks is negative, the LHS is greater than the value of collateral, but the lender monitors and decides to impose the borrowing constraint. When the shock is positive, the LHS is then less than the value of collateral, but the constraint does not bind because the lender does not monitor. We assume that the distribution of ϵ_t^M is such that when borrowing is much less than the value of collateral the probability of drawing a monitoring shock that leads to a binding constraint is 0. When borrowing exceeds the value of collateral by a large amount the probability of drawing a monitoring shock is such that the probability the lender audits goes to 1. The logistic function satisfies these assumptions, though other functions do it as well.[3]

Exogenous processes and shocks

The model is closed by specifying the process for the following 6 exogenous variables and their shocks:

$$\log A_t=(1-\rho_A(s_t))A^*(s_t)+\rho_A(s_t)\log A_{t-1}+\sigma_A(s_t)\varepsilon_{A,t} \tag{10}$$

$$\log E_t=(1-\rho_E(s_t))E^*(s_t)+\rho_E(s_t)\log E_{t-1}+\sigma_E(s_t)\varepsilon_{E,t} \tag{11}$$

$$\log P_t=(1-\rho_P(s_t))P^*(s_t)+\rho_P(s_t)\log P_{t-1}+\sigma_P(s_t)\varepsilon_{P,t} \tag{12}$$

$$\kappa_t=(1-\rho_A(s_t))\kappa^*(s_t)+\rho_\kappa(s_t)\kappa_{t-1}+\sigma_\kappa(s_t)\varepsilon_{\kappa,t} \tag{13}$$

$$\log d_t=\rho_d(s_t)\log d_{t-1}+\sigma_d(s_t)\varepsilon_{d,t} \tag{14}$$

$$r_t^*=(1-\rho_{r^*})\bar{r^*}+\rho_{r^*}r_{t-1}^*+\sigma_{r^*}\varepsilon_{r^*,t} \tag{15}$$

[3]The logistic function is also used by Kumhof et al. (2015) to model theoretically the transition to a default regime in their model.

For the TFP, relative price, expenditure and leverage shocks we allow for intercepts, autocorrelation and variances to all switch with the regime. The preference shock is mean zero so only the serial correlation and variance switches with the regime. The interest rate shock is interpreted as a world interest rate and hence it does not switch with the regime as the regime switching is modeled only for Mexican variables.

Model Timing

In the model, agents enter period t with knowledge of the lagged state variables (capital, debt, and past realization of exogenous shocks) and a probability distribution over the regime, $Pr[s_t|s_{t-1}, B^*_{t-1}, \lambda_{t-1}]$. They then learn the regime, s_t, which determines whether the constraint binds or not in period t. Next, the shocks to all exogenous processes in the model realize. Note here that these shocks are orthogonal to the realization of the regime. Agents then undertake decisions that pin down B^*_t, λ_t, which in turn imply a probability distribution over whether the constraint binds in period $t+1$. Figure 1 provides a graphical representation of the timing.

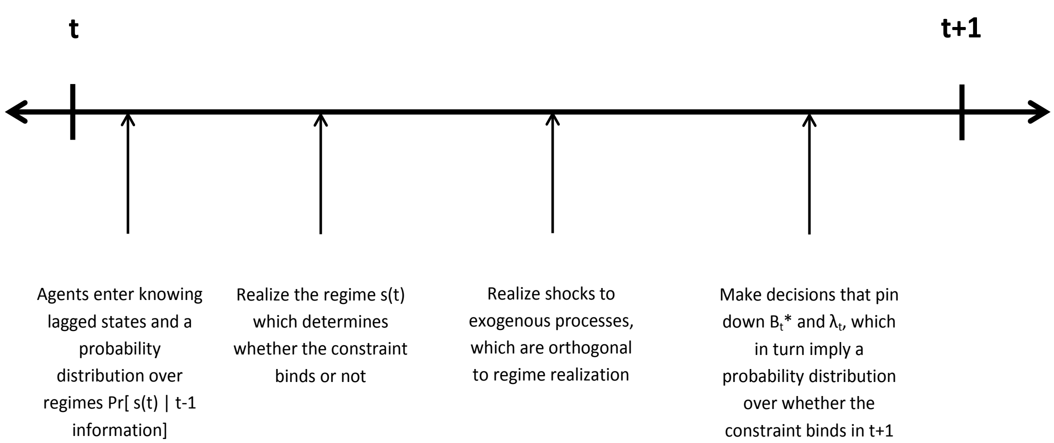

Figure 1: Intra-period timing

An implication of these timing assumptions is that agents may face a non-binding constraint, realize bad shocks, and borrow to smooth them, which would imply an increased probability of crisis tomorrow. In practice, since agents know how borrowing decisions affect the probability of a constraint binding in the future, they may increase borrowing to smooth some of the shock out, but will not increase borrowing by excessive amounts since precautionary effects are present in the model and agents make choices to avoid crisis states. [4]

2.3 First Order Conditions

Households maximize (1) subject to (2) and (5) and (6) by choosing C_t, B_t, K_t, V_t and H_t. The first-order conditions of this problem are the following:

$$v_t \left(C_t - \frac{H_t^{\omega}}{\omega} \right)^{-\rho} = \mu_t \tag{16}$$

$$(1-\alpha-\eta) A_t K_{t-1}^{\eta} H_t^{\alpha} V_t^{-\alpha-\eta} = P_t \left(1 + \phi r_t + \frac{\lambda_t}{\mu_t} \phi (1+r_t) \right) \tag{17}$$

$$\alpha A_t K_{t-1}^{\eta} H_t^{\alpha-1} V_t^{1-\alpha-\eta} = \phi W_t \left(r_t + \frac{\lambda_t}{\mu_t} (1+r_t) \right) + H_t^{\omega-1} \tag{18}$$

$$\mu_t = \lambda_t + \beta (1+r_t) \mathbb{E}_t \mu_{t+1} \tag{19}$$

$$\mathbb{E}_t \mu_{t+1} \beta \left(\begin{array}{c} 1 - \delta + \left(\frac{\iota}{2} \left(\frac{K_{t+1}}{K_t} \right)^2 - \frac{\iota}{2} \right) \\ + \eta A_{t+1} K_t^{\eta-1} H_{t+1}^{\alpha} V_{t+1}^{1-\eta-\alpha} \end{array} \right) = \mu_t \left(1 - \iota + \iota \left(\frac{K_t}{K_{t-1}} \right) \right) - \lambda_t \kappa_t q_t \tag{20}$$

The market prices for capital and labor are

$$q_t = 1 + \iota \left(\frac{K_t - K_{t-1}}{K_{t-1}} \right) \tag{21}$$

$$W_t = H_t^{\omega-1} \tag{22}$$

The budget constraint and the complementary slackness condition are

$$C_t + I_t = A_t K_{t-1}^{\eta} H_t^{\alpha} V_t^{1-\alpha-\eta} + S_t - P_t V_t - \phi r_t (W_t H_t + P_t V_t) - \frac{1}{(1+r_t)} B_t + B_{t-1} \tag{23}$$

$$B_t^* \lambda_t = 0. \tag{24}$$

The latter condition is key in our model. It combines information on the borrowing constraint in both regimes (5) and (6), as well on the switching between regimes 0 and 1 in (8) and (9). In the normal regime, the multiplier is zero and the borrowing cushion

[4] A difference between our timing and that of Mendoza (2010) is the separation of regime and shock realizations. In Mendoza's model, shocks to TFP in period t determine whether the constraint binds or not in that period, which makes shocks and realization of the regime correlated.

is unconstrained. In the crisis regime, the borrowing cushion is zero and the multiplier is constrained. The switch between regimes is then governed by an analogous of the traditional complementary slackness condition, which here is controlled by (8) and (9) and hence remains differentiable on its support.

Note finally from (19) that, when the collateral constraint binds, this small open economy faces an endogenous risk premium on debt measured by

$$\frac{\mu_t}{\beta \mathbb{E}_t \mu_{t+1}} = \frac{\lambda_t}{\beta \mathbb{E}_t \mu_{t+1}} + (1 + r_t).$$

Therefore, when the constraint binds, the interest rate is given by:

$$r_t = r_t^* + \psi \left(e^{\overline{B} - B_t} - 1 \right) + \frac{\lambda_t}{\beta \mathbb{E}_t \mu_{t+1}} \tag{25}$$

2.4 Competitive Equilibrium

A competitive equilibrium in our framework is a sequence of quantities $\{K_t, B_t, C_t, H_t, V_t, I_t, A_t, \kappa_t, Y_t, \lambda_t, \mu_t, B_t^*\}$ and prices $\{P_t, r_t, q_t, w_t\}$ that satisfy the household's first order conditions (13)-(17), the market prices for capital and labor (18)-(19), the market clearing condition (20), the definition of the borrowing cushion (6), the slackness condition (21), and the exogenous processes (9)-(12).

