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Mixed Frequency Structural Models: Identification, Estimation, and Policy Analysis *

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Abstract

In this paper we show analytically, with simulation experiments and with actual data that a mismatch between the time scale of a DSGE model and that of the time series data used for its estimation generally creates identification problems, introduces estimation bias and distorts the results of policy analysis. On the constructive side, we prove that the use of mixed frequency data, combined with a proper estimation approach, can alleviate the temporal aggregation bias, mitigate the identification issues, and yield more reliable policy conclusions. The problems and possible remedy are illustrated in the context of standard structural monetary policy models.

JEL Classification Codes: C32, C43, E32

Keywords: Structural VAR, DSGE models, temporal aggregation, mixed frequency data, identification, estimation, policy analysis

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1 Introduction

Researchers typically model economic decision making processes as if conducted at fixed specified intervals of time. However, as already mentioned in the literature (see Christiano and Eichenbaum (1987)), there is no reason to believe that the frequency at which economic agents make decisions coincides with the frequency at which time series are released. Christiano and Eichenbaum (1987) evaluate the consequences of the specification error that results when agents' true decision interval is finer than the data sampling interval. They show that the misalignment between agents' decision intervals and the data sampling frequency is not a secondary issue, and that temporal aggregation is important in practice, having the potential to account for results considered anomalous in the literature.[1]

Our first contribution is to provide a general treatment of the effects of temporal aggregation of DSGE models. With few exceptions, such as Christiano and Eichenbaum (1987), earlier literature addressed the temporal aggregation issue mostly in the context of reduced form ARMA and VARMA models (see, among the others, Brewer (1973), Wei (1981), Weiss (1984), Lutkepohl (1987), Marcellino (1998, 1999)). Taking standard monetary models as an example, we assume that their frequency is monthly, since most central banks take policy decisions once a month, while estimation is conducted with quarterly data. We show analytically that it is generally impossible to identify the structural parameters, and the estimated responses to the monetary shock can be rather different from the true ones.

Our main contribution is to demonstrate that the use of mixed frequency data can improve identification, alleviate the temporal aggregation bias, and get estimated policy responses closer to the actual ones. In a monetary policy context, this means, for example, estimating models using quarterly time series of GDP (given that this variable is only released quarterly) but monthly data on inflation and interest rate, rather than quarterly data on all the three variables. Intuitively, the identification gains come from enlarging the information set to better match the decision timing of the central bank, and this then generates improved estimation and policy analysis.

In this paper we therefore focus on the use of mixed-frequency data in a structural context. There is a small but growing literature which focuses on that. Giannone et al. (2009) develop a methodology to incorporate monthly information in quarterly DSGE models. However, the focus of their paper is different from ours. They consider quarterly DSGE based parameter estimates as given and they exploit monthly information only to obtain increasingly accurate early estimates and forecasts of the quarterly variables. Our goal is instead very different. We do not take the parameters as given, instead we want to

[1] The assumption that agents take decisions at a regular, fixed interval has also been questioned in the literature. Jorda (1999) analyzes the specification error that results when the agents' decision interval is random and does not coincide with the data sampling interval, with additional results provided in Jorda and Marcellino (2004).For simplicity and analytical tractability, we proceed under the assumption that agents make decisions at fixed time intervals.

analyze exactly what happens to these structural parameters when we consider different frequencies in the data.

Kim (2010) is the closest contribution to our analysis. He focuses on whether frequency misspecification of a New Keynesian model results in temporal aggregation bias of the Calvo parameter. For estimation, he proposes a data augmentation method, in a Bayesian framework. In our paper, we provide theoretical background to the analysis of Kim (2010), by showing analytically the mapping from a monthly structural model to the quarterly counterpart. Moreover, we highlight the causes of identification issues related to temporal aggregation and show how the use of mixed-frequency data can alleviate them.

