Rational Sunspots*

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April 4, 2016

Abstract

The instability of macroeconomic variables is usually ruled out by rational expectations. We propose a generalization of the rational expectations framework to estimate possible temporary unstable paths. Our approach yields drifting parameters and stochastic volatility. The methodology allows the data to choose between different possible alternatives: determinacy, indeterminacy and instability. We apply our methodology to US inflation dynamics in the '70s through the lens of a simple New Keynesian model. When unstable RE paths are allowed, the data unambiguously select them to explain the stagflation period in the '70s. Thus, our methodology suggests that US inflation dynamics in the '70s is better described by unstable rational equilibrium paths.

Keywords: Rational Expectations, Sunspots, Instability, Indeterminacy, Inflation, Monetary Policy.

JEL classification: E31, E52.

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*We thank Philippe Andrade, Paul Beaudry, Efrem Castelnuovo, John Cochrane, Valentina Gavazza, Martin Ellison, Luca Fanelli, Gaetano Gaballo, Tom Holden, Alejandro Justiniano, Albert Marcet, Leonardo Melosi, Davide Raggi, Luca Sala, Thomas Sargent, Paolo Surico, conference participants at the SED Meeting in Limassol, the SIE Meeting in Matera, the ESOBE Meeting in Vienna, and participants in many seminars for helpful comments on earlier versions. We are also grateful to Nicola Martinelli for I.T. support. This work acknowledges funding from the European Union's Seventh Framework Programme grant agreement number 612796, project MACFINROBODS "Integrated macro-financial modelling for robust policy design". A very preliminary version of this paper circulated under the title "Does inflation walk on unstable paths?". The views expressed in this paper are solely the responsibility of the authors and should not to be interpreted as reflecting the views of Sveriges Riksbank.

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"I should like to suggest that expectations, since they are informed predictions of future events, are essentially the same as the predictions of the relevant economic theory. [...] we call such expectations rational." (Muth, 1961, p. 316)

1 Introduction

The rational expectations assumption generally implies multiple equilibria, that is, an infinite number of rational expectations (RE henceforth) trajectories. Depending on the properties of the dynamic system at hand, these trajectories could be explosive or stable. After Muth's (1961) seminal contribution, then, the literature faced the problem of how to select an equilibrium out of many possible ones.[1] The stability criterion was accepted as a general consistency requirement to impose on a model of infinite horizons RE agents. Ruling out the possibility of unstable paths determined saddle path dynamic systems as the new standard in macroeconomics. Among the infinite RE equilibria in saddle path dynamics only one is stable: so that the stability criterion was enough to pin down a unique acceptable RE path. Blanchard and Kahn (1980) formalized this idea and conceptualized the solution algorithm on which dynamic macroeconomics is based.

Under the stability criterion, however, RE solutions have a hard time in explaining unstable behaviour in the data, such as hyperinflations or boom and bust episodes in asset markets. One possibility would be, instead, to associate the unstable behaviour in the data to an unstable trajectory in the model. Generally, expectations of future policy changes cannot rule out a temporary explosive path.[2] In more formal terms, RE expectations always admit what Gourieroux et al. (1982) call *asymptotically equivalent stationary* paths: that is, it is always possible to find non-stationary processes that are RE solutions and whose any time-path coincides with the corresponding time-path of the stationary solution after some date.[3] Therefore it is not clear from a theoretical perspective that one can rule out a priori the possibility of temporary explosive trajectories (see Cochrane, 2011 and below). We thus simply start from the acknowledgment that in the empirical analysis, it might be appropriate not to rule out this possibility and thus to consider the set of all possible RE equilibria.

[1] Sargent and Wallace (1973), Brock (1974), Phelps and Taylor (1977), Taylor (1977), Blanchard (1979), Blanchard and Kahn (1980) and Flood and Garber (1980) are some examples of this compelling debate in the literature, following Muth's contribution. See also the discussion in Burmeister et al. (1983).

[2] That would be the case, for example, in a Markov Switching context, where seemingly explosive paths could occur in a given regime.

[3] This idea is somewhat related with the rational bubble literature.

Hence, we propose a novel framework able to incorporate and to empirically test the possibility of explosive paths. First, we present a theoretical framework in which we clarify the role of multiple solutions in models with RE. To avoid confusion, it is important to stress from the outset which kind of multiplicity we are focusing on in this paper. In general, most readers would naturally think of multiplicity in this context as arising from the fact that any solution whose expectation error has zero conditional mean could be considered a RE solution. In other words, RE solutions are defined up to an arbitrary martingale process, because, for any given solution, it is always possible to construct another solution by adding a sunspot shock with zero conditional mean. Our framework, however, focuses on another source of multiplicity based on a generalization of the original RE assumption in Muth (1961). The original formulation of Muth (1961) stated that a rational expectations solution should be a function of present, past, and expected future values of the structural exogenous shock. Employing the method of undetermined coefficients, as in Muth (1961), we show that the set of admissible solutions is defined up to a free parameter that thus reveals a fundamental multiplicity of the RE solution, just defined as a function of the fundamental structural shock. The case of multiple solutions is the natural case in the original RE approach, even without any additive sunspot shock. Our framework parameterizes all the possible solutions and provides a way to compute them, by following the original insight of Blanchard (1979), who showed that all these solutions are a combination, defined by the value of this free parameter, of the forward-looking solution and the backward-looking one. While the eigenvalues describe the nature of the dynamic system (i.e. number of stable or unstable trajectories), the value of this parameter selects a particular solution among the infinitely many. We show that this parameter has an appealing interpretation: it shows how the infinite solutions differ in the way agents form their RE, or more precisely, in the way agents weight past data to calculate their RE. Given the stability properties of the dynamic system at hand provided by the eigenvalues, depending on the value of this parameter, the solution could be stable or unstable. We don't therefore rule out a priori the possibility that the system could be on an unstable path. Then, we assume that the parameter follows a stochastic process, driven by a non-fundamental (sunspot) shock. In our interpretation, the economy randomly switches among different solutions as expectations are then formed by randomizing across the infinite RE solutions. The chosen solution, however, is a Muth's (1961) solution, that is, it is always a 'fundamental' RE solution that just depends on the structural shocks: hence we refer to our sunspots as *rational sunspots.*

One the one hand, the idea is somewhat similar to the standard sunspot literature. While the standard sunspot shock randomizes over stable RE equilibrium paths under indeterminacy (i.e., an infinite number of stable trajectories), our sunspot randomizes over all the possible RE equilibrium paths expressed as a function of the structural shock, whatever the dynamics of the system that the eigenvalues dictate. On the other hand, we will show that our approach could also be seen as a different way to introduce sunspot shocks: while the literature so far has dealt with additive sunspots (Benhabib and Farmer, 1999; Lubik and Schorfheide, 2004), our sunspots are multiplicative. In our approach, sunspot shocks can be effective only when a fundamental error hits the economy, because sunspot disturbances interact with the fundamental ones. Given that our sunspots are multiplicative, our approach naturally implies that the solution exhibits drifting parameters and stochastic volatility. Moreover, our framework does not suffer from the identification problems underlined by Beyer and Farmer (2007) and by Cochrane (2011), because the likelihood function is a multivariate Normal in the case of determinacy and no sunspots, and not Normal otherwise.

Second, we develop an econometric strategy suited for our framework. Given that our sunspots are multiplicative and imply stochastic volatility, the likelihood is not Normal and we cannot use Gaussian methods. We thus proceed by estimating the model's parameters and the latent states using the Particle Learning approach of Carvalho, Johannes, Lopes and Polson (2010). This method relies on the assumption that the posterior distribution of the parameters depends on a set of sufficient statistics that are recursively updated. When we cannot use this assumption, we approximate the posterior distribution of the parameters using mixtures of Normals, as in Liu and West (2001). Finally, we use the sequential Bayes factor presented in West (1986) to compare the different models. The econometric strategy allows for the cases of determinacy, indeterminacy or explosiveness, without imposing them a priori. We then propose a methodology to let the data choose the preferred equilibria among all the possible ones, and thus to test the empirical validity of temporary unstable paths. By the same token, our approach could be seen as checking the validity of the stability criterion as usually imposed on the RE solutions.

In this paper, we apply our approach to explain the US inflation dynamics in the post-war sample. The Great Inflation of the '70s, and the subsequent Volcker disinflation, is among the most studied episodes of US monetary history. In an extremely influential article, Clarida et al. (2000) estimate an interest rate equation for the US and provide evidence that monetary policy was inadequately responding to inflation in the pre-Volcker period. They suggest that this monetary policy conduct could explain the different inflation behaviour between the Great Inflation period of the '70s and the so-called Great Moderation period of the late 80s and 90s. A simple New Keynesian model would predict that if monetary policy does not sufficiently react to inflation (i.e. the

Taylor principle is not satisfied) then there exists an infinite number of stable RE equilibria paths. Such indeterminacy of equilibria trajectories could explain the aggregate instability of the 70s through shifts in self-fulfilling agents' beliefs due to sunspot shocks. In a seminal contribution about the econometrics of indeterminate RE equilibria, Lubik and Schorfheide (2004) estimate a standard three-equations New-Keynesian model under both determinacy and indeterminacy. Their results provide support to the original Clarida et al. (2000) result in a multivariate context. Subsequently, other papers in the literature confirmed this narrative that identifies loose monetary policy as the cause of the Great Inflation period (e.g. Boivin and Giannoni, 2006; Benati and Surico, 2009; Mavroeidis, 2010; Castelnuovo et al., 2014; Castelnuovo and Fanelli, 2015).[4]

The New Keynesian literature, therefore, appeals to indeterminacy, induced by a dovish monetary policy, to explain the apparently explosive behaviour of inflation during the Great Inflation period, and to a hawkish one to explain the great Moderation. However, this has the rather paradoxical implication of appealing to a stable system to generate instability, as well as to an unstable system to ensure stability. From a theoretical perspective, a saddle path describes an unstable dynamic system, because there are infinite unstable trajectories while only one, that thus has measure zero, is stable. On the contrary, indeterminacy (i.e., a sink) has an infinite number of stable trajectories, so it is a stable dynamic system. Indeterminacy, however, opens up the possibility of rationalizing an explosive behaviour by randomizing among all these possible trajectories thanks to a sunspot shock. Nonetheless, a central bank that does not respect the Taylor principle is sure that the economy is on stable dynamics, while on the contrary satisfying the Taylor principle is potentially highly risky, because the probability of being on the unique stable path (among infinitely many unstable ones) is virtually zero. Macroeconomists assume agents are able to select this unique stable solution.

It seems to us it would be more natural to associate the unstable behaviour of inflation in the data to an unstable trajectory in the model. Cochrane (2011) argues that theory cannot rule out explosive inflation behaviour: *"economics does not rule out explosive inflation, so inflation remains indeterminate"* (abstract).[5] To select the

[4] Alternative possible explanations for the Great Inflation period put forward in the literature are stochastic volatility of the shocks (e.g., Justiniano and Primiceri, 2008; Villaverde et al., 2010) or escape dynamics (e.g., Sargent, 1999; Cho, et al., 2002; Sargent et al., 2006, Carboni and Ellison, 2009).

[5] *"In new-Keynesian models, higher inflation leads the Fed to set interest rates in a way that produces even higher future inflation. For only one value of inflation today will inflation fail to explode or, more generally, eventually leave a local region. Ruling out nonlocal equilibria, new-Keynesian modelers conclude that inflation today must jump to the unique value that leads to a locally bounded equilibrium path. But there is no reason to rule out nominal explosions or "nonlocal" nominal paths. Transversality conditions can rule out real explosions but not nominal explosions. Since the multiple nonlocal equilibria*

unique stable equilibrium, one needs to believe the assumption in the model that policy would stick to such a hawkish policy forever also on an explosive path, even though there are possible alternative policies that would allow the government to stop inflation or deflation.[6] In a sense, our strategy could be seen as taking Cochrane (2011) to the data, letting the data speak about their preferred solutions.[7]

We thus apply our framework to ask the following question: is there any evidence that inflation is described by unstable RE equilibria, at least temporarily in the 70s?

The seminal paper of Lubik and Schorfheide (2004, LS henceforth) is the natural benchmark against which to compare our results, so we will use both their econometric model and their data. If we impose the stability criterion on the estimation, that is, allowing just for determinacy or indeterminacy while ruling out instability, our econometric strategy recovers results that are practically identical to the one in LS. We interpret this finding as corroborating our estimation methodology. Our main result, however, is to provide evidence that the high inflation during the 1970s is better explained by unstable dynamics: the data seem to favour an unstable RE equilibrium rather than a stable one to explain the Great Inflation period.

From a policy perspective, our framework suggests a different interpretation of the Great Inflation. The latter was due to drifting expectations, *independently* from the stand of monetary policy. Although our estimates point to a passive monetary policy behaviour in the 70s, our framework implies that this is not the cause in itself of unstable inflation dynamics.

The paper proceeds as follows. Section 2 explains our approach by the means of a simple model. Section 3 presents the New Keynesian model we will use in the estimation. Section 4 explains our econometric strategy. Section 5 shows and comments on the empirical results, and Section 6 concludes.

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are valid, the new-Keynesian model does not determine inflation. " Cochrane (2011, p. 566)

[6] "*First, they require expectations that the government will follow the Taylor rule to explosive hyperinflations and deflations, beyond anything ever observed, and despite the presence of equilibrium-preserving stabilization policies such as the switch to a commodity standard, money growth, or non-Ricardian regime. Second, they require belief in a deep-seated monetary nonneutrality sufficient to send real rates to negative infinity or real money demand to infinity, though even the beginning of such events has never been observed. At a minimum, expectations of such events sound again like a weak foundation for what should be a simple question, the basic determination of the price level.*" Cochrane (2011, p. 590)

[7] "*Testing for determinacy is not as simple as testing the parameters of the Fed reaction function. Alas, no one has tried a test for determinacy in a more complex model*" (Cochrane, 2011, p. 593).

2 The Rational Sunspots Approach

We want to verify if rational expectations unstable paths can better explain the inflation dynamics. While unstable paths are usually ruled out by imposing the stability criterion as way of selecting equilibria, the possibility of a temporary walk on unstable paths is not necessarily in contrast with rational expectations. In what follows we explain our approach with a simple example and leave the more general matrix formulation of the problem to the Appendix.

2.1 A simple example

Consider the following model inspired by Cochrane (2011), including the Fisher equation (1) and the Taylor rule (2):

$$i_t = r + E_t \pi_{t+1} \quad (1)$$

$$i_t = r + \phi \pi_t + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \quad (2)$$

i_t is the nominal interest rate at time t, r is the real interest rate (assumed constant for simplicity), π_t is inflation and ε_t is a white noise exogenous shock. Finally, $E_t \pi_{t+1} = E(\pi_{t+1}|I_t)$ is the expected value of inflation at $t+1$, conditional on the information set available at time t.[8] The two equations above imply the following model:[9]

$$\pi_t = \frac{1}{\phi} E_t \pi_{t+1} + e_t, \quad (3)$$

where $e_t = -\frac{1}{\phi}\varepsilon_t$, so that e_t is a i.i.d shock $\sim N(0, \sigma_e^2)$. Equation (3) naturally has an infinite number of solutions, because one can find an infinite number of couples $(\pi_t, E_t \pi_{t+1})$ that satisfy it. To see it, simply rewrite (3) as: $E_t \pi_{t+1} = \phi \pi_t - \phi e_t$, and then the solutions are characterized by:

$$\pi_{t+1} = \phi \pi_t - \phi e_t + \eta_{t+1} \quad (4)$$

where $E_t \eta_{t+1} = 0$. Any process η_{t+1} such that the expectation error $\eta_{t+1} = \pi_{t+1} - E_t \pi_{t+1}$ has zero mean defines a different solution to (3). The constraint that the expected error in expectation should be zero (i.e., the error in expectation should not be correlated

[8] Coherently with the rational expectations hypothesis (in the strong form), we assume that the set I_t contains all the relevant information: all the present and past values of the endogenous and exogenous variables, and the structure of the model with its parameters.