3 Solving the Endogenous Switching Model

The key insight for mapping the model presented above into an endogenous regime-switching framework is to modify the slackness condition (24) so that the relevant variables are zero only in the relevant state. In particular, in the normal regime ($s_t = 0$), the borrowing constraint does not bind and $\lambda_t = 0$. In the crisis regime ($s_t = 1$), on the other hand, the borrowing constraint binds and $B_t^* = 0$.

To capture this feature in a regime switching framework, we introduce two state-dependent variables $\varphi(s_t)$ and $\nu(s_t)$, and re-write (24) as

$$\varphi(s_t) B_{ss}^* + \nu(s_t)(B_t^* - B_{ss}^*) + (1 - \varphi(s_t)) \lambda_{ss} + (1 - \nu(s_t))(\lambda_t - \lambda_{ss}) = 0. \tag{26}$$

In this modified slackness condition, $\varphi(0) = \nu(0) = 0$ when $s_t = 0$, and so the equation simplifies to $\lambda_t = 0$. While $\varphi(1) = \nu(1) = 1$ when $s_t = 1$, so that the equation simplifies to $B_t^* = 0$. This representation helps to preserve information in our perturbation approximation, since at first order, the above implies $d\lambda_t = 0$ for $s_t = 0$, and $dB_t^* = 0$ for $s_t = 1$, meaning that both variables are constant in the respective regimes.

3.1 Deterministic Steady State

Given the modified slackness condition (26), our perturbation method builds second-order Taylor expansions of the decision rules of the model equilibrium around a non-stochastic steady state. Defining a non-stochastic steady state in an endogenous regime-switching framework, however, is not trivial.

Definition: A steady state in our framework can be defined as a state in which ensues when all shocks are zero ($\varepsilon_{A,t} = \varepsilon_{P,t} = \varepsilon_{\kappa,t} = \varepsilon_{w,t} = \varepsilon_{r,t} = 0$) for all t, and the regime switching variables $\varphi(s_t)$, $a^*(s_t)$, $p^*(s_t)$, and $\kappa^*(s_t)$ are at their ergodic means across regimes associated with the steady state transition matrix:

$$P_{ss} = \begin{bmatrix} p_{00,ss} & p_{01,ss} \\ p_{10,ss} & p_{11,ss} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\exp\left(-\gamma_{0,1} B_{ss}^*\right)}{1+\exp\left(-\gamma_{0,1} B_{ss}^*\right)} & \frac{\exp\left(-\gamma_{0,1} B_{ss}^*\right)}{1+\exp\left(-\gamma_{0,1} B_{ss}^*\right)} \\ \frac{\exp\left(-\gamma_{1,1} \lambda_{ss}\right)}{1+\exp\left(-\gamma_{1,1} \lambda_{ss}\right)} & 1 - \frac{\exp\left(-\gamma_{1,1} \lambda_{ss}\right)}{1+\exp\left(-\gamma_{1,1} \lambda_{ss}\right)} \end{bmatrix}.$$

Note here that, since this matrix also depends on the steady state level of debt and multiplier, which in turn depend upon the ergodic means of the regime-switching variables, such state is the solution of a fixed point problem, which we describe in the Appendix.

The model has regime specific parameters that can affect the steady state of the economy in that regime. Namely, the switching parameters $\varphi(s_t)$, $\beta(s_t)$, $a(s_t)$, and $p(s_t)$ affect the level of the economy and matter for steady state calculations. Let $\xi = [\xi_0, \xi_1]$ denote the ergodic vector of P_{ss}. Then define the ergodic means of the switching parameters as

$$\bar{\varphi} = \xi_0 \varphi(0) + \xi_1 \varphi(1)$$

$$\bar{\beta} = \xi_0 \beta(0) + \xi_1 \beta(1)$$

$$\bar{a} = \xi_0 a(0) + \xi_1 a(1)$$

$$\bar{p} = \xi_0 p(0) + \xi_1 p(1).$$

The steady state of the economy depends on these ergodic means and satisfies the following equations in appendix.

In order to avoid circularity in finding the steady state, which in turn depends on the steady state of the transition probabilities, we first calibrate the steady state probabilities and then back out the associated parameters of the transition function. That is we assume

$$\begin{aligned}\gamma_{0,0} &= \log\left(\frac{1}{p_{00,ss}}-1\right)+\gamma_{0,1}B_{ss}^{*}\\ \gamma_{1,0} &= \log\left(\frac{1}{p_{11,ss}}-1\right)-\gamma_{1,1}\lambda_{ss}\end{aligned}$$

and we calibrate $p_{00,ss}$ and $p_{11,ss}$. We then estimate the γ.

The following table shows the steady state values for the variables in steady state. Note that these are the deterministic steady states associated with each model.

3.2 Second order approximation

Armed with the steady state of the endogenous regime-switching economy, we then construct a second-order approximations to the decision rules by taking derivatives of the equilibrium conditions. We relegate details of these derivations to the Appendix, but here we summarize.

For perturbation, we take the stacked equilibrium conditions $\mathbb{F}\left(\mathbf{x}_{t-1},\boldsymbol{\varepsilon}_t,\chi\right)$, and differentiate with respect to $\left(\mathbf{x}_{t-1},\boldsymbol{\varepsilon}_t,\chi\right)$. In general, regime-switching models, the first-order derivative with respect to $\mathbf{x}_{t-1}$ produces a complicated polynomial system denoted

$$\mathbb{F}_{\mathbf{x}}\left(\mathbf{x}_{ss},\mathbf{0},0\right)=0.$$

Often this system needs to be solved via Gröbner bases, which finds all possible solutions in order to check them for stability. In our case, all the regime switching parameters show up in the steady state, and we write $\boldsymbol{\theta}_t=\bar{\boldsymbol{\theta}}+\chi\hat{\boldsymbol{\theta}}\left(s_t\right)$ so the steady state can be solved. This is the *Partition Principle* of Foerster et al. (2016). Given these parameters, the regime switching in $\mathbb{F}_{\mathbf{x}}\left(\mathbf{x}_{ss},\mathbf{0},0\right)$ disappears and simplifies to the standard no-switching case that can be solved via a generalized eigenvalue procedure.

After solving the eigenvalue problem, the other systems to solve are

$$\begin{aligned}\mathbb{F}_{\boldsymbol{\varepsilon}}\left(\mathbf{x}_{ss},\mathbf{0},0\right) &= 0\\ \mathbb{F}_{\chi}\left(\mathbf{x}_{ss},\mathbf{0},0\right) &= 0\end{aligned}$$

and second order systems of the form (can apply equality of cross-partials)

$$\mathbb{F}_{\mathbf{i},\mathbf{j}}\left(\mathbf{x}_{ss},\mathbf{0},0\right)=0,\ \mathbf{i},\mathbf{j}\in\left\{\mathbf{x},\boldsymbol{\varepsilon},\chi\right\}.$$

Recalling that the decision rules take the the form

$$\mathbf{x}_t = h_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right)$$

$$\mathbf{y}_t = g_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right)$$

the second-order approximation takes the form

$$\mathbf{x}_t \approx \mathbf{x}_t + H_{s_t}^{(1)} S_t + \frac{1}{2} H_{s_t}^{(2)} \left(S_t \otimes S_t\right)$$

$$\mathbf{y}_t \approx \mathbf{y}_t + G_{s_t}^{(1)} S_t + \frac{1}{2} G_{s_t}^{(2)} \left(S_t \otimes S_t\right)$$

where $S_t = \left[\begin{array}{ccc} \left(\mathbf{x}_{t-1} - \mathbf{x}_{ss}\right)' & \boldsymbol{\varepsilon}_t' & 1 \end{array}\right]'$.