Finally, we work in a classical context, without employing Bayesian techniques. Therefore, from an econometric point of view, we provide a general Kalman filter based estimation method to deal with mixed frequency estimation in a classical maximum likelihood framework. Specifically, we adapt the method of Mariano and Murasawa (2010) to a structural context, and assess its finite sample properties in a set of Monte Carlo experiments. Bayesian estimation could be also considered, by combining our expression for the likelihood with the specification of prior distributions for the model parameters.

We then investigate how important these aggregation problems are in practice, and to what extent they influence the estimated parameters and structural relations across the variables. We use simulated and actual US data to estimate standard DSGE models with quarterly aggregated data, and compare the results to those obtained with monthly or mixed frequency data.

Overall, our empirical results support the theoretical findings and suggest that the extent of the temporal aggregation bias can be large, but substantially mitigated by the use of mixed frequency data. Specifically, with simulated data we are able to verify that the results obtained from the mixed frequency approach are very similar to those from the benchmark monthly model. Moreover, in the empirical small-scale New Keynesian example, we compare impulse responses obtained from a quarterly and a mixed-frequency model. Although typically the use of mixed monthly/quarterly data does not change the pattern of the responses, it can influence their persistence and magnitude.

To strengthen our empirical results, we also examine the mixed frequency version of a larger state-of-the-art model, developed at the quarterly level by Smets and Wouters (2007). We focus on the responses of some key variables to a variety of demand, supply and monetary policy shocks. Our mixed frequency estimation procedure still works well, despite the increasing computational challenges due to the high dimension of the model, and delivers interesting results. We confirm in this case the results obtained in the small scale New Keynesian example on the possible differences in the responses to structural shocks.

The paper is structured as follows. In Section 2, we focus on the temporal aggregation issues in a basic New Keynesian DSGE model. We provide analytical (Section 2), simulated (Sections 3 and 4) and empirical (Section 5) results to assess the relevance of

temporal aggregations issues. In Section 6, we summarize our main findings and conclude. The Appendix contains additional details.

2 The time scale problem in a DSGE model

In this section, we want to analyze the time scale problem in a structural context, namely in a new Keynesian model, which is the workhorse for the monetary policy analysis in the framework of dynamic stochastic general equilibrium (DSGE) models.

Our starting point is a basic New Keynesian model (see Galí (2008) for a comprehensive derivation of it). The New Keynesian Phillips curve (NKPC, thereafter) and the dynamic IS (DIS) constitute the non-policy block, the Taylor type monetary policy rule which describes how the nominal interest rate evolves over time closes the model.

We want to show that temporal aggregation generates two different problems. First, since it confounds parameters across equations, it is not always possible to identify the parameters of the high frequency model, once it has been aggregated at a lower frequency. Second, even when identification is not an issue and each parameter can be uniquely identified from a quarterly model, the common approach of considering the same structural model at a different frequency leads to different interpretations of the parameters values.

We first derive the mapping from the monthly specification to the equivalent quarterly counterpart of the same model. Then, we illustrate how the temporal aggregation bias can influence the estimates of the coefficients even when the model is uniquely identified. In a second step, we use a slightly more complicated version of the model to show how time aggregation raises also identification issues. Next, we show that the use of mixed frequency data can overcome both the temporal aggregation bias and the identification issue, allowing to identify the parameters of the underlying monthly model even when one variable can be only observed at quarterly frequency. Finally, we discuss estimation of the mixed frequency DSGE model.

As mentioned in the Introduction, an analysis on how to incorporate monthly information in estimated quarterly DSGE models has been conducted by Giannone et al. (2009). They focus on how to augment the quarterly model with monthly information to obtain a better forecasting performance. In this paper, we focus instead on the identification problems and estimation bias due to the mismatch between the time scale of the DSGE model and that of the data used for estimation.

2.1 A basic New Keynesian model: mapping from monthly to quarterly specification

In this subsection, we consider a very simple version of the New Keynesian model, a simplified version of the model analyzed by Clarida, Galí, Gertler (2000).