[9] This equation coincides also with equation (7) used in LS as a simple example to explain their approach.

with anything in the available information set) simply implies that the solution is characterized up to an arbitrary martingale process.[10]

The original formulation of Muth (1961) stated that a rational expectations solution should be a linear function of present, past, and expected future values of the structural exogenous shock. Muth (1961) (see also Blanchard, 1979) then employed the method of undetermined coefficients to derive the set of admissible solutions written as (see the Appendix[11]):

$$\pi_t = \phi\pi_{t-1} - \phi e_{t-1} + b e_t \tag{5}$$

Equation (5) represents all the solutions of equation (3) that are a function only of the history of the structural shocks. (3) thus admits an infinite number of solutions, and we can distinguish two cases: (i) an infinite number of stable solutions, if $|\phi| < 1$; (ii) a unique stable solution along with an infinite number of unstable solutions, if $|\phi| > 1$. All these solutions are parameterized by $b \in (-\infty, +\infty)$, because a particular value of b defines a particular solution. Following the terminology used by Blanchard (1979), two important solutions often considered in the literature are: (i) the pure forward looking solution corresponding to $b = 1$:

$$\pi_t^F = e_t; \tag{6}$$

(ii) the pure backward looking solution, corresponding to $b = 0$:

$$\pi_t^B = \phi\pi_{t-1}^B - \phi e_{t-1} = -\sum_{j=1}^{\infty} \phi^j e_{t-j} = -\frac{\phi}{(1-\phi L)} e_{t-1}. \tag{7}$$

Finally, it is easy to rewrite (5) as a linear combination of the forward (6) and backward looking (7) solutions as:[12]

$$\pi_t = (1-b)\,\pi_t^B + b\pi_t^F. \tag{8}$$

This equation reveals that each particular solution is a weighted combination of the backward and the forward one, and b is exactly the weight between these two solutions.

[10] It is straightforward to show that one can interpret η_t as a martingale difference process (see Pesaran, 1987).

[11] The Appendix shows that the undetermined coefficient solutions in Muth (1961) and Blanchard (1979) are given by:

$$\pi_t = \sum_{j=1}^{\infty} \phi^j (b-1) e_{t-j} + b e_t + \sum_{j=1}^{\infty} \frac{b}{\phi^j} E_t e_{t+j}.$$

[12] From (5):

$$\begin{aligned} \pi_t (1-\phi L) &= -\phi (1-b)\, e_{t-1} + b (1-\phi L)\, e_t \\ \pi_t &= (1-b) \left(-\frac{\phi}{(1-\phi L)} e_{t-1} \right) + b e_t. \end{aligned}$$

In this framework, it is natural to interpret b as the way agents form their expectations under the rational expectations hypothesis. One of the purposes of Muth 's (1961) original paper is to write the expectation at time t as an exponentially weighted average of past observations, because a previous paper (Muth, 1960) demonstrated that, under some assumptions, this is the optimal estimator. In the simple case of equation (3) the expectation (when $b \neq 0$) is given by:

$$E_t\pi_{t+1} = (b-1)\sum_{i=1}^{\infty}\left(\frac{\phi}{b}\right)^i \pi_{t+1-i}. \tag{9}$$

$E_t\pi_{t+1}$ is the product of two terms. First, $(b-1)$ measures how much the past is important in forming expectations in absolute terms: if $b = 1$, then the past does not matter. Second, the weights $\left(\frac{\phi}{b}\right)^i$ tell us how much agents look relatively more or less into the past. The lower is b, the more past terms are important in setting expectations. Then, b determines how the agents consider past observations in making forecasts both in *absolute* terms (b versus 1), and in *relative* terms.

The Taylor principle states that the central bank conducts an "active" policy if it moves the nominal interest rate more than proportionally with respect to inflation's variations, that is when $|\phi| > 1$. Otherwise, $|\phi| < 1$, monetary policy is labelled "passive". In this latter case, for every $b \in (-\infty, +\infty)$ the implied dynamics are stable. By contrast when the central bank conducts an active monetary policy, all the solutions are unstable but the forward looking one: equation (6). The literature normally imposes a *stability criterion*, ruling out non-explosive solutions. Therefore, it considers only two cases: (i) *determinacy*, when monetary policy is active and the agents choose the unique value of b, i.e. $b = 1$, that puts the system on the unique stable trajectory; (ii) *indeterminacy*, when monetary policy is passive and any value of b is consistent with a stable solution. The stability criterion thus does not solve the problem of selecting a unique equilibrium in this latter case. The literature then introduces an exogenous sunspot (i.e. non fundamental) shock and it assumes that the economy will choose randomly among infinite stable solutions depending on the realization of such shock. *"Sunspot equilibria can often be constructed by randomizing over multiple equilibria of a general equilibrium model, and models with indeterminacy are excellent candidates for the existence of sunspot equilibria since there are many equilibria over which to randomize."* Benhabib and Farmer (1999, p.390)

2.2 The rational sunspot

Thinking along the Benhabib and Farmer (1999) lines, there is a natural and simple way to introduce a sunspot shock in our setup: randomizing over b. We saw that: (i) there is an infinite number of equilibria; (ii) these equilibria are characterized by the infinite number of ways agents could form their expectations coherently with the Muth's rational expectations original formulation; (iii) these equilibria can be parametrized by b, as shown by equation (8). Therefore, it seems natural to consider b as the source of the multiple equilibria, by assuming that b_t is time varying and it follows a stochastic process.

Above we wrote the infinite solutions in two different ways: according to the martingale intuition, i.e. (4) and according to the undetermined coefficient method, i.e. (5). Comparing these two ways highlights the fact that there are two possible sources of multiplicity, because the expectation error can be written as $\eta_t(e_t, \zeta_t) = be_t + \zeta_t$ where ζ_t is any sunspot or non-fundamental error, as is usually assumed in the indeterminacy literature. The other source of multiplicity the literature does not consider is be_t. Our approach proposes a different way of introducing sunspots shock by randomizing over b. In other words, we introduce a multiplicative sunspot shock, rather than an additive one. Hence we will assume that $\eta_t(e_t, \zeta_t) = b_t(\zeta_t)e_t$ and that b_t follows a random walk process: $b_t = b_{t-1} + \zeta_t$, and $\zeta_t \sim i.i.d.N(0, \sigma_\zeta^2)$.

Our approach has a number of implications. Firstly, the interaction between the structural and the sunspot shock changes the nature of the solution. In particular, we consider solutions that satisfy the original Muth (1961) restrictions under undetermined coefficient, that are given by (see Appendix):

$$\pi_t = \theta_t \pi_{t-1} - \theta_t e_{t-1} + b_t e_t \tag{10}$$

with $\theta_t = \phi \dfrac{(1-b_t)}{(1-b_{t-1})}$ (with $b_{t-1} \neq 1$ otherwise FL solution). Since our sunspot shock satisfies the original Muth (1961) restrictions, we label it *rational sunspot.*

Secondly, the solution has the same form of (5) but it now implies drifting parameters and stochastic volatility. Drifting parameters naturally arise because agents change the way they form their expectation formation process each period, since:

$$E_t \pi_{t+1} = (b_t - 1) \sum_{i=1}^{\infty} \left(\frac{\phi}{b_t} \right)^i \pi_{t+1-i}. \tag{11}$$

Hence, even in the absence of a structural shock, the sunspot shock changes the structural dependence of π_t from its lagged value, because agents shift from one rational expectations equilibrium trajectory to another one, that implies different structural dynamics. Note that in (11) we restrict the weights to be just a function of the *current* realization of b_t and not of the past values of b (see Appendix).[13] A change in b_t is going

[13] We impose this condition by following Muth (1961) restrictions on the solution. It follows that we

to affect all the weights in (11) and not just the one in period t, where the sunspot is realized. In some periods the agents form their expectations with great trust in the past, while in some other periods they expect inflation to be more or less around its steady state (the forward looking solution in this simple case). Under *rational sunspots,* the sunspot is created not by an exogenous external additive element, but it is something related to the degree of freedom agents have in making forecasts, in line with the rational expectations hypothesis.

The sunspot shock also interacts with the structural shock through the term $b_t e_t$. The sunspot shock thus changes the way the economy reacts to the structural shock, possibly amplifying its effects on the economy. The emergence of stochastic volatility within the rational expectations framework is the direct consequence of assuming a multiplicative sunspot that makes the likelihood non Gaussian.

In other words, rational sunspots have the potential for an economic explanation of drifting parameters and stochastic volatility, without departing from the rational expectations hypothesis. The empirical research (Cogley and Sargent, 2005, Primiceri, 2005, Justiniano and Primiceri, 2008, and related literature) considers these as important features in explaining the dynamics of macroeconomic variables.

Thirdly, the forecast error is now the sum of two terms:

$$\eta_t = (\theta_t - \phi)(\pi_{t-1} - e_{t-1}) + b_t e_t = \frac{(E_{t-1}b_t - b_t)}{(1 - E_{t-1}b_t)} E_{t-1}\pi_t + b_t e_t. \tag{12}$$

The first term derives from the time varying coefficient θ in (10), because it depends on $\theta_t - E_{t-1}(\theta_t) = \theta_t - \phi$.[14] This term captures the fact that our sunspot shock changes the equilibrium trajectories agents choose by setting their expectations. The second term is the interaction term between our rational sunspot shock and the structural shock, and highlights the fact that a change in b also changes the impact response of the economy to the structural shock.

Fourthly, the difference between our rational sunspot shock and the usual sunspot shock in the literature is well explained by comparing the forecast error. The latter is given by $\eta_t = \frac{\zeta_t}{1-b_{t-1}} E_{t-1} y_t + b_t(\zeta_t)\omega_t$ in our case and by $\eta_t = M\omega_t + \zeta_t^{LS}$ in the standard case (e.g., LS). The key difference is that our sunspot is multiplicative rather than additive, so our approach could just be interpreted as another way of introducing sunspot shocks. One consequence of this assumption is that the likelihood is non Gaussian under

will not consider other possible solutions that Muth (1961) would label as "deviations from rationality". Note that this implies further restrictions on the considered solutions and thus works against our framework by tying our hands.

[14]Since $E_{t-1}\pi_t = \phi(\pi_{t-1} - e_{t-1})$, then $\eta_t = \pi_t - E_{t-1}\pi_t = (\theta_t - \phi)(\pi_{t-1} - e_{t-1})$ and given $\theta_t = \phi\frac{(1-b_t)}{(1-b_{t-1})}$, it follows: $\eta_t = \pi_t - E_{t-1}\pi_t = \left(\frac{b_{t-1} - b_t}{1 - b_{t-1}}\right)\phi(\pi_{t-1} - e_{t-1})$.

sunspots and Gaussian with no sunspots, and thus our setup does not suffer from the identification problem between a sunspot and a fundamental equilibrium, as in the case of additive sunspot (Beyer and Farmer, 2007).

2.3 Unstable paths

As explained above, we construct sunspots equilibria randomizing among the infinite rational expectations equilibria that are parametrized by b. The change in b, in terms of equation (9), can be interpreted as changes in the expectations formation process. When $|\phi| < 1$ the agents jump among stable self-fulfilling equilibria. Is this possibility only restricted to the case of passive policy? When the Taylor principle is respected there is only one stable solution and values of b_t different from one would pick an unstable trajectory, because of a temporary change in the expectations formation process. However, if b is time varying, theoretically it is not possible to rule out equilibria that are only temporarily unstable.[15] We simply start from the acknowledgment that in the empirical analysis, it would be appropriate to consider this possibility. Then, we want to allow temporary "walks along unstable paths" (i.e. in our simple example above, it would mean $|\phi| > 1$ and b_t different from one), by estimating the latent process for b_t and then ask to the data which kind of equilibria they preferred. Thus, we are not taking a stand a priori on the possible equilibria in our estimation strategy, by allowing for all the possible cases: indeterminacy, determinacy and instability. We then propose a methodology to let the data choose the preferred equilibria, and thus to test the empirical validity of these temporary unstable paths. This is what we turn to next, explaining our proposed methodology in a more general context.

2.4 The general solution

We consider the class of models that can be written in the form of Blanchard and Kahn (1980):

$$\begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \boldsymbol{\gamma} Z_t$$

where X_t is a $(n \times 1)$ vector of predetermined variables, and P_t is a $(m \times 1)$ vector of non-predetermined variables. The exogenous disturbances are collected in the $(\kappa \times 1)$ vector Z_t, that has a multivariate normal distribution: $Z_t \sim i.i.d.\ N(\mathbf{0}, \Sigma)$. The exogenous shocks in Z_t are called fundamental errors. Finally, A and $\boldsymbol{\gamma}$ are matrices with the parameters of the model.

[15] See the discussion of *asymptotically equivalent stationary* solutions in Gourieroux et al. (1982).

The matrix A can be written using the Jordan decomposition

$$A = C^{-1} J C$$

and we define the following set of block matrices:

$$C^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ (n \times n) & (n \times m) \\ B_{21} & B_{22} \\ (m \times n) & (m \times m) \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ (n \times n) & (n \times m) \\ C_{21} & C_{22} \\ (m \times n) & (m \times m) \end{bmatrix},$$

$$J = \begin{bmatrix} J_1 & \mathbf{0} \\ (n \times n) & (n \times m) \\ \mathbf{0} & J_2 \\ (m \times n) & (m \times m) \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 \\ (n \times \kappa) \\ \gamma_2 \\ (n \times \kappa) \end{bmatrix}$$

In the Appendix we show that the general solution is described by the following system:

$$X_t = (B_{11} J_1 C_{11} + B_{12} J_2 C_{21}) X_{t-1} + (B_{11} J_1 C_{12} + B_{12} J_2 C_{22}) P_{t-1} + \gamma_1 Z_{t-1} \tag{13}$$

$$\begin{aligned} C_{21} X_t + C_{22} P_t &= J_2 H_t (C_{21} X_{t-1} + C_{22} P_{t-1}) + \\ &+ H_t (C_{21}\gamma_1 + C_{22}\gamma_2) Z_{t-1} - \mathbf{b}_t J_2^{-1} (C_{21}\gamma_1 + C_{22}\gamma_2) Z_t \end{aligned} \tag{14}$$

where $\mathbf{b}_t$ is a $(m \times m)$ diagonal matrix:

$$\mathbf{b}_t = \begin{bmatrix} b_{1,t} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & b_{m,t} \end{bmatrix}$$

in which

$$b_{i,t} = b_{i,t-1} + \zeta_{i,t} \qquad \zeta_{i,t} \sim N(0, \sigma^2_{\zeta_i}) \quad \forall i$$

and $H_t = (I + \mathbf{b}_t)(I + \mathbf{b}_{t-1})^{-1}$. The shocks $\zeta_{i,t}$ are the rational sunspot shocks, as in the univariate example.