4 Estimating the Endogenous Switching Model

Our estimation procedure is full information Bayesian procedure. As usual, the posterior has no analytical solution so we use Markov-Chain Monte Carlo methods to sample from the posterior. A key obstacle in using these methods to sample from the posterior is the calculation of the value of the likelihood function which is needed at each step. We face two difficulties here. The first is the regime-switching model, and the second is the second-order solution that governs the decision rules in each regime. Our approach is to use the Unscented Kalman Filter (UKF) to compute approximations to the evaluation of the likelihood function. Since the Metroplis-Hastings algorithm we use for sampling is now standard in the literature, we omit a discussion of this procedure. The details of the construction of state space representation and the filtering steps for likelihood evaluation are reported in the appendix. Here we focus on the calibration of the parameters that are not estimated and the prior distribution of the estimated ones.

4.1 Calibrated Parameters

The calibration of the parameters that are not estimated follows Mendoza (2010). Consider first the steady state of the model in the non-binding regime. We normalize $a(0) = p(0) = 1$. Mendoza targets an annualized real rate of interest of 8.57%. In the non-binding regime, the steady state interest rate is $r_{ss} = r^* = \frac{1}{\beta} - 1$, and the debt level is $B_{ss} = \bar{B}$. Setting $\beta = 0.97959$ yields $r^* = 0.0208352$, which matches the target annualized rate. Mendoza also targets a debt-to-output ratio of -0.86 (based on quarterly output), which requires $\bar{B} = -1.7517$.

$$\bar{B} = \left(\frac{B}{Y}\right)_{ss} \Omega_w^{\frac{1}{1-\omega}} \left(\Omega_k^{\eta}\Omega_v^{1-\alpha-\eta}\right)^{\frac{\omega}{\alpha(1-\omega)}}$$

Now consider the steady state of the model in the case when only the binding regime occurs. In line with Mendoza's estimates on the Mexican sudden stop, we set $a(1) = -0.005$ and $p(1) = 0.005$, which, combined with ρ_a and ρ_p, lead to a roughly 5% decrease in TFP and a 5% increase in import prices. We set the interest rate elasticity $\psi_r = 0.001$, which implies that the real rate is increasing in debt. Table 1 summarizes this parameterization:

4.2 Priors for Estimated Parameters

The priors for the model parameters are specified to be fairly loose. A Beta distribution is used for the serial correlation coefficients to reflect a view that shocks are persistent.[5] The means of the TFP and relative price shock are normalized to 0, so that only the crisis regime constant term is estimated. We use a prior with negative mean for TFP, and positive mean for relative price in the crisis regime, though the prior standard deviation allows for a wide range of values. The mean for the κ process is centered at 0.2, which corresponds to the baseline case of Mendoza (2010). His tighter case (0.15) is a half standard deviation away from the mean, and his looser case of 0.3 is one standard deviation from the mean. The working capital and investment adjust cost parameters are also centered on the Mendoza (2010) calibration. We use a uniform prior for the logistic function parameter over a wide range of plausible variables that range from a very sharp function to a flat function.

5 Preliminary Estimation Results

Results reported here are for fewer shocks than described in the model section. Results for the 6 observable and 6 shock model will be added next.

Table 2 reports the posteriors of the estimated parameters. Overall the posteriors are tightly concentrated. The parameters of the logistic in particular are highly informative. The mean vales of these parameters implies that probability of a crisis increases with leverage, but there is a range of leverage ratios with elevated probability of crisis. Figure 2 plots

[5]Estimates with a uniform prior over (-1,1) yielded no posterior mass below zero.

the posterior of the logistic function to illustrate this range. The structural parameters κ, ι and ϕ are consistent with Mendoza (2010). We also find that the mean of the TFP process becomes negative in the crisis state.

The model provides an estimate of regime that the data is in. Figures 3 plots the smoothed estimate of the crisis probability. Figure 3 also includes shaded areas that represent the Reinhart-Rogoff currency crisis dates for Mexico. The model does well at picking up both the debt crisis of the 1980s, as well as the shorter-lived Tequila crisis in 1994-1995. The model can capture crisis events of different persistence because the transition probabilities depend on the state variables of the economy. The 1998 and 2008 Reinhart-Rogoff currency crisis episodes are not picked up, possibly because they did not originate in Mexico, but rather in the United States and other emerging markets.

The variance decompositions (Table 4) show that outside of the crisis state the fluctuations in real variables are largely driven by TFP and relative price shocks. During the crisis episodes leverage shocks are the most important. These shocks seem to have much less relevance during normal business cycle times.

6 Conclusions

TBC.

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Table 1: Calibrated Parameters

Parameter	Value
Discount Factor	$\beta = 0.97959$
Risk Aversion	$\rho = 2$
Labor Share	$\alpha = 0.592$
Capital Share	$\eta = 0.306$
Wage Elasticity of Labor Supply	$\omega = 1.846$
Capital Depreciation (8.8% Annually)	$\delta = 0.022766$
Interest Rate Intercept	$r^* = 0.0208352$
Interest Rate Elasticity	$\psi_r = 0.05$
Neutral Debt Level	$\bar{B} = -1.7517$
Mean of TFP Process, Normal Regime	$a(0) = 0$
Mean of Import Price Process, Normal Regime	$p(0) = 0$
Mean of Leverage Process, Normal Regime	$\kappa(0) = 0.15$
Persistence of TFP Process, Crisis Regime	$\rho_A(1) = 0$
Persistence of Import Price Process, Crisis Regime	$\rho_P(1) = 0$

Table 2: Prior Distribution

Parameter	Prior	Parameter	Prior (Mean, SD)
$\rho_x(0)$	Beta(0.7,0.2)	$\sigma_x(0)$	Inv.Gamma(0.01,0.05)
$\rho_x(1)$	Beta(0.7,0.2)	$\sigma_x(1)$	Inv.Gamma(0.01,0.05)
ρ_r	Beta(0.7,0.2)	σ_r	Inv.Gamma(0.001,0.02)
$A^*(1)$	Normal(-0.001,0.01)	σ_m	Inv.Gamma(0.005,0.001)
$E^*(0)$	Normal(0,0.01)	$E^*(1)$	Normal(0,0.01))
$P^*(1)$	Normal(0.001,0.01)	$\bar{r}^*$	Normal(0.003,0.01)
$\kappa^*(0)$	Normal(0.2,0.1)	$\kappa^*(1)$	Normal(0.2,0.1)
γ_0	Uniform(0,1000)	γ_1	Uniform(0,1000)
ι	Normal(2.75,0.1)	ϕ	Normal(0.25,0.1)

where x=(A, E, P, κ, d) and σ_m is the standard deviation of all the measurement errors.

Table 3: Estimation Results

Parameter	Posterior mean	q5	q95
$\sigma_w(0)$	0.0007	0.0001	0.0015
$\sigma_w(1)$	0.0438	0.0332	0.0496
$\sigma_a(0)$	0.0056	0.0043	0.0068
$\sigma_a(1)$	0.0091	0.0062	0.0123
$\sigma_p(0)$	0.0401	0.0338	0.0478
$\sigma_p(1)$	0.0487	0.0218	0.0766
$\sigma_\kappa(0)$	0.0012	0.0001	0.0030
$\sigma_\kappa(1)$	0.0248	0.0072	0.0419
σ_{m1}	0.0061	0.0048	0.0075
σ_{m2}	0.0127	0.0114	0.0142
σ_{m3}	0.0366	0.0271	0.0493
σ_{m4}	0.0035	0.0029	0.0041
$\gamma_{0,1}$	89.0076	73.2143	108.1845
$\gamma_{1,1}$	1.9676	0.0892	5.8921
$\rho_a(0)$	0.8134	0.7208	0.8843
$\rho_p(0)$	0.9637	0.9340	0.9876
$\rho_\kappa(0)$	0.6656	0.4152	0.8946
$\rho_a(1)$	0.7746	0.5543	0.8968
$\rho_p(1)$	0.9260	0.8258	0.9941
$\rho_\kappa(1)$	0.7804	0.6728	0.8872
a(1)	-0.0059	-0.0072	-0.0047
p(1)	0.0005	0.0000	0.0013
$\kappa(1)$	0.2305	0.2203	0.2440
ι	2.8233	2.8144	2.8360
ϕ	0.3036	0.2697	0.3217