The three equations describing the model are the following:

$$\pi_t = \beta E_t \pi_{t+1} + k y_t^* + \varepsilon_{st}, \quad (1)$$

$$y_t^* = E_t y_{t+1}^* - \tau \left(R_t - E_t \pi_{t+1} \right) + \varepsilon_{dt}, \quad (2)$$

$$R_t = \rho_r R_{t-1} + (1 - \rho_r) \left(\phi_\pi \pi_t + \phi_y y_t^* \right) + \varepsilon_{rt}, \quad (3)$$

where eq. (1) is the NKPC, eq. (2) the DIS and eq. (3) the policy rule, and π_t, y_t^* and R_t stand respectively for inflation rate, output growth and real interest rate. y_t^* is starred since it is not observable at a monthly frequency. For analytical tractability and without loss of generality, we assume that $\varepsilon_{st}, \varepsilon_{dt}$ and ε_{Rt} are uncorrelated, i.i.d. and normally distributed with mean equal to zero and variance respectively equal to σ_s^2, σ_d^2 and σ_R^2. Finally, k is a function of the Calvo parameter θ, which describes the price rigidity, and it is defined as $k = \frac{(1-\beta\theta)(1-\theta)}{\theta}$.

The model in eq. (1) - (3) can be written in matrix form as:

$$B_0 X_t^* = C X_{t-1}^* + D E_t X_{t+1}^* + \epsilon_t, \quad (4)$$

where $X_t^* = \begin{bmatrix} \pi_t & y_t^* & R_t \end{bmatrix}'$ and $\epsilon_t = \begin{bmatrix} \epsilon_{st} & \epsilon_{dt} & \epsilon_{rt} \end{bmatrix}'$, with $\epsilon_t \sim N(0, I_3)$.

The unique stable solution for this model is given by

$$A_0 X_t^* = A_1 X_{t-1}^* + \epsilon_t, \quad (5)$$

with A_0 and A_1 satisfying the two following conditions:

$$A_0 = B_0 - D A_0^{-1} A_1, \quad (6)$$

$$A_1 = C. \quad (7)$$

The matrices B_0, C, D, A_0 and A_1 are defined in Appendix 7.1.

The model is uniquely identified: all the parameters of the model in (1) - (3) appear in the data generating process defined in (5), and each set of parameters gives a unique value of A_0 and A_1 (for the proof, see Fucac, Waggoner and Zha (2007)).

We assume that agents' decision interval is in months, since the monetary authority typically takes decisions once a month.[2] If all the data were available at that frequency, the econometrician could simply estimate (5), with the restrictions determined by the structure of the economy described in the model, and obtain the estimates of all the parameters. But y_t^* is not observable, therefore the econometrician cannot estimate (5) directly.

The common strategy adopted in the literature is to estimate the model at quarterly frequency where all the data are available. The naive econometrician therefore simply

[2] In the Monte Carlo analysis we will also evaluate the consequences of a mispecified choice of temporal frequency.

estimates the following model:

$$A_0^N X_\tau = A_1^N X_{\tau-1} + \epsilon_\tau^N, \tag{8}$$

with $\epsilon_\tau^N \sim N(0, I)$, for $\tau = 1, 2, 3...$ where τ indicates quarters. In other words, what the econometrician does is to consider the same economy described by (5), and set the agents' decision interval equal to the sampling interval at which all the data are available. But this is obviously different from estimating (5) aggregated at quarterly level.

First, we consider the monthly process (5), and aggregate it at quarterly level. In order to aggregate the process we need to follow some steps, which we describe in Appendix 7.1. We obtain:

$$A_0^Q X_\tau = A_1^Q X_{\tau-1} + \epsilon_\tau^Q, \tag{9}$$

with $\epsilon_\tau^Q \sim N\left(0, \Sigma^Q\right)$, and Σ^Q is a diagonal matrix.

With this very basic model, we can identify all the parameters which define the monthly process from those in A_0^Q and A_1^Q. This allows us to isolate one of the two issues related to time aggregation, the temporal aggregation bias. In the next subsection we will show that with slightly more dynamics in the model the identification of the monthly parameters from A_0^Q and A_1^Q is no longer possible.