In general, if we have m non predetermined variables, the cardinality of the set of solutions is infinite to the power of m. However, as in the simple example, when the eigenvalues of the model are outside the unit circle, we can restrict the elements in $\mathbf{b}_t$ to ensure stability. In practice, to guarantee stability it is sufficient to impose the following *stability criterion*:

> *stability criterion*: for $i = 1...m$, if $|J_{2,i}| > 1$, then $b_{i,t} = 1 \ \forall t$,
>
> where $J_{2,i}$ is the i^{th} element in the main diagonal of J_2, and $b_{i,t}$ is the i^{th} element in the main diagonal of $\mathbf{b}_t$.

The criterion reduces the degrees of freedom in the matrix $\mathbf{b}_t$, and it downsizes the set of solutions. If, for example, there are $r \leq m$ number of eigenvalues outside the unit circle, the number of stable solutions is $\infty^{(m-r)}$. The limiting case is when the Blanchard-Kahn condition is satisfied, that is when the number of eigenvalues outside the unit circle is equal to the number of non predetermined variables: the criterion forces all the elements in the main diagonal of $\mathbf{b}$ to be equal to 1, and this is the unique stable solution. If the criterion is not satisfied, the dynamics of the variables are unstable.

3 Rational Sunspots at work: the Great Inflation and the New Keynesian Model

We apply our new methodology to inflation dynamics through the lens of the following prototypical New Keynesian model:

$$\begin{aligned} x_t &= E_t(x_{t+1}) - \tau(R_t - E_t(\pi_{t+1})) + g_t, && (15) \\ \pi_t &= \beta E_t(\pi_{t+1}) + \kappa(x_t - z_t), && (16) \\ R_t &= \rho_R R_{t-1} + (1-\rho_R)(\psi_1 \pi_t + \psi_2(x_t - z_t)) + \varepsilon_{R,t}, && (17) \end{aligned}$$

where x is output, π is inflation and R the nominal interest rate. π and R are expressed in deviation from the steady state, and x in deviation from the steady state trend path. The model admits 3 shocks: (i) a demand shock, g, that can be interpreted as a time-varying government spending shock or a preference shock; (ii) a shock to the marginal costs of production, z; (iii) a monetary policy shock, ε. The model and the notation are exactly the same as the one in the seminal paper by LS, that is the natural paper to compare the results of our methodology. The first equation is the New Keynesian IS curve (NKIS), that relates the dynamics of the output x_t to the real interest rate, given by the nominal interest rate, R_t, minus expected inflation, $E_t(\pi_{t+1})$. The dynamics of the inflation rate π_t are described by the second equation, the New Keynesian Phillips curve (NKPC). The NKIS and the NKPC come from the maximization problem of the households and the firms, and they are found loglinearizing, around the steady state, the respective first order conditions. A standard Taylor rule with inertia closes the model. It describes how the central bank conducts the monetary policy, moving the nominal interest rate R_t, in response to the deviations of inflation and output gap from their targets.

As in LS, we also suppose that the shocks in the NKIS and in the NKPC are autocorrelated, that is:

$$g_t = \rho_g g_{t-1} + \varepsilon_{g,t}; \qquad z_t = \rho_z z_{t-1} + \varepsilon_{z,t} \tag{18}$$

and we allow for non-zero correlation, ρ_{gz}, between the two innovations $\varepsilon_{g,t}$ and $\varepsilon_{z,t}$. The standard deviations of the zero-mean innovations $\varepsilon_{g,t}, \varepsilon_{z,t}$ and $\varepsilon_{R,t}$ are denoted σ_g, σ_z and σ_R, respectively.

The parameters of the model are also standard: $\beta \in (0,1)$ is the households' subjective discount factor, τ is the elasticity of intertemporal substitution in consumption, κ is the slope of the NKPC, that ultimately depends on the degree of nominal price stickiness and the labour supply elasticity, ρ_R is the inertial parameter in the Taylor rule while ψ_1 and ψ_2 measure the response of the nominal interest rate to the inflation and the output targets, respectively.

The model has five variables: three predetermined (R_t, g_t and z_t) and two non predetermined (x_t, π_t). Then, the matrix $\mathbf{b}_t$ has dimension two. We also know that among the five eigenvalues of the dynamic system, three of them are inside the unit circle (because ρ_g, ρ_z, and ρ_R are less than one in absolute value), and one is always outside the unit circle (for sensible values of the parameters, see Bullard and Mitra, 2002). The remaining eigenvalue can be inside or outside the unit circle, depending on the following condition (i.e. the Taylor principle):

$$\psi_1 > 1 - \frac{1-\beta}{\kappa}\psi_2. \tag{19}$$

The literature usually imposes the stability criterion to select valid equilibria and thus it distinguishes two possible cases. If (19) holds, the model has two eigenvalues greater than one in absolute value. This is the case of "determinacy": there is a unique *stable* RE equilibrium, i.e. the forward looking one, because the number of eigenvalues outside the unit circle is equal to the number of non predetermined variables. Otherwise, if (19) does not hold, there will be an infinite number of *stable* RE equilibria and this case is normally labelled "indeterminacy".

Note, however, that in both cases, due to the presence of at least one unstable eigenvalue, there is an infinite number of *unstable* RE equilibria that the literature usually does not consider because it imposes the stability criterion.

We can test the validity of the stability criterion in a particular sample comparing the relative performance of the New Keynesian model, under different hypotheses on the set of valid solutions: stability, for the cases both of determinacy and indeterminacy, and instability. Hence, we compare two assumptions: one in which the stability criterion is imposed, and one in which we also consider solutions excluded by the same criterion. The aim is to let the data speak about their preferred assumption.

3.0.1 Model M_S: the subset of stable solutions

When the stability criterion is imposed, we exclude unstable solutions. We label this case as model M_S, and the matrix $\mathbf{b}_t$ is:

$$\begin{aligned}\mathbf{b}_t &= \begin{bmatrix} b_{1,t} & 0 \\ 0 & 1 \end{bmatrix} \\ b_{1,t} &= \begin{cases} 1 & \text{if } \psi_1 > 1 - \frac{1-\beta}{\kappa}\psi_2 \\ b_{1,t-1} + \zeta_t & \zeta_t \sim N(0, \sigma_\zeta^2) \quad \text{otherwise.} \end{cases}\end{aligned}$$

The south east element in $\mathbf{b}_t$ is imposed to be 1 because, in the matrix A of the Blanchard - Kahn canonical form, there is always one "explosive" eigenvalue. For the first element, $b_{1,t}$, instead, we distinguish the two cases described above. $b_{1,t}$ is automatically posed equal to one, when the Taylor principle is satisfied, because the eigenvalue is outside the unit circle and we need to select the forward looking solution. $b_{1,t}$, instead, follows a random walk driven by a sunspot shock, when the Taylor principle is not satisfied, because the eigenvalue is then inside the unit circle and thus there is an infinite number of stable solutions.

3.0.2 Model M_U: a subset of unstable solutions

In this case, stability criterion is not imposed. We define the matrix $\mathbf{b}_t$ as:

$$\begin{aligned}&\mathbf{b}_t = b_{1,t} I \\ &b_{1,t} = b_{1,t-1} + \zeta_t \qquad \zeta_t \sim N(0, \sigma_\zeta^2)\end{aligned}$$

The set of solutions considered does not contain the stable set allowed in M_S. The only intersection between the two cases is the forward looking solution when $b_{1,t} = 1$, that is the unique possibility for the stability criterion to hold in this case. The next section explains the method used to compare the two assumptions just presented.

4 Econometric Strategy

We estimate the parameters of the New Keynesian model (15) - (17), and the latent process $b_{1,t}$ using Bayesian methods. The assumption of a time varying $b_{1,t}$ implies that the likelihood of the model is non Gaussian. For this reason we make our inference using particle filtering. In the estimation of non linear or non Gaussian DSGE models it is common to use a particle filter just to approximate the likelihood function within a MCMC approach, being the Kalman filter non available (see Fernandez-Villaverde and Rubio-Ramirez, 2005). We depart from this tradition, and use a Sequential Monte Carlo method with parameter learning, based on Carvalho, Johannes, Lopes and Polson (2010), that allows us to make sequential inference on the parameters and on the latent process $b_{1,t}$. Chen, Petralia and Lopes (2010) show how this approach is a valid alternative to MCMC in estimating DSGE models. In our case, the technique is particularly useful in order to understand how the inference evolves over time, and to compare the stable (M_S) and the unstable model (M_U).

4.1 The method

The purpose is to approximate the joint posterior distribution:

$$f\left(\vartheta_{0:T}, b_{0:T}, \varphi, \nu | y_{1:T}\right) \tag{20}$$

where ϑ_t is a vector with all the latent processes except b_t, φ is the vector with all the parameters except the variances of the shocks, that are collected in ν, and y_t is a vector with the observed data at time t. The subscript $j : h$ indicates the history of a variable from time j to h, for example $y_{1:t} = \{y_1, y_2, ...y_t\}$.

In Monte Carlo simulations the target distribution is numerically approximated by a sufficiently large number of draws (particles) from the same distribution. Since we are not able to draw directly from (20), we use an importance sampling technique: the idea is to draw the particles from another distribution $q\left(\vartheta_{0:T}, b_{0:T}, \varphi, \nu\right)$, called importance distribution, and to approximate the target density (20) by assigning appropriate weights to each particle. If the support of $f\left(\vartheta_{0:T}, b_{0:T}, \varphi, \nu | y_{1:T}\right)$ is included in the support of $q\left(\vartheta_{0:T}, b_{0:T}, \varphi, \nu\right)$, then for each particle i the appropriate weight is:

$$w^{(i)} = \frac{f\left(\vartheta_{0:T}^{(i)}, b_{0:T}^{(i)}, \varphi^{(i)}, \nu^{(i)} | y_{1:T}\right)}{q\left(\vartheta_{0:T}^{(i)}, b_{0:T}^{(i)}, \varphi^{(i)}, \nu^{(i)}\right)} \tag{21}$$

Notice that we can specify the problem in a recursive way: under standard assumptions[16] the weights can be recursively updated:

$$w_t^{(i)} \propto w_{t-1}^{(i)} \frac{f\left(y_t | \vartheta_t^{(i)}, b_t^{(i)}, \varphi^{(i)}, \nu^{(i)}\right) f\left(\vartheta_t^{(i)}, b_t^{(i)} | \vartheta_{t-1}^{(i)}, b_{t-1}^{(i)}, \varphi^{(i)}, \nu^{(i)}\right)}{q\left(\vartheta_t^{(i)}, b_t^{(i)}, \varphi^{(i)}, \nu^{(i)}\right)} \tag{22}$$

so that, if we have an approximation of the target distribution at time $t-1$, we can obtain the approximation at t drawing from the so called importance transition density $q\left(\vartheta_t^{(i)}, b_t^{(i)}, \varphi^{(i)}, \nu^{(i)}\right)$, and using equation (22).

[16] We assume that the latent processes are Markov chains, and that the standard dependence structure for state space models applies.

The design of the particle filter ultimately consists in choosing a convenient importance distribution. In the rest of the Section we describe the main aspects of our choice, and we refer to the Appendix for the details and the algorithm.

The optimal choice for the importance transition density would be the conditional distribution of the unknowns, given the observed data[17], but it is not available, the model being non Gaussian. However, we can get close to the optimal density noting that, conditional on b_t and on the parameters, the model is linear and Gaussian, and the optimal transition density for ϑ_t simply corresponds to the standard posterior distribution computed with the Kalman filter. To get particles for b_t, instead, we can simply use its prior distribution.

The parameters of the model are also estimated recursively. We divided the parameters in two sets, collecting the variances in ν and all the other parameters in φ. The reason is that for ν we are able to characterize the posterior distribution using a set of sufficient statistics s_t. In the Particle Learning approach $s_t = S(s_{t-1}, \vartheta_t, b_t, \vartheta_{t-1}, b_{t-1}, y_t)$ is a random variable that can be recursively updated, and it can be added to the latent vector. For each draw of ϑ_t, b_t, we can update the sufficient statistics at time t, and obtain a draw of ν from its posterior distribution.

We are, however, not able to use the Particle Learning procedure for the parameters in φ, and we use the method proposed by Liu and West (2001), approximating the posterior of φ using mixtures of Normals. Liu and West (2001) build their approach on the auxiliary particle filter proposed by Pitt and Shephard (1999), in which an importance density function for the predictive distribution is used to preselect the particles that have the best forecasting ability. This resample - propagate scheme leads to higher efficiency.

Our econometric strategy is summarized by the importance transition density:

$$\underbrace{q\left(y_t|\vartheta_{t-1}^{(i)}, b_{t-1}^{(i)}, m^{(i)}, \nu^{(i)}\right)}_{\text{Predictive distribution}} \quad \underbrace{q\left(b_t|b_{t-1}^{(i)}, \nu^{(i)}\right)}_{\text{Prior of } b_t} \quad \underbrace{N(\varphi; m^{(i)}, h^2\Sigma)}_{\text{Normal distribution for } \varphi} \cdot \\ \cdot \underbrace{q\left(\vartheta_t|\vartheta_{t-1}^{(i)}, b_t^{(i)}, \varphi^{(i)}, \nu^{(i)}, y_t\right)}_{\text{Optimal transition density for } \vartheta_t} \underbrace{f\left(\nu^{(i)}|s_t^{(i)}\right)}_{\text{Posterior of } \nu} \tag{23}$$

where $m^{(i)}$, h and Σ are the parameters of the Normal distribution, as in the Liu and West (2001) filter (see the Appendix). The essence of the estimation algorithm can be read from our proposal density: at each time we use the importance density for the predictive distribution to preselect the best particles in terms of forecasting ability; then,

[17] This importance distribution is optimal in the sense that it minimizes the variance of our Monte Carlo estimator.

we proceed propagating each particle by drawing a value of b_t from its prior distribution, and a value for the parameters in φ using the Normal distribution suggested by Liu and West (2001); given these values we can draw the other latent states in ϑ_t from the optimal transition, and we update the sufficient statistics; finally, we draw the parameters in ν from its posterior distribution.

In the practical implementation, additional problems can occur, related to particle degenerations, as it is common in the particle filtering literature. For this reason, sometimes an additional resampling step is needed. We refer to the Appendix for the detailed algorithm.

4.2 Sequential model monitoring

Sequential Monte Carlo methods can be used to compare different models over time, checking which specification is preferred in terms of predictive likelihood. This is achieved using the sequential Bayes factor by West (1986): at each time we can compute the predictive likelihood for each of the two models M_S and M_U, and the likelihood ratio:

$$H_t = \frac{f(y_t|y_{0:t-1}, M_S)}{f(y_t|y_{0:t-1}, M_U)} \ . \tag{24}$$

We asses the relative predictive performance of the most recent κ observations, computing the so called cumulative Bayes factor: $W_t(\kappa) = H_t H_{t-1} ... H_{t-\kappa+1}$, where the parameter κ controls the length of the window in which the two models are compared.