Table 4: Variance Decomposition

			C	I	r	Y
World Interest Rate Shock	$\varepsilon_{w,t}$	Non-Binding	0.0001	0.0128	0.0066	0.0000
Technology Shock	$\varepsilon_{a,t}$	Non-Binding	0.3087	0.2670	0.6390	0.3158
Import Price Shock	$\varepsilon_{p,t}$	Non-Binding	0.6817	0.3777	0.1971	0.6814
Leverage Shock	$\varepsilon_{\kappa,t}$	Non-Binding	0.0095	0.3424	0.1572	0.0027
World Interest Rate Shock	$\varepsilon_{w,t}$	Binding	0.0074	0.0044	0.3701	0.0145
Technology Shock	$\varepsilon_{a,t}$	Binding	0.0106	0.0003	0.0004	0.0705
Import Price Shock	$\varepsilon_{p,t}$	Binding	0.0124	0.0002	0.0003	0.0630
Leverage Shock	$\varepsilon_{\kappa,t}$	Binding	0.9696	0.9951	0.6291	0.8520

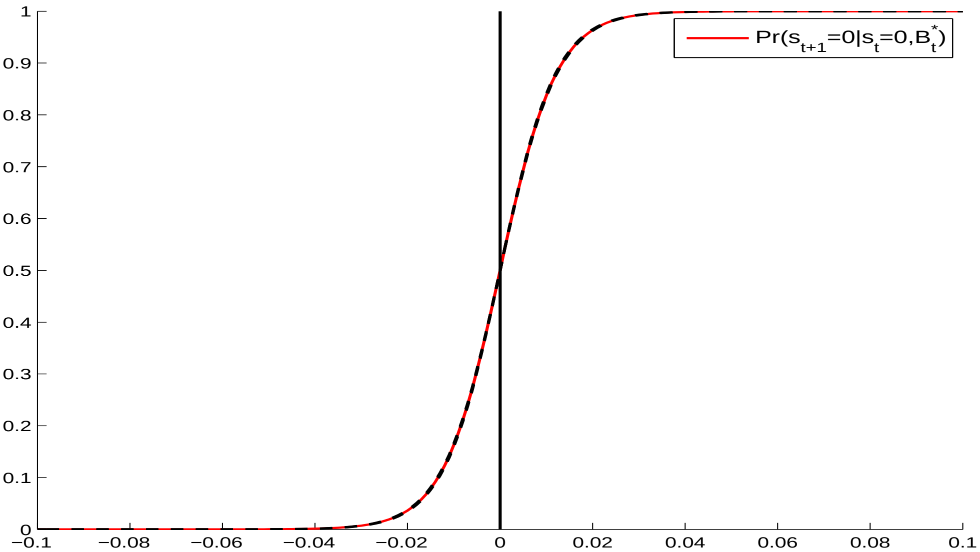

Figure 2: Transition probability of non-binding regime conditional on nonbinding

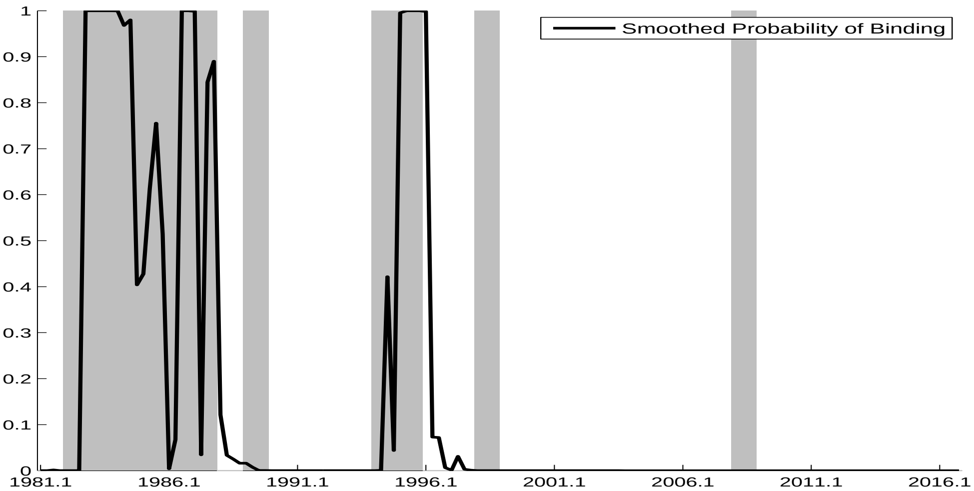

Figure 3: Smoothed Probability of Binding Regime and Reinhart-Rogoff Currency Crisis (Shaded areas)

A Details of the Solution Method

This Appendix gives detail about two aspects of the solution method. First, the definition and solution for the steady state of the endogenous regime-switching economy. Second, the perturbation method that generates Taylor expansions to the solution of the economy around the steady state.

A.1 Steady State

The model has two features that make defining the steady state non-standard. First, as is common in regime-switching models, the switching parameters $\varphi(s_t)$, $a^*(s_t)$, $p^*(s_t)$, and $\kappa^*(s_t)$ all affect the level of the economy directly, and will thus matter for steady state calculations. Solution methods such as Foerster et al. (2016) define the steady state by using the ergodic means of these parameters across regimes. However, in our case the transition matrix P is endogenous, making the ergodic distribution problematic, since it depend on economic variables that in turn depend on the ergodic means. Our solution method for the steady state proceeds in two steps.

A.1.1 Step 1: Find Variables given P_{ss}

To find the steady state, we first assume that P_{ss} is known. Let $\xi = [\xi_0, \xi_1]$ denote the ergodic vector of P_{ss}. Then define the ergodic means of the switching parameters as

$$\bar{\varphi} = \xi_0 \varphi(0) + \xi_1 \varphi(1) \tag{A.1}$$

$$\bar{a}^* = \xi_0 a^*(0) + \xi_1 a^*(1) \tag{A.2}$$

$$\bar{p}^* = \xi_0 p^*(0) + \xi_1 p^*(1) \tag{A.3}$$

$$\bar{\zeta}_0 = \xi_0 \zeta_0(0) + \xi_1 \zeta_0(1) \tag{A.4}$$

$$\bar{\zeta}_1 = \xi_0 \zeta_1(0) + \xi_1 \zeta_1(1) \tag{A.5}$$

The steady state of the economy depends on these ergodic means, and satisfies the following equations

$$\left(C_{ss} - \frac{H_{ss}^{\omega}}{\omega}\right)^{-\rho} = \mu_{ss} \tag{A.6}$$

$$(1-\alpha-\eta) A_{ss} K_{ss}^{\eta} H_{ss}^{\alpha} V_{ss}^{-\alpha-\eta} = P_{ss}\left(1 + \phi r_{ss} + \frac{\lambda_{ss}}{\mu_{ss}} \phi (1 + r_{ss})\right) \tag{A.7}$$

$$\alpha A_{ss} K_{ss}^{\eta} H_{ss}^{\alpha-1} V_{ss}^{1-\alpha-\eta} = \phi W_{ss}\left(r_{ss} + \frac{\lambda_{ss}}{\mu_{ss}}(1+r_{ss})\right) + H_{ss}^{\omega-1} \tag{A.8}$$

$$\mu_{ss} = \lambda_{ss} + \beta \frac{(1+r_{ss})}{(1+\tau_{ss}^{B})}\mu_{ss} \tag{A.9}$$

$$\mu_{ss}\beta\left(\begin{array}{c} 1-\delta+\left(\frac{\iota}{2}\left(\frac{k_{ss}}{K_{ss}}\right)^2-\frac{\iota}{2}\right) \\ +\eta A_{ss} K_{ss}^{\eta-1} H_{ss}^{\alpha} V_{ss}^{1-\eta-\alpha} \end{array}\right) = \mu_{ss}\left(1-\iota+\iota\left(\frac{K_{ss}}{K_{ss}}\right)\right) - \lambda_{ss}\kappa q_{ss} \tag{A.10}$$

$$q_{ss} = 1+\iota\left(\frac{K_{ss}-K_{ss}}{K_{ss}}\right) \tag{A.11}$$

$$W_{ss} = H_{ss}^{\omega-1} \tag{A.12}$$

$$C_{ss} + I_{ss} = A_{ss} K_{ss}^{\eta} H_{ss}^{\alpha} V_{ss}^{1-\alpha-\eta} - P_{ss}V_{ss} - \phi r_{ss}(W_{ss}H_{ss} + P_{ss}V_{ss}) - \frac{(1+\tau_{ss}^{B})}{(1+r_{ss})}B_{ss} + B_{ss} - T_{ss} \tag{A.13}$$

$$I_{ss} = \delta K_{ss} + (K_{ss}-K_{ss})\left(1+\frac{\iota}{2}\left(\frac{K_{ss}-K_{ss}}{K_{ss}}\right)\right) \tag{A.14}$$

$$B_{ss}^{*} = \frac{(1+\tau_{ss}^{B})}{(1+r_{ss})}B_{ss} - \phi(1+r_{ss})(W_{ss}H_{ss}+P_{ss}V_{ss}) + \kappa q_{ss} K_{ss} \tag{A.15}$$

$$\bar{\varphi} B_{ss}^{*} + (1-\bar{\varphi})\lambda_{ss} = 0 \tag{A.16}$$

$$T_{ss} = \tau_{ss}^{B} B_{ss} \tag{A.17}$$

$$\tau_{ss}^{B} = \bar{\zeta}_0 + \bar{\zeta}_1\left(\frac{B_{ss}}{Y_{ss}}\right) \tag{A.18}$$

$$r_{ss} = r^{*} + \psi_r\left(e^{\overline{B}-B_{ss}} - 1\right) \tag{A.19}$$

$$\log A_{ss} = (1-\rho_A(s_t))\bar{a}^{*} + \rho_A(s_t)\log A_{ss} \tag{A.20}$$

$$\log P_{ss} = (1-\rho_P(s_t))\bar{p}^{*} + \rho_P(s_t)\log P_{ss} \tag{A.21}$$

$$k_{ss} = K_{ss} \tag{A.22}$$

$$Y_{ss} = A_{ss} K_{ss}^{\eta} H_{ss}^{\alpha} V_{ss}^{1-\alpha-\eta} \tag{A.23}$$

$$\Phi_{ss}^{by} = \frac{B_{ss}}{Y_{ss}} \tag{A.24}$$

We can partially solve for some of these directly

$$A_{ss} = \exp \bar{a}^{*} \tag{A.25}$$

$$P_{ss} = \exp \bar{p}^{*} \tag{A.26}$$

$$q_{ss} = 1 \tag{A.27}$$

Suppose know r_{ss} and τ^B_{ss}

$$\Omega_v = \frac{P_{ss}\left(1 + \phi r_{ss} + \phi\left(1 + r_{ss}\right)\left(1 - \beta\frac{(1+r_{ss})}{(1+\tau^B_{ss})}\right)\right)}{(1 - \alpha - \eta)} \tag{A.28}$$

$$\Omega_h = \frac{1 + \phi\left(r_{ss} + \left(1 + r_{ss}\right)\left(1 - \beta\frac{(1+r_{ss})}{(1+\tau^B_{ss})}\right)\right)}{\alpha} \tag{A.29}$$

$$\Omega_k = \frac{1}{\eta}\left(\frac{1 - \left(1 - \beta\frac{(1+r_{ss})}{(1+\tau^B_{ss})}\right)\kappa}{\beta} - 1 + \delta\right) \tag{A.30}$$

$$H_{ss} = \left(\frac{A_{ss}}{\Omega_k^{\eta}\Omega_h^{\alpha}\Omega_v^{1-\alpha-\eta}}\right)^{\frac{1}{\alpha(\omega-1)}} \tag{A.31}$$

$$Y_{ss} = \Omega_h H^{\omega}_{ss} \tag{A.32}$$

$$V_{ss} = \frac{\Omega_h}{\Omega_v} H^{\omega}_{ss} \tag{A.33}$$

$$K_{ss} = \frac{\Omega_h}{\Omega_k} H^{\omega}_{ss} \tag{A.34}$$

$$W_{ss} = H^{\omega-1}_{ss} \tag{A.35}$$

$$I_{ss} = \delta K_{ss} \tag{A.36}$$

$$k_{ss} = K_{ss} \tag{A.37}$$

$$B_{ss} = \bar{B} - \log\left(1 + \frac{r_{ss} - r^*}{\psi_r}\right) \tag{A.38}$$

$$C_{ss} = Y_{ss} - I_{ss} - P_{ss}V_{ss} - \phi r_{ss}\left(W_{ss}H_{ss} + P_{ss}V_{ss}\right) - \frac{\left(1 + \tau^B_{ss}\right)}{\left(1 + r_{ss}\right)}B_{ss} + B_{ss} - T_{ss} \tag{A.39}$$

$$C_{ss} = Y_{ss} - \left(1 + \phi r_{ss}\right)P_{ss}V_{ss} - \delta K_{ss} - \phi r_{ss}W_{ss}H_{ss} + \left(1 - \frac{\left(1 + \tau^B_{ss}\right)}{\left(1 + r_{ss}\right)} - \tau^B_{ss}\right)B_{ss} \tag{A.40}$$

$$\mu_{ss} = \left(C_{ss} - \frac{H^{\omega}_{ss}}{\omega}\right)^{-\rho} \tag{A.41}$$

$$\lambda_{ss} = \mu_{ss}\left(1 - \beta\frac{\left(1 + r_{ss}\right)}{\left(1 + \tau^B_{ss}\right)}\right) \tag{A.42}$$

$$T_{ss} = \tau^B_{ss}B_{ss} \tag{A.43}$$

$$B^*_{ss} = \frac{\left(1+\tau^B_{ss}\right)}{\left(1+r_{ss}\right)} B_{ss} - \phi\left(1+r_{ss}\right)\left(W_{ss}H_{ss} + P_{ss}V_{ss}\right) + \kappa K_{ss} \tag{A.44}$$

$$\Phi^{by}_{ss} = \frac{B_{ss}}{Y_{ss}} \tag{A.45}$$

and then r_{ss} and τ_B solve

$$\tau^B_{ss} = \bar{\zeta}_0 + \bar{\zeta}_1\left(\frac{B_{ss}}{Y_{ss}}\right) \tag{A.46}$$

$$\bar{\varphi} B^*_{ss} + \left(1-\bar{\varphi}\right)\lambda_{ss} = 0 \tag{A.47}$$

A.1.2 Steady State Solution, Step 2: Check P_{ss}

Given the variables B^*_{ss} and λ_{ss}, have a new value

$$P_{ss} = \begin{bmatrix} p_{00,ss} & p_{01,ss} \\ p_{10,ss} & p_{11,ss} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\exp(\gamma_{00}-\gamma_{01}B^*_{ss})}{1+\exp(\gamma_{00}-\gamma_{01}B^*_{ss})} & \frac{\exp(\gamma_{00}-\gamma_{01}B^*_{ss})}{1+\exp(\gamma_{00}-\gamma_{01}B^*_{ss})} \\ \frac{\exp(\gamma_{10}-\gamma_{11}\lambda_{ss})}{1+\exp(\gamma_{10}-\gamma_{11}\lambda_{ss})} & 1 - \frac{\exp(\gamma_{10}-\gamma_{11}\lambda_{ss})}{1+\exp(\gamma_{10}-\gamma_{11}\lambda_{ss})} \end{bmatrix}, \tag{A.48}$$

which can be checked against the guess in step 1. The steady state solves $\left\| P^{(i+1)}_{ss} - P^{(i)}_{ss} \right\| <$ *tolerance* for successive iterations i.

A.2 Perturbation

A.2.1 Equilibrium Conditions

The 19 equilibrium conditions are written as

$$\mathbb{E}_t f\left(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_t, \mathbf{x}_{t-1}, \chi\boldsymbol{\varepsilon}_{t+1}, \boldsymbol{\varepsilon}_t, \boldsymbol{\theta}_{t+1}, \boldsymbol{\theta}_t\right) = 0 \tag{A.49}$$

where the variables are separated into the predetermined variables $\mathbf{x}_{t-1}$ and the non-predetermined variables $\mathbf{y}_t$. There are 4 predetermined variables

$$\mathbf{x}_{t-1} = \left[K_{t-1}, B_{t-1}, A_{t-1}, P_{t-1}\right] \tag{A.50}$$

and 15 non-predetermined variables

$$\mathbf{y}_t = \left[C_t, H_t, V_t, I_t, k_t, r_t, q_t, W_t, \mu_t, \lambda_t, B^*_t, \tau^B_t, T_t, Y_t, \Phi^{by}_t\right] \tag{A.51}$$

with 4 shocks

$$\boldsymbol{\varepsilon}_t = \left[\varepsilon_{r,t}, \varepsilon_{w,t}, \varepsilon_{A,t}, \varepsilon_{P,t}\right] \tag{A.52}$$

and 6 switching variables

$$\boldsymbol{\theta}_t = \left[\varphi\left(s_t\right), a^*\left(s_t\right), p^*\left(s_t\right), \zeta_0\left(s_t\right), \zeta_1\left(s_t\right), \gamma\left(s_t\right), \rho_A\left(s_t\right), \rho_P\left(s_t\right)\right]. \tag{A.53}$$