5 Empirical Results

5.1 Data and subsamples

To compare our results with the seminal work by LS, we estimate the New Keynesian model (15) - (17) on the same quarterly postwar data for inflation, output and nominal interest rate used by LS, as available from the AER website. Inflation and interest rates are annualized, and the HP filter is used to get a measure of the output gap.[18]

Figure 1 plots the inflation series. As it is clear, from the mid Sixties until the end of the 70s, the U.S. experienced a period of price instability, also known as "Great Inflation". Then, the Volcker disinflation took place and prices came back under control: inflation became low, as did the volatility of prices and of other macroeconomic variables.

[18]As from footnote 9 at p. 202 in LS: (i) output is log real per capita GDP HP detrended over the period 1955:I to 1998:IV; (ii) inflation is annualized percentage change of CPI-U; (iii) Nominal interest rate is the average Federal Funds Rate in percent.

By contrast to the previous period, these times are known as the "Great Moderation". One popular explanation of this shift through the lens of the New Keynesian model (e.g., Clarida et al, 2000) ascribes it to the shift in the monetary policy conducted: from a passive (i.e., (19) not satisfied) to an active (i.e., (19) satisfied) monetary policy. As we previously underlined, this interpretation excludes a priori unstable paths, even though inflation exceeded 15%. Here we want to answer the following question: would the data prefer an explanation of the Great Inflation based on a stable system with sunspot shocks, as in LS, or one based on unstable dynamics?

Again we closely follow LS in considering two subsamples: the pre-Volcker period, from 1960:I to 1979:II, and a post-82 period from 1982:IV to 1997:IV.[19]

5.2 Priors

Table 1 collects the prior distributions for the parameters. We chose them in accordance with LS, in the same spirit that we chose the model specification and the data. While we refer to LS for the detailed description of each prior, we focus on the few differences we have.

The variance of the parameter ψ_1 is 0.25 in LS, while we increase it to 1. This parameter determines if the Taylor principle is respected, and when we impose stability (model M_S) it draws the line between determinacy and indeterminacy. In our sequential estimation we found it useful to have a wider prior, in order to not to weight too much the first part of the sample in determining which of the two alternatives is preferred.

We specify the prior for the variance covariance matrix of the shock $\varepsilon_{g,t}$ and $\varepsilon_{g,t}$ as an Inverse Wishart with scale matrix and degrees of freedom as in Table 1. The Inverse Wishart prior allows us to update the posterior of the parameters using sufficient statistics, as in the Particle Learning approach described above. This is a big advantage in terms of the efficiency of our particle filter. On the other hand, our choice is very similar to the one of LS in terms of mean and variances of the three parameters involved (σ_g, σ_z and ρ_{gz}).

The variance of our sunspot shock is distributed as an Inverse Gamma with mean and variance both equal to 0.005. This value is lower than the one in LS because our sunspot shock enters in a multiplicative way.

Finally, the process $b_{1,t}$ at $t = 0$ is supposed to be Normally distributed, with mean 1, and variance 0.005, in accordance to the prior of the sunspot shock.

[19] As in LS, we exclude the Volcker disinflation period where monetary policy is characterized by nonborrowed-reserve targeting rather than by an interest rate rule.

Table 1

Prior Distributions			
Parameter	Density	Mean	Variance
ψ_1	Gamma	1.1	1
ψ_2	Gamma	0.25	0.15^2
ρ_R	Beta	0.5	0.2^2
π^*	Gamma	4	4
r^*	Gamma	2	1
κ	Gamma	0.5	0.2
τ^{-1}	Gamma	2	0.5^2
ρ_g	Beta	0.7	0.1^2
ρ_z	Beta	0.7	0.1^2
σ_R^2	Inverse Gamma	0.31^2	0.16^2
σ_b^2	Inverse Gamma	0.005	0.005
Variance Covariance	Density	Scale	Degrees of freedom
Σ_{gz}	Inverse Wishart	$3\begin{bmatrix} 0.4^2 & 0 \\ 0 & 1.2^2 \end{bmatrix}$	5

5.3 Estimation results

Table 2 reports the estimates of the parameters in the two subsamples. For each subsample, Table 2 shows the estimates for both the stable (M_S) and the unstable (M_U) model and, for comparison, the correspondent estimates in the paper by LS (see Table 3, p. 206).

5.3.1 Great Inflation subsample

The model under stability: M_S Let us first analyze the results for the model under stability (model M_S) where we impose the stability criterion. Contrary to LS, however, our methodology allows us not to impose a determinate or an indeterminate equilibrium prior to the estimation, but lets the data choose which one to select during the estimation. Despite this, Table 2 shows that under stability (model M_S) our methodology recovers results very similar to LS. We interpret this finding as corroborating our estimation methodology.

The point estimates of the policy rule parameters are very close and statistically indistinguishable from the ones in LS, as visualized in Figure 2, that displays our prior and posterior distributions and the 90% intervals in LS.[20]

[20] The 90 percent intervals do not overlap only for the slope of the Phillips Curve, κ, and of the

Table 2						
Estimates						
	Pre- Volcker 1960:I - 1979:II			Post-82 1982:IV - 1997:IV		
Parameter	M_S	M_U	LS	M_S	M_U	LS
ψ_1	0.77 [0.68 0.87]	0.31 [0.19 0.49]	0.77 [0.64 0.91]	2.18 [1.33 3.41]	0.42 [0.12 1.08]	2.19 [1.38 2.99]
ψ_2	0.2 [0.13 0.31]	0.22 [0.16 0.34]	0.17 [0.04 0.30]	0.33 [0.14 0.72]	0.44 [0.30 0.70]	0.30 [0.07 0.51]
ρ_R	0.69 [0.61 0.76]	0.54 [0.47 0.66]	0.60 [0.42 0.78]	0.85 [0.79 0.89]	0.78 [0.72 0.83]	0.84 [0.79 0.89]
π^*	1.83 [1.34 2.38]	4.03 [2.52 5.94]	4.28 [2.21 6.21]	3.73 [3.20 4.32]	2.86 [2.19 3.59]	3.43 [2.84 3.99]
r^*	1.41 [1.16 1.86]	1.42 [1.07 2.11]	1.13 [0.63 1.62]	3.51 [2.88 4.22]	2.72 [1.94 3.49]	3.01 [2.21 3.80]
κ	0.12 [0.09 0.17]	0.09 [0.07 0.12]	0.77 [0.39 1.12]	0.53 [0.31 0.90]	0.18 [0.13 0.25]	0.58 [0.27 0.89]
τ^{-1}	3.38 [2.54 4.21]	3.07 [2.49 3.59]	1.45 [0.85 2.05]	1.47 [0.96 2.40]	2.46 [1.71 3.42]	1.86 [1.04 2.64]
ρ_g	0.74 [0.70 0.77]	0.76 [0.73 0.79]	0.68 [0.54 0.81]	0.85 [0.77 0.91]	0.75 [0.68 0.8]	0.83 [0.77 0.89]
ρ_z	0.82 [0.78 0.85]	0.84 [0.79 0.87]	0.82 [0.72 0.82]	0.77 [0.63 0.88]	0.74 [0.66 0.80]	0.85 [0.77 0.93]
ρ_{gz}	0.12 [0.09 0.17]	0.06 [0.04 0.09]	0.14 [−0.4 0.71]	0.03 [0.01 0.06]	0.005 [−0.015 0.027]	0.36 [0.06 0.67]
σ_R	0.21 [0.19 0.25]	0.16 [0.14 0.19]	0.23 [0.19 0.27]	0.17 [0.14 0.21]	0.12 [0.10 0.14]	0.18 [0.14 0.21]
σ_g	0.20 [0.18 0.24]	0.16 [0.14 0.19]	0.27 [0.17 0.36]	0.14 [0.11 0.18]	0.14 [0.11 0.17]	0.18 [0.14 0.23]
σ_z	0.82 [0.69 1.00]	0.62 [0.54 0.74]	1.13 [0.95 1.30]	0.57 [0.49 0.69]	0.52 [0.40 0.71]	0.64 [0.52 0.76]
σ_ς	0.05 [0.04 0.06]	0.07 [0.04 0.09]	0.20 [0.12 0.27]	–	0.04 [0.03 0.06]	–

90% credibility interval in brackets

Hence, in accordance with the literature, our method also points to indeterminacy as the most plausible explanation of the Great Inflation period once the stability criterion is imposed on the model. It suggests that the Fed did not respect the Taylor principle, and thus movements in inflation (and output) were due to shifts in expectations due to sunspots shocks. The estimated standard deviation of the sunspots for M_s shock is lower (one fourth) than the one estimated by LS. The same is true for the standard of the technology shock, σ_z, which is significantly lower in our estimates. This is because our sunspot is a multiplicative sunspots shock that interacts and amplifies the structural shocks, rather than an additive one as in LS's approach. Hence, these standard deviations are not really comparable due to the different assumption about how the sunspot

elasticity of intertemporal substitution, τ^{-1}.

affects the model.

Figure 3 displays the transmission mechanism of the structural shocks, by showing the generalized impulse response functions (GIRFs) and their 90% intervals of R, x and π to the structural shocks: to the monetary policy shock in the first row, to the demand shock in the second row and to the supply shock in the third row.[21] These GIRFs are very similar in shape to the ones of a determinate equilibrium.[22] Note that the technology shock is the only one that moves output and inflation in opposite directions, as required to explain the stagflation episode during the last part of the Great Inflation period. This explains why both for M_S and LS, the standard deviation of the technology shocks is much bigger than the other shocks.

Recall that the non-linear multiplicative sunspot shock affects the model only in the presence of a structural shock. Hence, to understand how the sunspots shock affects the transmission mechanism of our model, we plot in Figure 4 the GIRFs for two different values of b: 1.3 and 1.5, respectively. The sunspot shock interacts with the structural shocks, amplifying the effects of the latter, thereby acting similarly as a stochastic volatility shifter (Justiniano and Primiceri, 2008). We interpret this shock as a shift in the way people form expectations after a structural shock hits the economy.

We think that one of the most interesting aspects of our methodology is the estimated path for $b_{1,t}$ that measures of how much expectations deviate from the standard forward looking rational expectations solution. Recall that when $b_{1,t} = 1$ then expectations are selecting the forward looking solution, otherwise they are selecting a combination of the backward and forward looking ones. Figure 5 shows the estimated path for $b_{1,t}$ in the case of M_s, and the corresponding sequential estimate of the policy parameter ψ_1. Figure 5 clearly depicts the challenge faced by the New Keynesian model in this subsample: to explain the stable output and inflation paths in the first part of the subsample and then the stagflation in the second part of the subsample, where output and inflation move in opposite directions, and inflation accelerates. Up the first oil shock, the estimate of $b_{1,t}$ points toward expectations aligned on the "standard" forward looking solution and, correspondingly, ψ_1 is estimated to satisfy the Taylor principle. Until to that point the data would favour a determinate stable model. However, such a model cannot explain the data in the second part of the subsample. Then, the data switch to favour the only alternative model available under stability: a model with a sunspot shock. The extra degree of freedom provided by the sunspot makes the data choose the indeterminate model both in our and in LS's estimation, despite the fact that the structural dynamics

[21] To construct the GIRFs, we take each IRFs generated by each particle (which is just a vector of parameter values and b) and then we average across the IRFs of the 500,000 particles.

[22] Thees are also very similar to the IRFs in LS under their prior 2.

of the model are a priori at odds with the data, as we argued above. $b_{1,t}$ drifts away from 1, when inflation starts to grow in the data.

Of course, another plausible possibility explored in the literature to make a stable determinate model able to match such behaviour in the data would be to have a stochastic volatility model, where the standard deviation of technology shocks increases in the second part of the subsample (e.g. Justiniano and Primiceri, 2008). Our multiplicative sunspot shock yields a similar effect, as explained above, but the sunspot shock occurs only if the model is indeterminate given our assumptions on imposing the stability criterion.

The model under instability: M_U The model M_U instead does not impose the stability criterion, and hence it makes the data consider also *unstable* rational expectations trajectories. The estimation points towards an even smaller reaction of monetary policy to inflation.[23] However, it should by now be clear to the reader that this does not imply indeterminacy as usually intended in the literature, that is, an infinite number of *stable* trajectories. It does imply another sort of indeterminacy, in the sense that we let the data choose among an infinite number of rational expectations trajectories. However, these trajectories will all be unstable in our case, irrespective of whether the Taylor principle is satisfied or not. (19) is a condition for one eigenvalue to be inside or outside the unit circle, but whatever the value of ψ_1, there is always an unstable eigenvalue. As explained in Section 3, in the case $M_{U,}$ we do not impose the stability criterion with respect to this unstable eigenvalue, that is, we do not force the model to the forward looking solution. It follows that, despite the parameter estimates being very similar between the two cases M_U and M_S, M_U gives a completely different interpretation about the instability of that period. Independently from the Fed policy, the dynamics of M_U are structurally unstable.

Figure 6 shows the GIRFs in this case. Again a supply shock generates stagflation. Most importantly, however, stagflation could also now be generated by a monetary policy shock. In particular, a contractionary monetary policy shock can be inflationary: inflation drops on impact but then starts rising and it is above steady state after 4 quarters. Interestingly, a somewhat similar behaviour is highlighted in LS under their preferred Prior 1: *"an increase in the nominal interest rate can have a slightly inflationary effect"* (p. 207, see Figure 3, p. 208 and the discussion at p. 207-208 therein). They conclude that *"before 1979 indeterminacy substantially altered the propagation*

[23] The other parameter estimates are very similar between the two cases. M_U implies a slightly smaller standard deviation of the technology shock and a slightly higher standard deviation of the sunspot shock.

*of shocks" (*abstract). Similarly, instability in our framework substantially alters the transmission mechanism. However, in our case, output remains below steady state, so that a monetary policy shock could generate an opposite response of output and inflation. In LS case, instead, inflation and output move in the same way after a monetary policy shock: after dropping on impact, they *both* become slightly positive. Our framework therefore seems to be able to provide a transmission mechanism more prone to accommodate stagflation.

This is also true regarding the effects of the sunspot shock. The impulse response function to a sunspot shock in the case of the indeterminate model in LS does imply (again) that output and inflation move in the same direction (see Figure 2, p. 207), not in an opposite one, as in a stagflation episode. Intuitively, if a sunspot shock leads to a self-fulfilling increase in inflation, then the real interest rate decreases, due to the passive monetary policy, and thus output increases, rather than decreases. Thus the structural dynamics implied by an indeterminate stable model do not seem to be well suited to explain stagflation episodes after an additive sunspot shock. In our setup, instead, the non-linear sunspot shock amplifies the responses of the model to a structural shock, as explained above and shown in Figure 7.

The data could, however, still choose a stable forward looking solution when $b_{1,t}$ is estimated to equal one. Figure 8 shows the estimated path for the latent process $b_{1,t}$ under model M_U. Similarly as before, it initially fluctuates around 1 (the 90 percent interval is centered around 1) and then it drifts from 1 (now outside the 90 percent interval), exactly when inflation starts increasing and drifting away from its steady state value from 3% to 15%. If we allow unstable paths, the estimation then unambiguously selects those to explain the data in this period.