These variables are partitioned into some that affect the steady state, $\boldsymbol{\theta}_{1,t}$, and some that do not, $\boldsymbol{\theta}_{2,t}$. The partition in this case is

$$\boldsymbol{\theta}_{1,t} = \left[\varphi\left(s_t\right), a^*\left(s_t\right), p^*\left(s_t\right), \zeta_0\left(s_t\right), \zeta_1\left(s_t\right)\right] \tag{A.54}$$

$$\boldsymbol{\theta}_{2,t} = \left[\gamma\left(s_t\right), \rho_A\left(s_t\right), \rho_P\left(s_t\right)\right] \tag{A.55}$$

For solving the model, the functional forms are

$$\boldsymbol{\theta}_{1,t+1} = \bar{\boldsymbol{\theta}}_1 + \chi\hat{\boldsymbol{\theta}}_1\left(s_{t+1}\right) \tag{A.56}$$

$$\boldsymbol{\theta}_{1,t} = \bar{\boldsymbol{\theta}}_1 + \chi\hat{\boldsymbol{\theta}}_1\left(s_t\right) \tag{A.57}$$

$$\boldsymbol{\theta}_{2,t+1} = \boldsymbol{\theta}_2\left(s_{t+1}\right) \tag{A.58}$$

$$\boldsymbol{\theta}_{2,t} = \boldsymbol{\theta}_2\left(s_t\right) \tag{A.59}$$

$$\mathbf{x}_t = h_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right) \tag{A.60}$$

$$\mathbf{y}_t = g_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right) \tag{A.61}$$

$$\mathbf{y}_{t+1} = g_{s_{t+1}}\left(\mathbf{x}_t, \chi\boldsymbol{\varepsilon}_{t+1}, \chi\right) \tag{A.62}$$

$$p_{s_t,s_{t+1},t} = \pi_{s_t,s_{t+1}}\left(\mathbf{y}_t\right) \tag{A.63}$$

Using these in the equilibrium conditions and being more explicit about the expectation operator, given $\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right)$ and s_t, the

$$F_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right) = \int \sum_{s'=0}^{1} \pi_{s_t,s'}\left(g_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right)\right) f \begin{pmatrix} g_{s_{t+1}}\left(h_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right), \chi\boldsymbol{\varepsilon}', \chi\right), \\ g_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right), \\ h_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right), \\ \mathbf{x}_{t-1}, \chi\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}_t, \\ \bar{\boldsymbol{\theta}} + \chi\hat{\boldsymbol{\theta}}\left(s'\right), \bar{\boldsymbol{\theta}} + \chi\hat{\boldsymbol{\theta}}\left(s_t\right) \end{pmatrix} d\boldsymbol{\mu}\boldsymbol{\varepsilon}' = 0 \tag{A.64}$$

Stacking these conditions for each regime produces

$$\mathbb{F}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right) = \begin{bmatrix} F_{s_t=1}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right) \\ F_{s_t=2}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right) \end{bmatrix} \tag{A.65}$$

A.2.2 Generating Approximations

For perturbation, we take the stacked equilibrium conditions $\mathbb{F}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right)$, and differentiate with respect to $\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right)$. In general regime-switching models, the first-order derivative with respect to $\mathbf{x}_{t-1}$ produces a complicated polynomial system denoted

$$\mathbb{F}_{\mathbf{x}}\left(\mathbf{x}_{ss}, \mathbf{0}, 0\right) = 0. \tag{A.66}$$

Often this system needs to be solved via Gröbner bases, which finds all possible solutions in order to check them for stability. In our case with endogenous probabilities, the standard stability checks fail, so we will focus on finding a single solution and ignore the possibility of indeterminacy from multiple solutions, a common simplification in the regime-switching literature with and without endogenous switching (e.g. Farmer et al., 2011; Foerster, 2015; Maih, 2015; Lind, 2014). In the literature which computes global solutions to non-regime switching occasionally binding constraint models (e.g. Benigno et al. (2013), Mendoza (2010)) there are no proofs of uniqueness and the focus is also on computing a single solution without concern for the possibility of other solutions. To find a single solution to our model we guess at a set of policy functions for regime $s_t = 1$, which collapses the equilibrium conditions $\mathbb{F}_{\mathbf{x}}\left(\mathbf{x}_{ss}, \mathbf{0}, 0; s_t = 0\right)$ into a fixed-regime eigenvalue problem, and solve for the policy functions for $s_t = 0$. Then, using this solution as guesses, we solve for regime $s_t = 0$ under the fixed-regime eigenvalue problem, and iterate on this procedure to convergence.

After solving the iterative eigenvalue problems, the other systems to solve are

$$\mathbb{F}_{\boldsymbol{\varepsilon}}\left(\mathbf{x}_{ss}, \mathbf{0}, 0\right) = 0 \tag{A.67}$$

$$\mathbb{F}_{\chi}\left(\mathbf{x}_{ss}, \mathbf{0}, 0\right) = 0 \tag{A.68}$$

and second order systems of the form (can apply equality of cross-partials)

$$\mathbb{F}_{\mathbf{i},\mathbf{j}}\left(\mathbf{x}_{ss}, \mathbf{0}, 0\right) = 0,\ \mathbf{i}, \mathbf{j} \in \left\{\mathbf{x}, \boldsymbol{\varepsilon}, \chi\right\}. \tag{A.69}$$

Recall the decision rules have the form

$$\mathbf{x}_t = h_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right) \tag{A.70}$$

$$\mathbf{y}_t = g_{s_t}\left(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi\right) \tag{A.71}$$

and so the second-order approximation takes the form

$$\mathbf{x}_t \approx \mathbf{x}_{ss} + H_{s_t}^{(1)} S_t + \frac{1}{2} H_{s_t}^{(2)} \left(S_t \otimes S_t \right) \tag{A.72}$$

$$\mathbf{y}_t \approx \mathbf{y}_{ss} + G_{s_t}^{(1)} S_t + \frac{1}{2} G_{s_t}^{(2)} \left(S_t \otimes S_t \right) \tag{A.73}$$

where $S_t = \left[\begin{array}{ccc} (\mathbf{x}_{t-1} - \mathbf{x}_{ss})' & \boldsymbol{\varepsilon}_t' & 1 \end{array} \right]'$.

B Estimation Procedure

B.1 State Space Representation

For likelihood estimation, the state space representation is

$$\mathcal{X}_t = \mathcal{H}_{s_t} \left(\mathcal{X}_{t-1}, \epsilon_t \right)$$

$$\mathcal{Y}_t = \mathcal{G}_{s_t} \left(\mathcal{X}_t, \mathcal{U}_t \right),$$

where $\mathcal{Y}_t$ is the vector of observables variables:

$$\mathcal{Y}_t = \left[\begin{array}{cccc} \Delta y_t & \Delta c_t & \Delta i_t & r_t \end{array} \right]'.$$

Given s_t and $\boldsymbol{\varepsilon}_t$, we can construct a first order approximation to $\Delta \mathbf{y}_t$ by

$$\begin{aligned} \Delta \mathbf{y}_t &= \mathbf{y}_t - \mathbf{y}_{t-1} \\ &= G_{s_t}^{(1)} \left[\begin{array}{ccc} \hat{\mathbf{x}}_{t-1}' & \boldsymbol{\varepsilon}_t & 1 \end{array} \right]' - \mathbf{y}_{t-1} \end{aligned}$$

and the first order approximation to $\mathbf{x}_t$ is

$$\mathbf{x}_t = \mathbf{x}_{ss} + H_{s_t}^{(1)} \left[\begin{array}{ccc} \hat{\mathbf{x}}_{t-1}' & \boldsymbol{\varepsilon}_t & 1 \end{array} \right]'$$