It is possible to compare the relative fit of the stable (M_S) and unstable (M_U) models by computing the Sequential Bayes factor as in West (1986). Figure 9 shows twice the natural logarithm of the Sequential Bayes factor (as suggested by Kass and Raftery, 1995) together with the path of inflation in the sample for a 10 year window. Model M_S is at the numerator and model M_U is at the denominator, so that a value of zero of the logarithm of the cumulative Bayes factor means that the two models have the same performance in terms of predictive likelihood; while a positive value means that M_S is preferred (and vice versa for negative values). The advantage of the Sequential Bayes factor, with respect to the conventional measures in Bayesian Econometrics, is that we can compare two models over time, and verify the sub-periods in which a model performs better than another in terms of predictive likelihood. In our specific case, as expected, the unstable model is much preferred from the '70s onwards, when inflation starts drifting away reaching high values. According to the Kass and Raftery (1995, p.

777) classification, there is "very strong" evidence in favour of M_U from the beginning of the '70s. In particular, the Sequential Bayes factor reaches a very low level from first quarter of 1975 onwards.

Our methodology allows the data to choose between different possible alternatives: determinacy, indeterminacy and instability. When the data are allowed this possibility, they unambiguously select the unstable model to explain the stagflation period in the '70s.

5.3.2 Post-82 Subsample

In the second subsample, our estimates under stability again reproduce the same results as in LS. There is no difference between our parameter estimates and the ones in LS, again signalling the reliability of our estimation methodology (see Figure 10). The Taylor principle is satisfied and hence the data choose the unique determinate forward looking solution under M_S, there is no sunspot shock and the process for $b_{1,t}$ degenerates to the value of one.

In the case of model M_U, on the other hand, the estimation yields results similar to the Great Inflation subsample (see Table 2). The Taylor principle is not satisfied, so the model is either on unstable paths or on one of the stable trajectories under indeterminacy, when $b_{1,t}$ is equal to 1. Figure 11 shows that $b_{1,t}$ is always inside the 90% probability interval for the whole period, meaning that the estimation cannot exclude a stable trajectory.[24]

Comparing the two models as in the previous case using the Sequential Bayes factor presents mixed evidence (see Figure 12). While the stable model is mostly preferred, there is no strong evidence in favour of either one of the models.

6 Conclusion

We propose a novel way to introduce sunspots in a RE model to take into account the possibility of unstable trajectories, through a simple generalization of the standard Muth (1961) RE framework. First, we show how all the possible solutions could be parameterized by one single parameter that has a natural interpretation as the way agents weight past data to form their RE. Then, we introduce rational sunspots by assuming that this parameter is a random variable, so that agents select one of the possible RE

[24]Note that, under M_U, the model will never recover a constant value of $b_{1,t}$ equal to 1, as it does under M_S when the Taylor principle is satisfied and the stability criterion is imposed. The estimation under M_U instead models $b_{1,t}$ as a random walk and thus subject to shocks. A path (and 90 percent interval) as in Figure 8 is statistically indistinguishable from $b_{1,t} = 1$.

fundamental solutions. Third, we propose an empirical methodology that allows the data to choose among the different RE alternatives: determinacy, indeterminacy and instability. Finally, we apply this approach to the data to explain US inflation dynamics in the Great Inflation and Great Moderation period. The empirical evidence suggests that the Great Inflation in the U.S. can be explained by temporary unstable paths, while the usual practice of excluding a priori unstable solutions seems not to be supported by the data. When allowed, the data unambiguously select the unstable model to explain the stagflation period in the '70s. Our framework provides a different interpretation of the Great Inflation from a policy perspective. Despite our estimates point to a passive monetary policy behaviour in the 70's, our framework implies that this is not the cause in itself of unstable inflation dynamics, that was instead due to drifting expectations, independently from the stand of monetary policy.

Our analysis therefore suggests that unstable paths can be empirically relevant, also within the context of rational expectations. This result may call for a rethinking of the stability criterion as the selection mechanism among all the possible RE paths, and for theoretically considering the possibility that RE could push the economy to walk along unstable paths, at least temporarily.

This line of research is still in its infancy and can be expanded in many directions. First, one direction would be to endogenize the process for the rational sunspot. The process for the drifting expectations is taken as exogenous in this paper (as in the sunspot literature) and then estimated on the data. However, we would like to be able to say something about why agents RE start to drift, by endogenizing this expectation formation process and then estimating it on the data, in a spirit similar to the escape dynamics literature put forward by Sargent. Second, another possible extension would be to consider processes for b that admit only temporary explosion as the *asymptotically equivalent stationary* paths proposed in Gourieroux et al. (1982) or the Markov-Switching framework. Third, there are many potential application of our framework, notably, but not exclusively, finance, where boom and bust episodes of asset prices (stock, houses, etc..) is a pervasive phenomenon.

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Appendix

The solution for the simple model

Consider equation (3) in the paper:

$$\pi_t = \frac{1}{\phi} E_t \pi_{t+1} + e_t \tag{25}$$

$$e_t = -\frac{1}{\phi}\varepsilon_t \qquad \varepsilon_t \sim i.i.d.N(0,\sigma_\varepsilon^2) \tag{26}$$

In this appendix we treat the general case with a time varying b_t (then, the case with b_t constant is simply obtained). Suppose that

$$b_t = b_t(\zeta_t) \tag{27}$$

where ζ_t is a random variable, called sunspot shock, orthogonal to the fundamental shocks e_s $(s = 1, 2, ...)$ and such that $E_t \zeta_t = 0$ $\forall t$.

Following Muth (1961) and Blanchard (1979) we guess the solution for model (25):

$$\pi_t = \sum_{j=1}^{\infty} u_{j,t} e_{t-j} + b_t e_t + \sum_{j=1}^{\infty} c_{j,t} E_t e_{t+j} \tag{28}$$

where $u_{j,t}$, b_t and $c_{j,t}$ are coefficients to be determined. Hence verify using undetermined coefficients:

$$\begin{aligned} \pi_t &= \frac{1}{\phi} E_t \pi_{t+1} + e_t \\ \sum_{j=1}^{\infty} u_{j,t} e_{t-j} + b_t e_t + \sum_{j=1}^{\infty} c_{j,t} E_t e_{t+j} &= \frac{1}{\phi} E_t \left(\sum_{j=1}^{\infty} u_{j,t+1} e_{t+1-j} + b_{t+1} e_{t+1} + \sum_{j=1}^{\infty} c_{j,t+1} E_t e_{t+1+j} \right) + e_t \end{aligned}$$

that is:

$$\begin{aligned} & u_{1,t} e_{t-1} + u_{2,t} e_{t-2} + u_{3,t} e_{t-3} + ... + b_t e_t + c_{1,t} E_t e_{t+1} + c_{2,t} E_t e_{t+2} + ... \\ = & \frac{1}{\phi} E_t \left(u_{1,t+1} e_t + u_{2,t+1} e_{t-1} + u_{3,t+1} e_{t-2} + ... + b_{t+1} e_{t+1} + c_{1,t+1} e_{t+2} + c_{2,t+1} e_{t+3} + ... \right) + e_t \end{aligned}$$

equal coefficients to find an expression for the $u's$:

$$\begin{aligned} e_t &: \qquad b_t = \frac{1}{\phi} E_t u_{1,t+1} + 1 \Rightarrow E_t u_{1,t+1} = \phi(b_t - 1) \\ e_{t-1} &: \qquad u_{1,t} = \frac{1}{\phi} E_t u_{2,t+1} \Rightarrow E_t u_{2,t+1} = \phi u_{1,t} \\ &\vdots \\ e_{t-j} &: \qquad u_{j,t} = \frac{1}{\phi} E_t u_{j+1,t+1} \Rightarrow E_t u_{j+1,t+1} = \phi u_{j,t} \end{aligned}$$

and for the $c's$:

$$\begin{aligned}
e_{t+1} &: & c_{1,t} &= \frac{1}{\phi} E_t b_{t+1} \\
e_{t+2} &: & c_{2,t} &= \frac{1}{\phi} E_t c_{1,t+1} \\
&\vdots \\
e_{t+j+1} &: & c_{j+1,t} &= \frac{1}{\phi} E_t c_{j,t+1}
\end{aligned}$$

These equations need an assumption on the stochastic process governing b_t to be satisfied. Otherwise, in general the system can not be solved.

Note that if b_t is constant, then the solution could be written as:

$$\pi_t = \sum_{j=1}^{\infty} \phi^j (b-1) e_{t-j} + b e_t + \sum_{j=1}^{\infty} \frac{b}{\phi^j} E_t e_{t+j}. \tag{29}$$

Random walk process for b_t

Assume that b_t is following a random walk process as $b_t = b_{t-1} + \zeta_t$, with $\zeta_t \sim i.i.d.N(0, \sigma_\zeta^2)$. Then $E_{t+1} b_{t+1} = b_t$. Hence:

$$r_t: \qquad b_t = \frac{1}{\phi} E_t u_{1,t+1} + 1 \Rightarrow E_t u_{1,t+1} = \phi(b_t - 1)$$

However, given $E_t u_{1,t+1} = \phi(b_t - 1)$ what can we say about $u_{1,t+1}$? Assuming that $u_{1,t+1} = F(b_{t+1})$, the problem then is to find the function F such that $E_t u_{1,t+1} = \phi(b_t - 1)$, given the stochastic process for b_t. Assuming that F is linear then we are looking for a linear function such that $E_t (a_1 b_{t+1} + a_0) = \phi(b_t - 1)$, that is: $a_1 E_t b_{t+1} + a_0 = \phi b_t - \phi \Rightarrow a_1 b_t + a_0 = \phi b_t - \phi \Rightarrow$

$$\begin{aligned}
a_1 &= \phi \\
a_0 &= -\phi
\end{aligned}$$

so

$$u_{1,t+1} = \phi b_{t+1} - \phi \tag{30}$$

Equal coefficients that multiply e_{t-1}:

$$E_t u_{2,t+1} = \phi u_{1,t}$$

Then, $u_{1,t} = \phi b_t - \phi$ needs to be equal to $\frac{1}{\phi} E_t u_{2,t+1}$. Following the same reasoning, assuming $u_{2,t+1}$ is a linear function of b_{t+1}, we need to solve for

$$E_t (a_1 b_{t+1} + a_0) = \phi u_{1,t} = \phi^2 b_t - \phi^2.$$

Then, it must be

$$\begin{aligned} a_1 &= \phi^2 \\ a_0 &= -\phi^2 \end{aligned}$$

so that:

$$u_{2,t+1} = \phi^2 b_{t+1} - \phi^2 \tag{31}$$

generally

$$u_{j,t} = \phi^j b_t - \phi^j \tag{32}$$

Having solved for the $u's$ let's solve for the $c's$. This is easy since $E_{t+1}b_{t+1} = b_t$:

$$e_{t+1}: \qquad c_{1,t} = \frac{1}{\phi} E_t b_{t+1} = \frac{1}{\phi} b_t \tag{33}$$

Following the method implemented above we obtain, in general:

$$c_{j,t} = \frac{1}{\phi^j} b_t \tag{34}$$

Equations (32) and (34) are the coefficients of equation (28), written as function of b_t. Equation (28) is a solution for model (25) only if it satisfies these restrictions.

In our case, because the exogenous shocks are $i.i.d.$ with zero mean, the sum $\sum_{j=1}^{\infty} c_{j,t} E_t e_{t+j}$ in equation (28) is zero. Then, substituting equation (32) we have:

$$\pi_t = (b_t - 1) \sum_{j=1}^{\infty} \phi^j e_{t-j} + b_t e_t \tag{35}$$

so that we have the pure forward looking solution when $b_t = 1$ (equation, 6 in the paper):

$$\pi_t^F = e_t = -\frac{1}{\phi} \varepsilon_t$$

and the pure backward looking solution when $b_t = 0$:

$$\begin{aligned} \pi_t^B &= -\sum_{j=1}^{\infty} \phi^j e_{t-j} = -\phi \sum_{j=1}^{\infty} \phi^j e_{t-j-1} - \phi e_{t-1} \\ &= \phi \pi_{t-1}^B - \phi e_{t-1} = \phi \left(\pi_{t-1}^B - \pi_{t-1}^F \right), \end{aligned}$$

that corresponds to equation (7). Note that equation (35) can be rewritten as:

$$\pi_t = (1 - b_t) \pi_t^B + b_t \pi_t^F \tag{36}$$

that is, each particular solution depends on b_t, and it can be written as a linear combination of the backward and the forward one.

The recursive formulation

We first report the important equations:

$$\begin{aligned} \pi_t &= (1-b_t)\pi_t^B + b_t\pi_t^F && (37)\\ \pi_t^B &= \phi\pi_{t-1}^B - \phi\pi_{t-1}^F && (38)\\ \pi_t^F &= e_t && (39) \end{aligned}$$

substituting π_t^B and π_t^F in the first equation we obtain

$$\pi_t = \phi(1-b_t)\pi_{t-1}^B - \phi(1-b_t)e_{t-1} + b_t e_t \quad (40)$$

Multiply for $(1-b_t)$ equation (38) and substitute in the last equation to find π_t^B:

$$\begin{aligned} (1-b_t)\pi_t^B &= \phi(1-b_t)\pi_{t-1}^B - \phi(1-b_t)e_{t-1}\\ (1-b_t)\pi_t^B &= \pi_t - b_t e_t\\ \pi_t^B &= \frac{\pi_t - b_t e_t}{(1-b_t)} \end{aligned}$$

Use this expression, lagged, in (40) to derive the complete set of solutions for model (25), when $b_{t-1} \neq 1$:

$$\pi_t = \alpha_t\pi_{t-1} - \alpha_t e_{t-1} + b_t e_t \quad (41)$$

with $\alpha_t = \phi\dfrac{(1-b_t)}{(1-b_{t-1})}$. The particular case of $b_t = b$ constant is obtainable offsetting the sunspot shocks, that is imposing $\sigma_\zeta^2 = 0$. The coefficient α_t becomes:

$$\alpha_t = \phi\frac{1-b_{t-1}}{1-b_{t-1}} = \phi$$

and π_t is described by equation (5):

$$\pi_t = \phi\pi_{t-1} + \varepsilon_{t-1} - \frac{b}{\phi}\varepsilon_t$$

Expectations as a weighted average of past observations

Under the rational expectations hypothesis, the expected value in model (25), can be written as a weighted average of the past observations (see Muth, 1961):

$$\begin{aligned} E_t\pi_{t+1} &= \sum_{i=1}^{\infty} V_{i,t}\pi_{t+1-i} = && (42)\\ &= V_{1,t}\pi_t + V_{2,t}\pi_{t-1} + V_{3,t}\pi_{t-2} + ... \end{aligned}$$

where we need to determine the coefficients $V_{i,t}$. Using equation (36) we have:

$$\begin{aligned} E_t\pi_{t+1} &= V_{1,t}\left[(b_t-1)\sum_{j=1}^{\infty}\phi^j e_{t-j}+b_t e_t\right]+ \\ &\quad +V_{2,t}\left[(b_{t-1}-1)\sum_{j=1}^{\infty}\phi^j e_{t-j-1}+b_{t-1}e_{t-1}\right]+ \\ &\quad +V_{3,t}\left[(b_{t-2}-1)\sum_{j=1}^{\infty}\phi^j e_{t-j-2}+b_{t-2}e_{t-2}\right]+... \end{aligned}$$

Rearrange:

$$\begin{aligned} E_t\pi_{t+1} &= V_{1,t}b_t e_t + \\ &\quad +\left[V_{1,t}\left(b_t-1\right)\phi+V_{2,t}b_{t-1}\right]e_{t-1}+ \\ &\quad +\left[V_{1,t}\left(b_t-1\right)\phi^2+V_{2,t}\left(b_{t-1}-1\right)\phi+V_{3,t}b_{t-2}\right]e_{t-2}+ \\ &\quad +... \end{aligned}$$

Then, bring equation (36) one step ahead,

$$\begin{aligned} E_t\pi_{t+1} &= (b_t-1)\sum_{j=1}^{\infty}\phi^j e_{t-j+1} = \\ &= (b_t-1)\left[\phi e_t+\phi^2 e_{t-1}+\phi^3 e_{t-2}+...\right] \end{aligned}$$

and compare coefficients:

$$e_t: \qquad \begin{aligned} (b_t-1)\,\phi &= V_{1,t}b_t \\ V_{1,t} &= \frac{(b_t-1)}{b_t}\phi \end{aligned}$$

$$e_{t-1}: \qquad \begin{aligned} (b_t-1)\,\phi^2 &= \left[V_1\left(b_t-1\right)\phi+V_2 b_{t-1}\right] \\ V_2 &= \frac{(b_t-1)}{b_t b_{t-1}}\phi^2 \end{aligned}$$

$$e_{t-2}: \qquad \begin{aligned} (b_t-1)\,\phi^3 &= \left[V_{1,t}\left(b_t-1\right)\phi^2+V_{2,t}\left(b_{t-1}-1\right)\phi+V_{3,t}b_{t-2}\right] \\ V_{3,t} &= \frac{(b_t-1)}{b_t b_{t-1}b_{t-2}}\phi^3 \end{aligned}$$

in general:

$$V_{i,t}=\frac{(b_t-1)}{\prod_{0}^{i-j=1} b_{t-i}}\phi^i$$

and when $b_t = b$ constant:

$$V_i=\frac{(b-1)}{b^i}\phi^i$$

The multivariate case

b constant

We show how to compute the complete set of solutions of a system with rational expectations.