Therefore, the state equation is

$$\mathcal{X}_t = \left[\begin{array}{c} \mathbf{x}_t \\ \mathbf{y}_t \\ \Delta \mathbf{y}_t \end{array} \right] = \left[\begin{array}{c} \mathbf{x}_{ss} + H_{s_t}^{(1)} \left[\begin{array}{ccc} \hat{\mathbf{x}}_{t-1}' & \boldsymbol{\varepsilon}_t & 1 \end{array} \right]' \\ \mathbf{y}_{ss} + G_{s_t}^{(1)} \left[\begin{array}{ccc} \hat{\mathbf{x}}_{t-1}' & \boldsymbol{\varepsilon}_t & 1 \end{array} \right]' \\ G_{s_t}^{(1)} \left[\begin{array}{ccc} \hat{\mathbf{x}}_{t-1}' & \boldsymbol{\varepsilon}_t & 1 \end{array} \right]' - \mathbf{y}_{t-1} \end{array} \right]$$

and the observation equation is

$$
\mathcal{Y}_t = \begin{bmatrix} \Delta y_t \\ \Delta c_t \\ \Delta i_t \\ r_t \end{bmatrix} = D \begin{bmatrix} \hat{\mathbf{x}}_t \\ \mathbf{y}_t \\ \Delta \mathbf{y}_t \end{bmatrix} + \mathcal{U}_t
$$

where D denotes a selection matrix. Therefore, in matrix form, we have:

$$
\begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \Delta \mathbf{y}_t \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{ss} + H^{(1)}_{\chi,s_t} \\ \mathbf{y}_{ss} + G^{(1)}_{\chi,s_t} \\ G^{(1)}_{\chi,s_t} \end{bmatrix} + \begin{bmatrix} H^{(1)}_{x,s_t} & 0 & 0 \\ G^{(1)}_{x,s_t} & 0 & 0 \\ G^{(1)}_{x,s_t} & -I & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_{t-1} \\ \mathbf{y}_{t-1} \\ \Delta \mathbf{y}_{t-1} \end{bmatrix} + \begin{bmatrix} H^{(1)}_{\varepsilon,s_t} \\ G^{(1)}_{\varepsilon,s_t} \\ G^{(1)}_{\varepsilon,s_t} \end{bmatrix} \boldsymbol{\varepsilon}_t
$$

and

$$
\begin{bmatrix} \Delta y_t \\ \Delta c_t \\ \Delta i_t \\ r_t \end{bmatrix} = S \Delta \mathbf{y}_t + \mathcal{U}_t
$$

which can be denoted as

$$
\mathcal{X}_t = A_{s_t} + B_{s_t} \mathcal{X}_{t-1} + C_{s_t} \boldsymbol{\varepsilon}_t
$$

$$
\mathcal{Y}_t = D \mathcal{X}_t + E \mathcal{U}_t
$$

B.2 Filtering

The second-order approximation of the Regime Switching DSGE model with pruning takes the following form

$$
\mathcal{Y}_t = \begin{bmatrix} \Delta y_t \\ \Delta c_t \\ \Delta i_t \\ r_t \\ \frac{TB}{Output} \end{bmatrix} = D \begin{bmatrix} \hat{\mathbf{x}}^f_t \\ \hat{\mathbf{x}}^s_t \\ \mathbf{y}_t \\ \mathbf{y}_{t-1} \end{bmatrix} + \mathcal{U}_t \tag{B.1}
$$

where D denotes a selection matrix and $\mathcal{U}_t$ denotes measurement errors.

$$
\mathcal{X}_t = \begin{bmatrix} \hat{\mathbf{x}}_t^f \\ \hat{\mathbf{x}}_t^s \\ \mathbf{y}_t \\ \mathbf{y}_{t-1} \end{bmatrix} = \mathcal{H}_{s_t}\left(\mathcal{X}_{t-1}, \varepsilon_t\right)
$$

$$
= \begin{bmatrix} H_{s_t}^{(1)} \begin{bmatrix} \hat{\mathbf{x}}_{t-1}^{f\prime} & \boldsymbol{\varepsilon}'_t & 1 \end{bmatrix}' \\ H_{s_t}^{(1)} \begin{bmatrix} \hat{\mathbf{x}}_{t-1}^{s\prime} & 0 & 0 \end{bmatrix}' + \frac{1}{2} H_{s_t}^{(2)} \left(\begin{bmatrix} \hat{\mathbf{x}}_{t-1}^{f\prime} & \boldsymbol{\varepsilon}'_t & 1 \end{bmatrix}' \otimes \begin{bmatrix} \hat{\mathbf{x}}_{t-1}^{f\prime} & \boldsymbol{\varepsilon}'_t & 1 \end{bmatrix}' \right) \\ \mathbf{y}_{ss} + G_{s_t}^{(1)} \begin{bmatrix} \hat{\mathbf{x}}_{t-1}^{f\prime} & \boldsymbol{\varepsilon}'_t & 1 \end{bmatrix}' + G_{s_t}^{(1)} \begin{bmatrix} \hat{\mathbf{x}}_{t-1}^{s\prime} & 0 & 0 \end{bmatrix}' \\ + \frac{1}{2} G_{s_t}^{(2)} \left(\begin{bmatrix} \hat{\mathbf{x}}_{t-1}^{f\prime} & \boldsymbol{\varepsilon}'_t & 1 \end{bmatrix}' \otimes \begin{bmatrix} \hat{\mathbf{x}}_{t-1}^{f\prime} & \boldsymbol{\varepsilon}'_t & 1 \end{bmatrix}' \right) \\ \mathbf{y}_{t-1} \end{bmatrix} \tag{B.2}
$$

where ε_t is the set of exogenous shocks.

The UKF uses the unscented transformation to calculate the state mean and covariance matrix. It propagates the deterministically chosen sigma-points through the nonlinear function. The transformed points are used to calculate the mean and covariance matrix. As Julier and Uhlmann (1997) note, the key approximation taken to develop the UKF is that the prediction density and the filtering density are both Gaussian.

The filter starts by combining the state vector and exogenous disturbances into a single vector $\mathcal{X}_{t-1}^a = [\mathcal{X}_{t-1}, \epsilon_t]'$ with the following mean and covariance matrix conditional on $y_{1:t-1}$ and regime s_{t-1}:

$$
\mathcal{X}_{t-1}^a(s_{t-1}) = \begin{bmatrix} \mathcal{X}_{t-1|t-1}(s_{t-1}) \\ 0_\epsilon \end{bmatrix} \tag{B.3}
$$

$$
P_{t-1}^a(s_{t-1}) = \begin{bmatrix} P_{t-1|t-1}^x(s_{t-1}) & 0 \\ 0 & I \end{bmatrix} \tag{B.4}
$$

where L is the number of state variables and exogenous shocks.

The sigma-points $\mathcal{X}_{i,t-1}^a(s_{t-1})$ that consist of the sigma-points for state variables $\mathcal{X}_{i,t-1}^x(s_{t-1})$ and the sigma-points for exogenous shocks $\mathcal{X}_{i,t-1}^\varepsilon(s_{t-1})$ are chosen as follows:

$$
\begin{aligned}
\mathcal{X}_{0,t-1}^a(s_{t-1}) &= \mathcal{X}_{t-1}^a(s_{t-1}) \\
\mathcal{X}_{i,t-1}^a(s_{t-1}) &= \mathcal{X}_{t-1}^a(s_{t-1}) + \left(h\sqrt{P_{t-1}^a(s_{t-1})}\right)_i \text{ for } i = 1 \dots L \\
\mathcal{X}_{i,t-1}^a(s_{t-1}) &= \mathcal{X}_{t-1}^a(s_{t-1}) - \left(h\sqrt{P_{t-1}^a(s_{t-1})}\right)_{i-L} \text{ for } i = L+1 \dots 2L
\end{aligned} \tag{B.5}
$$

where $h = \sqrt{3}$. The weights for the sigma-points are given by:

$$\begin{aligned} w_0 &= \frac{h-L}{2h} \\ w_i &= \frac{1}{2h} \text{ for } i = 1 \ldots 2L \end{aligned} \tag{B.6}$$