Consider a system with rational expectations written in the form of Blanchard and Kahn (1980):

$$\begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \boldsymbol{\gamma} Z_t \tag{43}$$

X_t is a $(n \times 1)$ vector of predetermined variables and P_t is a $(m \times 1)$ vector of non predetermined variables. $Z_t \sim i.i.d.\ N(\mathbf{0}, \Sigma)$ is a $(\kappa \times 1)$ vector of exogenous random variables.

Use the Jordan form to rewrite A

$$A = C^{-1} J C.$$

In the main diagonal of J there are the eigenvalues of A, ordered by increasing absolute value. We decompose the matrices C^{-1}, J, C and $\boldsymbol{\gamma}$ as follows:

$$\begin{aligned} C^{-1} &= \begin{bmatrix} B_{11} & B_{12} \\ (n \times n) & (n \times m) \\ B_{21} & B_{22} \\ (m \times n) & (m \times m) \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ (n \times n) & (n \times m) \\ C_{21} & C_{22} \\ (m \times n) & (m \times m) \end{bmatrix}, \\ J &= \begin{bmatrix} J_1 & \mathbf{0} \\ (n \times n) & (n \times m) \\ \mathbf{0} & J_2 \\ (m \times n) & (m \times m) \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 \\ (n \times \kappa) \\ \gamma_2 \\ (n \times \kappa) \end{bmatrix}. \end{aligned}$$

Define

$$\begin{bmatrix} Y_t \\ Q_t \end{bmatrix} = C \begin{bmatrix} X_t \\ P_t \end{bmatrix},$$

and rewrite equation (43) in terms of $\begin{bmatrix} Y_t \\ Q_t \end{bmatrix}$:

$$\begin{bmatrix} E_t Y_{t+1} \\ E_t Q_{t+1} \end{bmatrix} = \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix} \begin{bmatrix} Y_t \\ Q_t \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} Z_t \,. \tag{44}$$

Now consider the second block of equation (44),

$$Q_t = J_2^{-1} E_t Q_{t+1} + \Omega_t \tag{45}$$

where $\Omega_t = -J_2^{-1}(C_{21}\gamma_1 + C_{22}\gamma_2)Z_t$. The system (45) has m disjoined equations, and each of them admits an infinite number of solutions because of the presence of an expected value. Defining $q_{i,t}$ as the i^{th} element of Q_t, and $\omega_{i,t}$ the corresponding disturbance, we write all the solutions of the generic row of equation (45) as

$$q_{i,t} = \sum_{j=1}^{\infty} u_{i,j}\omega_{i,t-j} + b_i\omega_{i,t} + \sum_{j=1}^{\infty} c_{i,j}E_t\omega_{i,t+j} \ . \tag{46}$$

Using matrices instead of scalars the solutions can be rewritten as

$$Q_t = \sum_{j=1}^{\infty} \mathbf{u}_j\Omega_{t-j} + \mathbf{b}\Omega_t + \sum_{j=1}^{\infty} \mathbf{c}_j E_t\Omega_{t+j} \tag{47}$$

where $\mathbf{u}_j$, $\mathbf{b}$ and $\mathbf{c}_j$ are diagonal matrices of coefficients to be determined. Bring equation (47) one step ahead

$$E_tQ_{t+1} = \sum_{j=1}^{\infty} \mathbf{u}_j\Omega_{t+1-j} + E_t\mathbf{b}\Omega_{t+1} + \sum_{j=1}^{\infty} \mathbf{c}_j E_t\Omega_{t+1+j}$$

and substitute in equation (45)

$$Q_t = J_2^{-1}\sum_{j=2}^{\infty} \mathbf{u}_j\Omega_{t+1-j} + J_2^{-1}\mathbf{u}_1\Omega_t - \Omega_t + J_2^{-1}E_t\mathbf{b}\Omega_{t+1} + J_2^{-1}\sum_{j=1}^{\infty} \mathbf{c}_j E_t\Omega_{t+1+j} \ . \tag{48}$$

We find the coefficients comparing the matrices of equation (47) to the ones of equation (48):

$$\begin{aligned}
&\mathbf{b} = J_2^{-1}\mathbf{u}_1 + I \Longrightarrow \boxed{\mathbf{u}_1 = J_2\mathbf{b} - J_2} \\
&\mathbf{u}_1 = J_2^{-1}\mathbf{u}_2 \Longrightarrow \boxed{\mathbf{u}_{j+1} = J_2\mathbf{u}_j} \quad j = 1...\infty \\
&\boxed{\mathbf{c}_1 = J_2^{-1}\mathbf{b}} \\
&\mathbf{c}_2 = J_2^{-1}\mathbf{c}_1 \Longrightarrow \boxed{\mathbf{c}_{j+1} = J_2^{-1}\mathbf{c}_j} \quad j = 1...\infty
\end{aligned}$$

The matrices $\mathbf{u}_j$ and $\mathbf{c}_j$ are functions of $\mathbf{b}$ and J_2, and since J_2 is given, the complete set of solutions is parametrized by $\mathbf{b}$. There are two particular cases: the pure backward looking solution, corresponding to $\mathbf{b} = \mathbf{0}$, that implies $\mathbf{c}_j = \mathbf{0}$ and $\mathbf{u}_j = J_2^j$, $j = 1...\infty$; the pure forward looking solution corresponding to $\mathbf{b} = I$, that implies $\mathbf{u}_j = \mathbf{0}$ and $\mathbf{c}_j = J_2^{-j}$, $j = 1...\infty$. The backward looking solution can be written as follows:

$$Q_t^B = \sum_{j=1}^{\infty} \mathbf{u}_j\Omega_{t-j} \tag{49}$$

$$\begin{aligned}
Q_t^B &= \sum_{j=1}^{\infty} J_2^j\Omega_{t-j} = -J_2\Omega_{t-1} - J_2^2\Omega_{t-2} - J_2^3\Omega_{t-3} - .... \\
Q_t^B &= -J_2\Omega_{t-1} + J_2\left[-J_2\Omega_{t-2} - J_2^2\Omega_{t-3} - J_2^3\Omega_{t-4} - ...\right] \\
Q_t^B &= J_2Q_{t-1}^B - J_2\Omega_{t-1} \qquad \text{(A7)}
\end{aligned}$$

The forward looking solution is

$$Q_t^F = b\Omega_t + \sum_{j=1}^{\infty} \mathbf{c}_j E_t \Omega_{t+j} = I\Omega_t + J_2^{-1} E_t \Omega_{t+1} + J_2^{-2} E_t \Omega_{t+2} + ...$$

and since $E_t\Omega_{t+j} = \mathbf{0} \ \ \forall j \geq 1$, we obtain

$$Q_t^F = \Omega_t \ . \tag{50}$$

Following Blanchard (1979) we write any other solution as a linear combination of the backward and the forward looking solutions. In compact form

$$Q_t = \boldsymbol{\lambda} Q_t^B + (I - \boldsymbol{\lambda}) Q_t^F \tag{51}$$

where $\boldsymbol{\lambda} = I - \mathbf{b}$ is a diagonal matrix. The elements in the main diagonal of $\mathbf{b}$ are such that $\mathbf{b} = \mathbf{0} \Rightarrow Q_t = Q_t^B$, and $\mathbf{b} = I \Rightarrow Q_t = Q_t^F$.

Substitute the equations (A7) and (50) in equation (51)

$$\begin{aligned} Q_t &= \boldsymbol{\lambda}\left(J_2 Q_{t-1}^B - J_2 \Omega_{t-1}\right) + (I - \boldsymbol{\lambda})\Omega_t \\ &= \boldsymbol{\lambda} J_2 Q_{t-1}^B - \boldsymbol{\lambda} J_2 Q_{t-1}^F + J_2 Q_{t-1}^F - J_2 Q_{t-1}^F + (I - \boldsymbol{\lambda})\Omega_t \ . \end{aligned}$$

In the last passage we have added and subtracted $J_2 Q_{t-1}^F$. Since both J_2 and $\boldsymbol{\lambda}$ are diagonal matrices the commutative property holds and we can write

$$\begin{aligned} Q_t &= J_2\left(\boldsymbol{\lambda} Q_{t-1}^B + (I - \boldsymbol{\lambda}) Q_{t-1}^F\right) - J_2\Omega_{t-1} + (I - \boldsymbol{\lambda})\Omega_t \\ Q_t &= J_2 Q_{t-1} - J_2 \Omega_{t-1} + \mathbf{b}\Omega_t \end{aligned} \tag{52}$$

Equation (52) represents the infinite number of solutions for Q_t parametrized by $\mathbf{b}$. The complete set of solutions for model (43) is found using the definition of Q_t and the first n rows of the model written with the Jordan matrices:

$$X_t = \left(B_{11}J_1C_{11} + B_{12}J_2C_{21}\right) X_{t-1} + \left(B_{11}J_1C_{12} + B_{12}J_2C_{22}\right) P_{t-1} + \gamma_1 Z_{t-1} \tag{53}$$

$$\begin{aligned} C_{21}X_t + C_{22}P_t = J_2(C_{21}X_{t-1} + C_{22}P_{t-1}) \quad &+ (C_{21}\gamma_1 + C_{22}\gamma_2)Z_{t-1} + \\ &- \mathbf{b}J_2^{-1}(C_{21}\gamma_1 + C_{22}\gamma_2)Z_t \end{aligned} \tag{54}$$

In the paper we focus on the case in which the matrix A has at least n eigenvalues inside the unit circle. This means that the model admits at least one stable solution. If this condition is not satisfied the equations (53) and (54) continue to represent the complete set of solutions that are all unstable.

Adding sunspots

Add the hypothesis that each element in the main diagonal of $\mathbf{b}$ is described by the following stochastic process:

$$b_{i,t} = b_{i,t-1} + \zeta_{i,t}$$

with $\zeta_{i,t} \sim i.i.d.N(0, \sigma^2_{\zeta_i})$, $i = 1, 2, ...m$. With this hypothesis equation (46) becomes

$$q_{i,t} = \sum_{j=1}^{\infty} u_{i,t}^{(j)} \omega_{i,t-j} + b_{i,t}\omega_{i,t}$$

and its solution is:

$$\begin{aligned} q_{i,t} &= \alpha_{i,t} q_{i,t} + \alpha_{i,t}\omega_{i,t-1} + b_{i,t}\omega_{i,t} \\ \alpha_{i,t} &= J_{2,i}\frac{(1 - b_{i,t})}{(1 - b_{i,t-1})} \end{aligned}$$

where $J_{2,i}$ is the i^{th} eigenvalue in the main diagonal of J_2. Putting in matrix form the system with these m disjoined equations, we obtain the following system, analogous to equation (52):

$$Q_t = J_2 (I + \mathbf{b}_t) (I + \mathbf{b}_{t-1})^{-1} Q_{t-1} - J_2 (I + \mathbf{b}_t) (I + \mathbf{b}_{t-1})^{-1} \Omega_{t-1} + \mathbf{b}_t \Omega_t$$

Finally, the solution is represented by the following system:

$$X_t = (B_{11}J_1C_{11} + B_{12}J_2C_{21}) X_{t-1} + (B_{11}J_1C_{12} + B_{12}J_2C_{22}) P_{t-1} + \gamma_1 Z_{t-1}$$

$$\begin{aligned} C_{21}X_t + C_{22}P_t &= J_2 (I + \mathbf{b}_t) (I + \mathbf{b}_{t-1})^{-1} (C_{21}X_{t-1} + C_{22}P_{t-1}) + \\ &+ (I + \mathbf{b}_t) (I + \mathbf{b}_{t-1})^{-1} (C_{21}\gamma_1 + C_{22}\gamma_2)Z_{t-1} - \mathbf{b}_t J_2^{-1}(C_{21}\gamma_1 + C_{22}\gamma_2)Z_t \end{aligned}$$

The particle filter

We want to approximate the target density:

$$f(\vartheta_{0:T}, b_{0:T}, \varphi, \nu_{0:T}|y_{1:T})$$

where we use a little abuse of notation, and we denote ν_t as the inference on the parameters in ν at time t. We assume: (i) the latent processes ϑ_t and b_t are Markov chains; (ii) given ϑ_t and b_t, y_t is conditionally independent on y_s $\forall t$ and $s \neq t$. Then, we can write the target density in a recursive way:

$$\begin{aligned} & f(\vartheta_{0:t}, b_{0:t}, s_{0:t}, \varphi, \nu_{0:t}|y_{1:t}) \propto f(\vartheta_{0:t}, b_{0:t}, s_{0:t}, \varphi, \nu_{0:t}, y_t|y_{1:t-1}) \\ = \; & f(y_t|\vartheta_t, b_t, \varphi, \nu_t) f(\vartheta_t, b_t, s_t, \nu_t|\vartheta_{t-1}, b_{t-1}, s_{t-1}\varphi, , \nu_{t-1}) f(\vartheta_{0:t-1}, b_{0:t-1}, s_{0:t-1}, \varphi, \nu_{0:t-1}|y_{1:t-1}) \\ = \; & f(y_t|\vartheta_t, b_t, \varphi, \nu_t) f(\nu_t|s_t) f(\vartheta_t|b_t, \vartheta_{t-1}, \varphi, \nu_{t-1}) \cdot \\ & \quad \cdot f(b_t|b_{t-1}, \nu_{t-1}) f(\vartheta_{0:t-1}, b_{0:t-1}, s_{0:t-1}, \varphi, \nu_{0:t-1}|y_{1:t-1}) \end{aligned}$$

where we use the fact that $f\left(s_t|\vartheta_t, b_t, \vartheta_{t-1}, b_{t-1}, s_{t-1}, \varphi, \nu_{t-1}\right) = 1$. Following Liu and West (2001), we suppose that the target density at $t-1$ is approximated as

$$f\left(\vartheta_{0:t-1}, b_{0:t-1}, s_{0:t-1}, \varphi, \nu_{0:t-1}|y_{1:t-1}\right) \approx \sum_{i=1}^{N} w_{t-1}^{(i)} N\left(m^{(i)}; h^2\Sigma\right)$$

where $m^{(i)} = a\varphi^{(i)} + (1-a)\varphi$, $\Sigma = Var(\varphi)$ and $h = \sqrt{(1-a^2)}$. In our case we have $a = 0.99$. Then, the target density for each particle i is

$$f\left(y_t|\vartheta_t^{(i)}, b_t^{(i)}, \varphi^{(i)}, \nu_t^{(i)}\right) f\left(\nu_t^{(i)}|s_t^{(i)}\right) f\left(\vartheta_t^{(i)}|\vartheta_{t-1}^{(i)}, b_t^{(i)}, \varphi^{(i)}, \nu_{t-1}^{(i)}\right) f\left(b_t|b_{t-1}^{(i)}, \nu_{t-1}^{(i)}\right) N\left(m^{(i)}; h^2\Sigma\right)$$

and the importance transition density is

$$q\left(y_t|\vartheta_{t-1}^{(i)}, b_{t-1}^{(i)}, m^{(i)}, \nu^{(i)}\right) f\left(\nu^{(i)}|s_t^i\right) q\left(\vartheta_t|\vartheta_{t-1}^{(i)}, b_t^{(i)}, \varphi^{(i)}, \nu^{(i)}, y_t\right) q\left(b_t|b_{t-1}^{(i)}, \nu^{(i)}\right) N(\varphi; m^{(i)}, h^2\Sigma)\ .$$

The algorithm

In practice the estimation is implemented through the following algorithm:

`For` $t = 1...T$:

0 `Compute` $\varphi = E(\varphi)$ `and` $\Sigma = Var(\varphi)$. `For` $i = 1...N$ `put`

$$\begin{aligned} m^{(i)} &= a\varphi^{(i)} + (1-a)\varphi \\ g(b_{t-1}^{(i)}) &= E(b_t|b_{t-1} = b_{t-1}^{(i)}) \end{aligned}$$

`For` $i = 1...N$

1 `Compute weights:` $\tilde{w}_t^{(i)} \propto w_{t-1}^{(i)} q\left(y_t|\vartheta_{t-1}^{(i)}, g(b_{t-1}^{(i)}), m^{(i)}, \nu^{(i)}\right)$

2 `Resample` $\left\{\tilde{\vartheta}_{t-1}^{(i)}\right\}_{i=1}^{N} \left\{\tilde{b}_{t-1}^{(i)}\right\}_{i=1}^{N} \left\{\tilde{s}_{t-1}^{(i)}\right\}_{i=1}^{N} \left\{\tilde{m}^{(i)}\right\}_{i=1}^{N} \left\{\nu^{(i)}\right\}_{i=1}^{N}$ `according to` $\tilde{w}_t^{(i)}$

3 `Propagate:`

  `(i) draw` $\varphi^{(i)}$ `from` $N(\varphi; \tilde{m}^{(i)}, h^2\Sigma)$

  `(ii) draw` $\tilde{b}_t^{(i)}$ `from` $q\left(b_t|\tilde{b}_{t-1}^{(i)}, \nu^{(i)}\right)$

  `(iii) draw` $\tilde{\vartheta}_t^{(i)}$ `from` $q\left(\vartheta_t|\tilde{\vartheta}_{t-1}^{(i)}, \tilde{b}_t^{(i)}, \varphi^{(i)}, \nu^{(i)}, y_t\right)$

4 `Compute new weights:` $w_t^{(i)} = \dfrac{f(y_t|\tilde{\vartheta}_{t-1}^{(i)}, \tilde{b}_t^{(i)}, \varphi^{(i)}, \nu^{(i)})}{q(y_t|\tilde{\vartheta}_{t-1}^{(i)}, g(\tilde{b}_{t-1}^{(i)}), \tilde{m}^{(i)}, \nu^{(i)})}$

5 `Update sufficient statistics` $\tilde{s}_t^{(i)} = S(\tilde{s}_{t-1}^{(i)}, \tilde{\vartheta}_t^{(i)}, y_t)$

6 `Draw` $\nu^{(i)}$ `from` $f\left(\nu^{(i)}|\tilde{s}_t^i\right)$

7 `Final draws:` $\left\{\vartheta_t^{(i)}\right\}_{i=1}^{N} \left\{b_t^{(i)}\right\}_{i=1}^{N} \left\{s_t^{(i)}\right\}_{i=1}^{N} \left\{\varphi^{(i)}\right\}_{i=1}^{N} \left\{\nu^{(i)}\right\}_{i=1}^{N}$ using $w_t^{(i)}$

The weights at step 4 are computed as

$$w_t^{(i)} = \frac{f\left(y_t|\tilde{\vartheta}_t^{(i)}, \tilde{b}_t^{(i)}, \varphi^{(i)}, \nu^{(i)}\right) f\left(\tilde{\vartheta}_t^{(i)}|\tilde{\vartheta}_{t-1}^{(i)}, \tilde{b}_t^{(i)}, \varphi^{(i)}, \nu^{(i)}\right)}{q\left(y_t|\tilde{\vartheta}_{t-1}^{(i)}, g(\tilde{b}_{t-1}^{(i)}), \tilde{m}^{(i)}, \nu^{(i)}\right) q\left(\tilde{\vartheta}_t^{(i)}|\tilde{\vartheta}_{t-1}^{(i)}, \tilde{b}_t^{(i)}, \varphi^{(i)}, \nu^{(i)}, y_t\right)} \tag{55}$$

Consider the density $q\left(\tilde{\vartheta}_t^{(i)}|\tilde{\vartheta}_{t-1}^{(i)},\tilde{b}_t^{(i)},\varphi^{(i)},\nu^{(i)},y_t\right)$ in the denominator: it can be rewritten as

$$q\left(\tilde{\vartheta}_t^{(i)}|\tilde{\vartheta}_{t-1}^{(i)},\tilde{b}_t^{(i)},\varphi^{(i)},\nu^{(i)},y_t\right)=\frac{f\left(y_t|\tilde{\vartheta}_t^{(i)},\tilde{b}_t^{(i)},\varphi^{(i)},\nu^{(i)}\right)f\left(\tilde{\vartheta}_t^{(i)}|\tilde{\vartheta}_{t-1}^{(i)},\tilde{b}_t^{(i)},\varphi^{(i)},\nu^{(i)}\right)}{f\left(y_t|\tilde{\vartheta}_{t-1}^{(i)},\tilde{b}_t^{(i)},\varphi^{(i)},\nu^{(i)}\right)}$$

that substituted in (55) gives the expression in the algorithm.

Figures

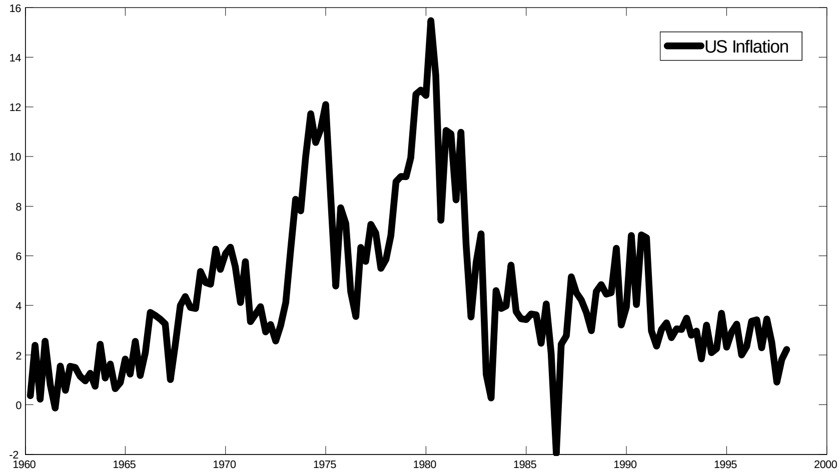

Figure 1: CPI inflation, quarterly data. Sample: 1955Q1 - 2006Q4

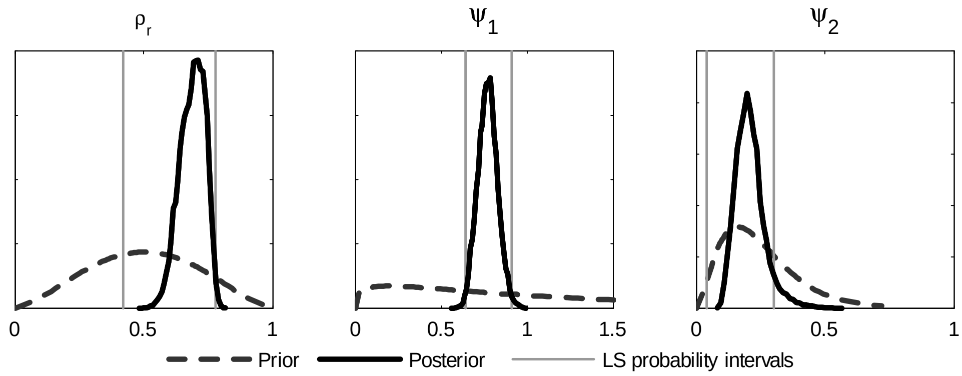

Figure 2: M_S : Comparison between the posterior distributions of the policy parameters and the probability intervals of LS.

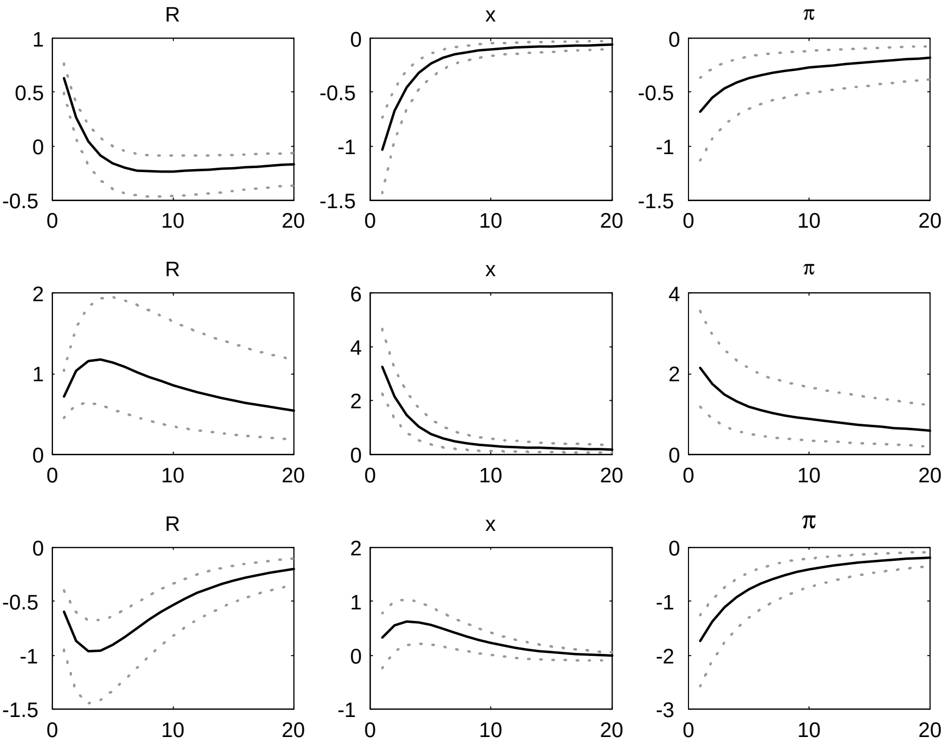

Figure 3: Generalized Impulse Response Function in the M_S model.

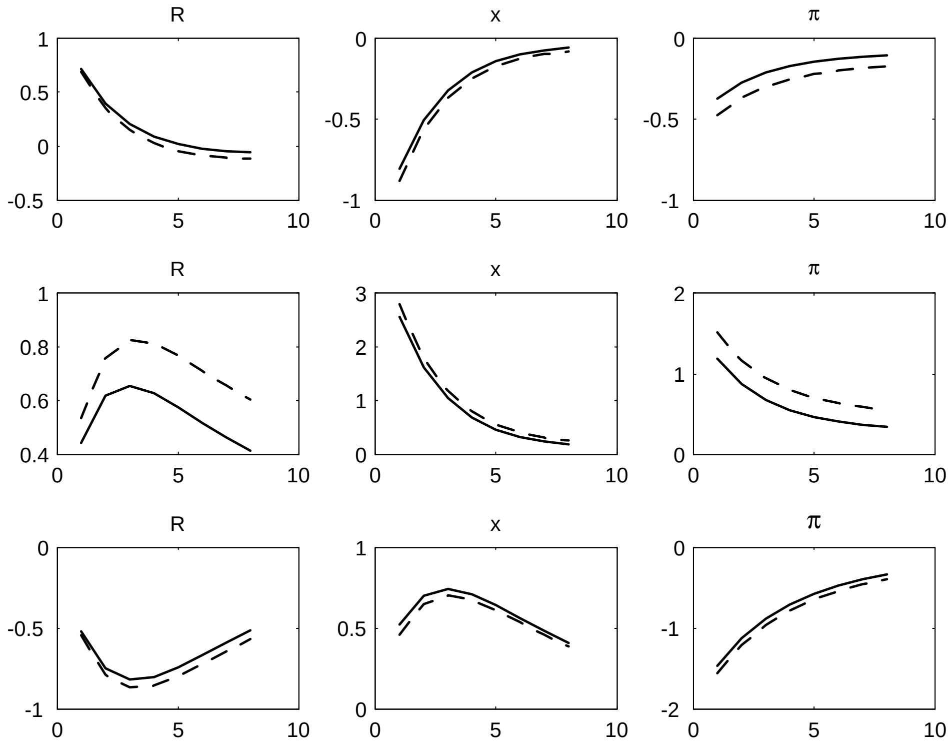

Figure 4: Generalized Impulse Response Function in the M_S model: solid line: $b_1 = 1.3$, dashed line: $b_1 = 1.5$.