The sigma-points and the assigned weights are used to calculate the expected mean and covariance by propagating sigma-points through transition equations and taking weighted average:

$$\mathcal{X}_{i,t|t-1}(s_{t-1}, s_t) = H_{s_t}(\mathcal{X}^x_{i,t-1}(s_{t-1}), \mathcal{X}^\varepsilon_{i,t-1}(s_{t-1})) \tag{B.7}$$

$$\mathcal{X}_{t|t-1}(s_{t-1}, s_t) = \sum_{i=0}^{2L} w_i \mathcal{X}_{i,t|t-1}(s_{t-1}, s_t) \tag{B.8}$$

$$\begin{aligned} &P^x_{t|t-1}(s_{t-1}, s_t) \\ &= \sum_{i=0}^{2L} w_i [\mathcal{X}_{i,t|t-1}(s_{t-1}, s_t) - \mathcal{X}_{t|t-1}(s_{t-1}, s_t)][\mathcal{X}_{i,t|t-1}(s_{t-1}, s_t) - \mathcal{X}_{t|t-1}(s_{t-1}, s_t)]^T \end{aligned} \tag{B.9}$$

$$\mathcal{Y}_{t|t-1}(s_{t-1}, s_t) = D\mathcal{X}_{t|t-1}(s_{t-1}, s_t) \tag{B.10}$$

By the above conditions, we get the Gaussian approximation predictive density $p(\mathcal{X}_t|\mathcal{Y}_{1:t-1}, s_{t-1}, s_t) = N(\mathcal{X}_{t|t-1}(s_{t-1}, s_t), P^x_{t|t-1}(s_{t-1}, s_t))$. The predictions are then updated using the standard Kalman filter updating rule:

$$\begin{aligned} P^y_{t|t-1}(s_{t-1}, s_t) &= DP^x_{t|t-1}(s_{t-1}, s_t)D^T + R \\ P^{xy}_{t|t-1}(s_{t-1}, s_t) &= P^x_{t|t-1}(s_{t-1}, s_t)D^T \\ K_t(s_{t-1}, s_t) &= P^{xy}_{t|t-1}(s_{t-1}, s_t)(P^y_{t|t-1}(s_{t-1}, s_t))^{-1} \\ \mathcal{X}_{t|t}(s_{t-1}, s_t) &= \mathcal{X}_{t|t-1}(s_{t-1}, s_t) + K_t(s_{t-1}, s_t)(\mathcal{Y}_t - \mathcal{Y}_{t|t-1}(s_{t-1}, s_t)) \\ P^x_{t|t}(s_{t-1}, s_t) &= P^x_{t|t-1}(s_{t-1}, s_t) - K_t(s_{t-1}, s_t)P^y_{t|t-1}(s_{t-1}, s_t)K^T_t(s_{t-1}, s_t) \end{aligned} \tag{B.11}$$

The updating step gives $p(\mathcal{X}_t|\mathcal{Y}_{1:t}, s_{t-1}, s_t) = N(\mathcal{X}_{t|t}(s_{t-1}, s_t), P^x_{t|t}(s_{t-1}, s_t))$. As a by-product of the filter, we can get the density of $\mathcal{Y}_t$ conditional on $\mathcal{Y}_{1:t-1}$, s_t, and s_{t-1}

$$p(\mathcal{Y}_t|\mathcal{Y}_{1:t-1}, s_{t-1}, s_t; \theta) = N(\mathcal{Y}_{t|t-1}(s_{t-1}, s_t), P^y_{t|t-1}(s_{t-1}, s_t)) \tag{B.12}$$

Since the Unscented Kalman filter with regime switches creates a large number of nodes over each iteration where the filtered mean and covariance matrix need to be evaluated, we implement the following collapsing procedure suggested by Kim and Nelson (1999)

$$\mathcal{X}_{t|t}(s_t = j) = \frac{1}{\Pr(s_t = j|\mathcal{Y}_{1:t})}\Big\{\sum_{i=1}^{M}\Pr(s_{t-1} = i, s_t = j|\mathcal{Y}_{1:t})\mathcal{X}_{t|t}(s_{t-1} = i, s_t = j)\Big\} \tag{B.13}$$

$$\begin{aligned} P^x_{t|t}(s_t = j) = &\frac{1}{\Pr(s_t = j|\mathcal{Y}_{1:t})}\Big\{\sum_{i=1}^{M}\Pr(s_{t-1} = i, s_t = j|\mathcal{Y}_{1:t})[P^x_{t|t}(s_{t-1} = i, s_t = j) \\ &+ (\mathcal{X}_{t|t}(s_t = j) - \mathcal{X}_{t|t}(s_{t-1} = i, s_t = j))(\mathcal{X}_{t|t}(s_t = j) - \mathcal{X}_{t|t}(s_{t-1} = i, s_t = j))^T]\Big\} \end{aligned} \tag{B.14}$$

where $\Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t})$ and $\Pr(s_t|\mathcal{Y}_{1:t})$ are obtained from the following standard Hamilton filter

$$\Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t-1}) = \Pr(s_t|s_{t-1})\Pr(s_{t-1}|\mathcal{Y}_{1:t-1}) \tag{B.15}$$

$$\Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t}) = \frac{p(\mathcal{Y}_t|s_t, s_{t-1}, \mathcal{Y}_{1:t-1})\Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t-1})}{\sum_{s_t}\sum_{s_{t-1}} p(\mathcal{Y}_t|s_t, s_{t-1}, \mathcal{Y}_{1:t-1})\Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t-1})} \tag{B.16}$$

$$\Pr(s_t|\mathcal{Y}_{1:t}) = \sum_{s_{t-1}}\Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t}) \tag{B.17}$$

Finally, we can get the conditional marginal likelihood,

$$p(\mathcal{Y}_t|\mathcal{Y}_{1:t-1};\theta) = \sum_{s_t}\sum_{s_{t-1}} p(\mathcal{Y}_t|s_t, s_{t-1}, \mathcal{Y}_{1:t-1})\Pr(s_t, s_{t-1}|\mathcal{Y}_{1:t-1}) \tag{B.18}$$

B.3 Smoothing

Once we run through the UKF for $t = 1, \ldots, T$, we can also get the smoothed joint probability $\Pr(s_t, s_{t+1}|\mathcal{Y}_{1:T})$, $\Pr(s_t|\mathcal{Y}_{1:T})$, $x_{t|T}(s_t, s_T)$, and $P^x_{t|T}(s_t, s_T)$:

$$\begin{aligned} &\Pr(s_t, s_{t+1}|\mathcal{Y}_{1:T}) = \frac{\Pr(s_{t+1}|\mathcal{Y}_{1:T})\Pr(s_t|\mathcal{Y}_{1:t})\Pr(s_{t+1}|s_t)}{\Pr(s_{t+1}|\mathcal{Y}_{1:t})} \\ &\Pr(s_t|\mathcal{Y}_{1:T}) = \sum_{s_{t+1}}\Pr(s_t, s_{t+1}|\mathcal{Y}_{1:T}) \\ &\mathcal{X}_{t|T}(s_t, s_{t+1}) = \mathcal{X}_{t|t}(s_t) + \tilde{K}_t(s_t, s_{t+1})(\mathcal{X}_{t+1|T}(s_{t+1}) - \mathcal{X}_{t+1|t}(s_t, s_{t+1})) \\ &P^x_{t|T}(s_t, s_{t+1}) = P^x_{t|t}(s_t) - \tilde{K}_t(s_t, s_{t+1})(P^x_{t+1|T}(s_{t+1}) - P^x_{t+1|T}(s_t, s_{t+1}))\tilde{K}_t(s_t, s_{t+1})^T \end{aligned} \tag{B.19}$$

Given the above smoothing algorithm, we implement the collapsing procedures similar to those in the filtering steps:

$$\mathcal{X}_{t|T}(s_t = j) = \frac{1}{\Pr(s_t = j|\mathcal{Y}_{1:T})}\Big\{\sum_{j=1}^{M}\Pr(s_t = i, s_{t+1} = j|\mathcal{Y}_{1:T})\mathcal{X}_{t|T}(s_t = i, s_{t+1} = j)\Big\} \tag{B.20}$$

$$
\begin{aligned}
P^x_{t|T}(s_t = j) = \frac{1}{\Pr(s_t = j|\mathcal{Y}_{1:T})}\Big\{ \sum_{j=1}^{M} \Pr(s_t = i, s_{t+1} = j|\mathcal{Y}_{1:T})[P^x_{t|T}(s_t = i, s_{t+1} = j) \\
+ (\mathcal{X}_{t|T}(s_t = j) - \mathcal{X}_{t|T}(s_t = i, s_{t+1} = j))(\mathcal{X}_{t|T}(s_t = j) - \mathcal{X}_{t|T}(s_t = i, s_{t+1} = j))^T]\Big\}
\end{aligned}
\tag{B.21}
$$