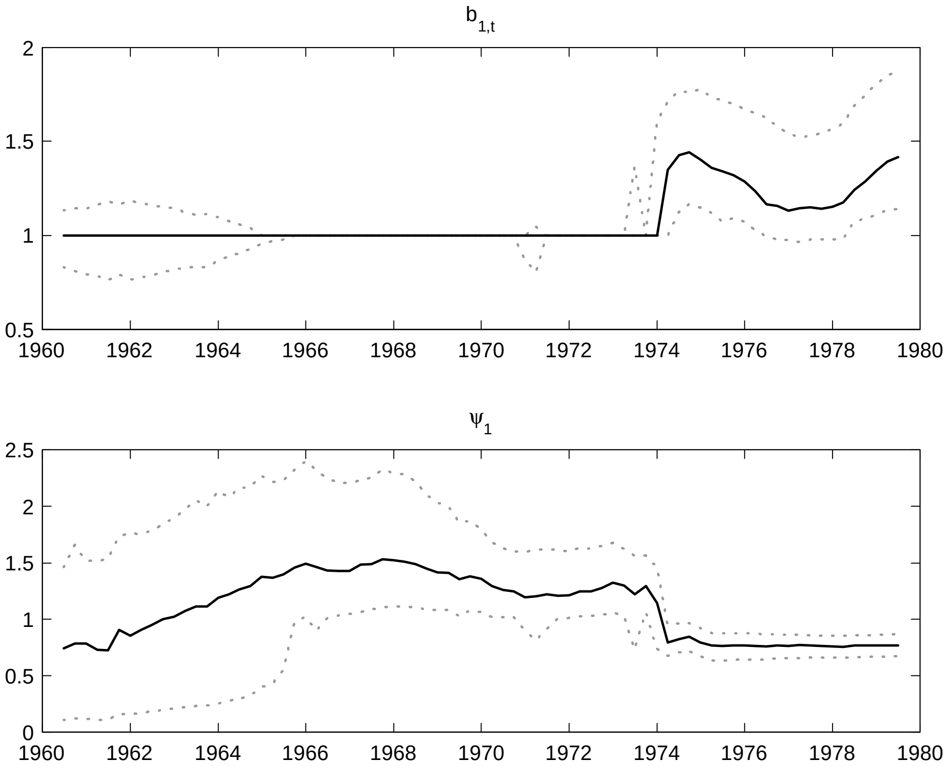

Figure 5: Estimated path for $b_{1,t}$ for the stable model M_s in the Great Inflation subsample (first panel); sequential inference on the parameter ψ_1 (second panel).

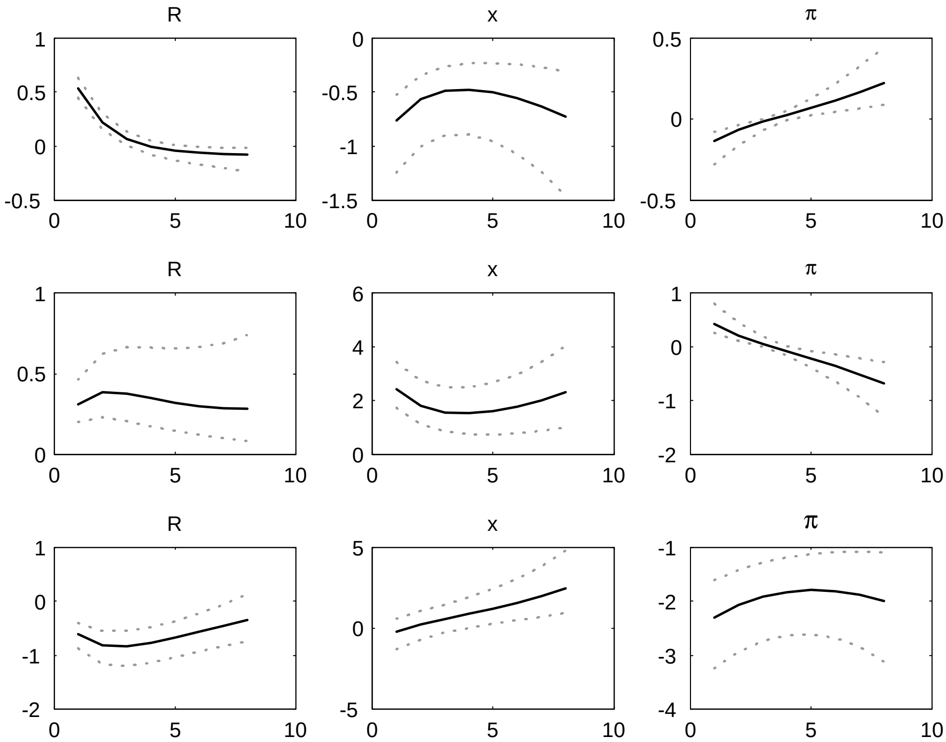

Figure 6: Generalized Impulse Response Function in the M_U model.

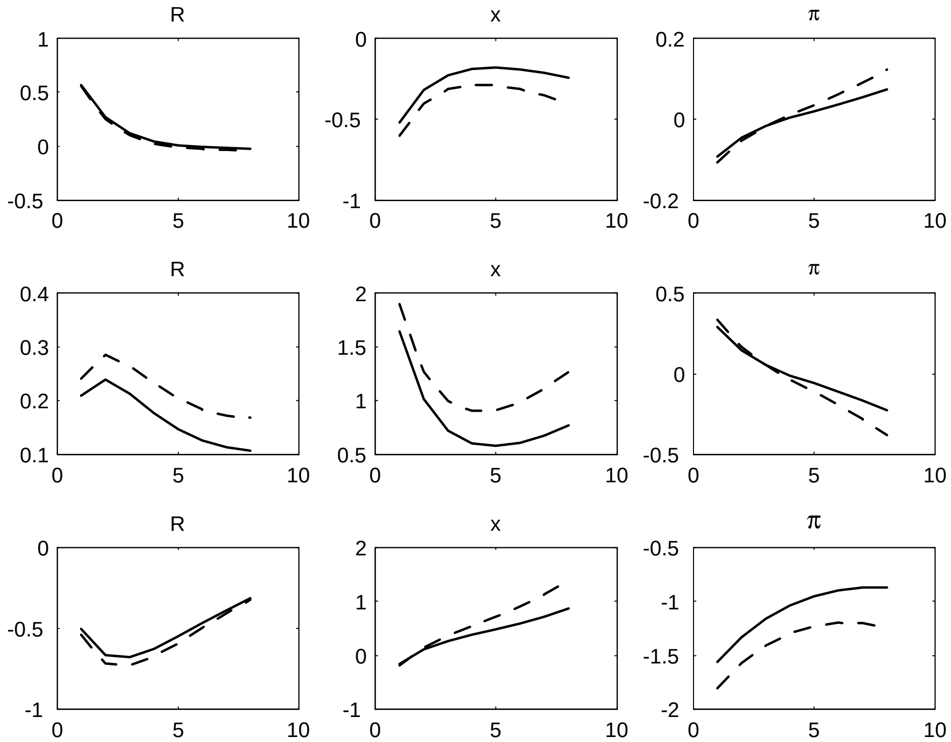

Figure 7: Generalized Impulse Response Function in the M_U model: solid line: $b_1 = 1.3$, dashed line: $b_1 = 1.5$.

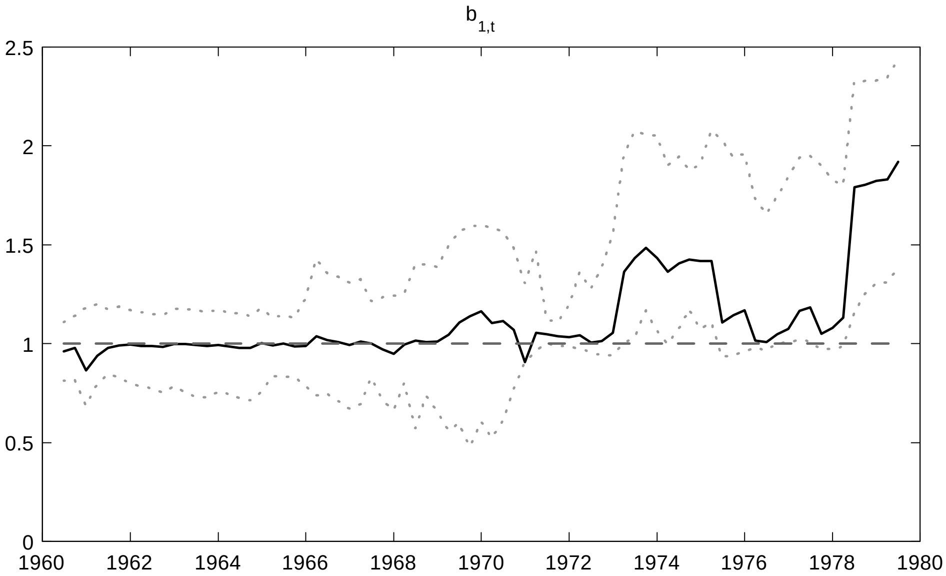

Figure 8: Estimated path of $b_{1,t}$ for the unstable model M_U in the Great Inflation subsample

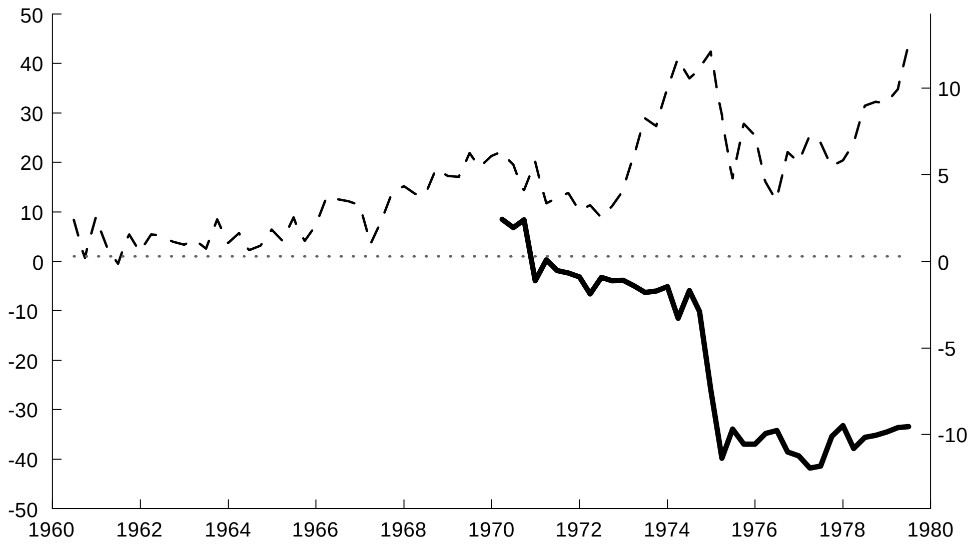

Figure 9: Comparing M_S - M_U, Great Inflation period. The panels show $2\ln(W_t)$ (solid line, scale on the left axis) and the inflation rate (dashed line, scale on the right axis)

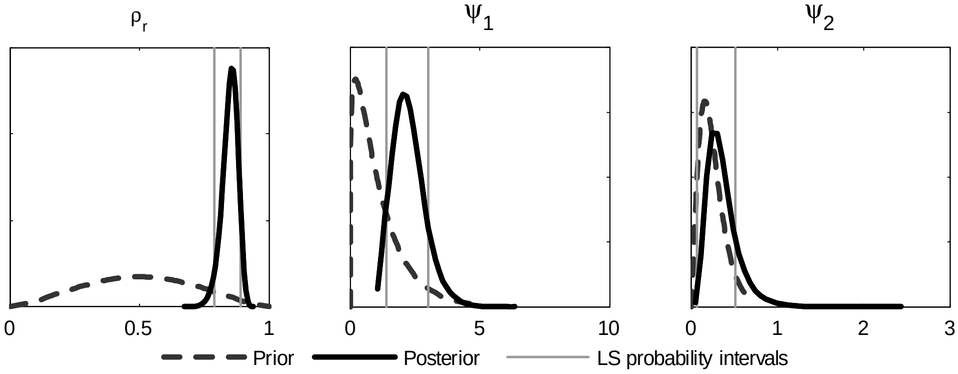

Figure 10: M_U : Comparison between the posterior distributions of the policy parameters and the probability intervals of LS.

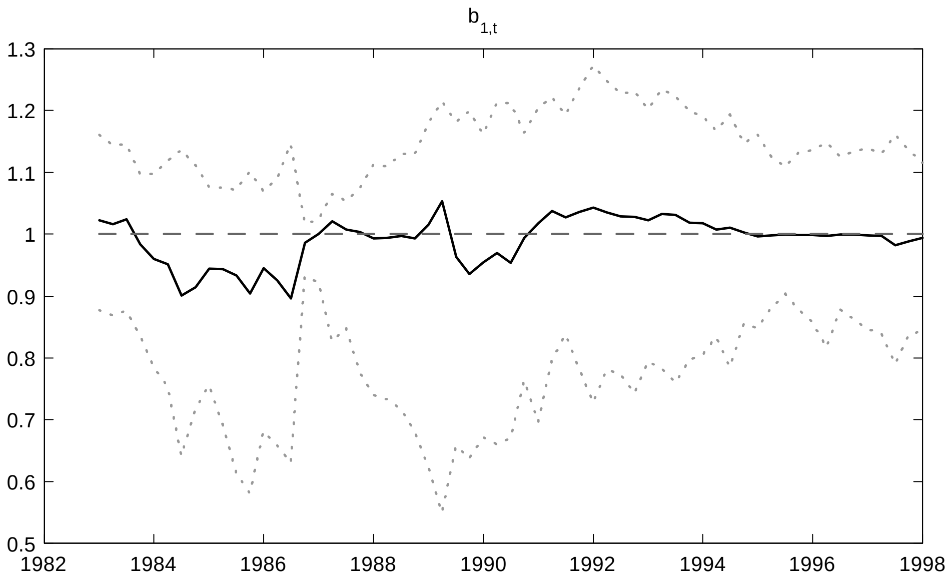

Figure 11: Estimated path of $b_{1,t}$ for the unstable model M_U in the Post-82 subsample

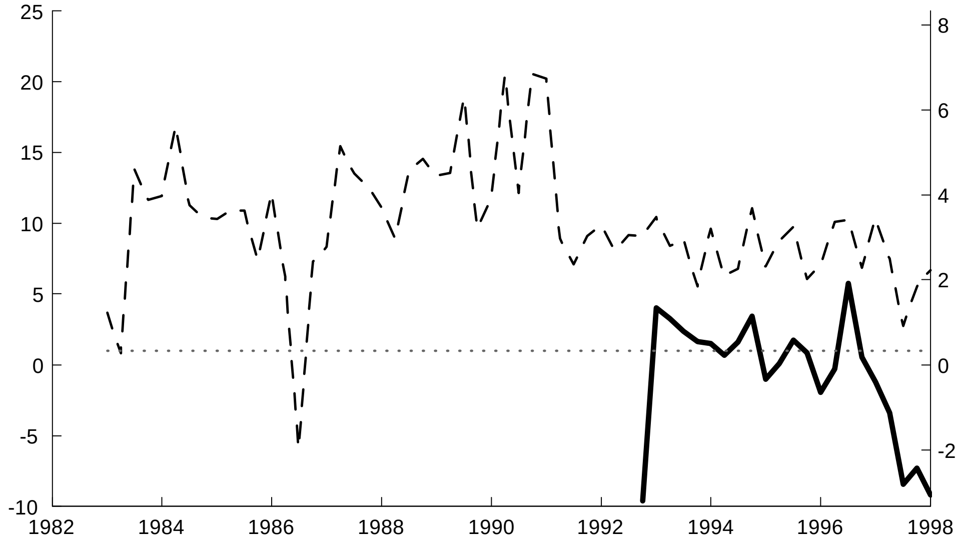

Figure 12: Comparing M_S - M_U, Post-82 sub-sample. The panels show $2\ln(W_t)$ (solid line, scale on the left axis) and the inflation rate (dashed line, scale on the right axis)