The naive estimated model (8) obtained by considering agents acting at a quarterly frequency is different from the correct quarterly aggregated model (9). If we focus on the dynamics of the two models, (8) and (9), we see that they have the same zero restrictions in the matrices:

$$A_0^i = \begin{bmatrix} X & X & X \\ 0 & X & X \\ X & X & X \end{bmatrix}, A_1^i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & X \end{bmatrix}, \Sigma^i = \begin{bmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{bmatrix}.$$

However, the only non-zero element of A_1^i, which describes the dynamics of the model is different in the two approaches: while in the naive case $A_1^N(3,3) = \frac{1}{\sigma_R^N}\rho_R^N$, in the quarterly aggregated model it is $A_1^Q(3,3) = \frac{\mathbf{1}}{\boldsymbol{\sigma}_R}\frac{\rho_R^3}{\Psi}$, where Ψ is a non-linear function of the other structural parameters. Furthermore, we can see that ρ_R^N is different from ρ_R^3, which is usually considered the quarterly counterpart of a monthly ρ_r. This is an example of how the naive econometrician is therefore interpreting the coefficients in a different way from the one who considers time aggregation. Hence, it is not surprising that the estimation of these two different models gives rise to discrepancies in the estimated parameters.

Kim (2010), with a slightly different model setup, performed an empirical analysis showing a temporal aggregation bias in the estimated structural parameters[3]. Our analysis provides a theoretical justification for his results.

[3]Specifically, he found that the estimated Calvo parameter implies different average price duration when based on monthly rather than quarterly data.

2.2 A second New Keynesian model: aggregation and loss of identification

As we have already mentioned, the aggregation of a structural process at a lower frequency always leads to non-linear combinations of the parameters, which generally prevents the identification of the disaggregated process. In this subsection, we aim at illustrating the identification issues related to temporal aggregation. We do so by analyzing an extended version of the New Keynesian model in (1) - (3). The model equations are:

$$\pi_t = \beta E_t \pi_{t+1} + k y_t^* + \varepsilon_{st}, \tag{10}$$

$$y_t^* = E_t y_{t+1}^* - \tau \left(R_t - E_t \pi_{t+1}\right) + p y_{t-1}^* + \varepsilon_{dt}, \tag{11}$$

$$R_t = \rho_r R_{t-1} + (1-\rho_r)\left(\phi_\pi \pi_t + \phi_y y_t^*\right) + \varepsilon_{rt}. \tag{12}$$

Hence, the NKPC and monetary policy rule remain the same as before, while the DIS changes: y_t^* depends, among other things, not only on the expected future output but also on its value in the previous period, y_{t-1}^*. In other words, the dynamic of the DIS is more complex. For more details on this DIS formulation, see Furher and Rudebusch (2004).[4]

Similarly to the previous example, the model can be rewritten first in a matrix form and then in a reduced form like:

$$A_0 X_t^* = A_1 X_{t-1}^* + \epsilon_t, \tag{13}$$

with the same constraints on A_0 and A_1.[5] We normalize σ_d to one, to achieve identification. The reason is that we want to start with a uniquely identified process at monthly level, in such a way that we can disentangle the identification issues coming from temporal aggregation.

Since, again, in X_t^* we have y_t^* which is not observable every month, we aggregate (13), so that in the aggregated process we just have observations of y_t^* which are available (i.e. we have only observations at $t, t-3, t-6$,...).

What we obtain is

$$A_0^Q X_\tau = A_1^Q X_{\tau-1} + \epsilon_\tau^Q, \tag{14}$$

with $\epsilon_\tau^Q \sim N\left(0, \Sigma^Q\right)$.

Differently from the model analyzed in Section 2.1, not all the parameters which describe the monthly structural model ($\beta, k, \tau, p, \rho_r, \phi_\pi, \phi_y, \sigma_s$,and σ_r) can be uniquely identified from A_0^Q, A_1^Q and Σ^Q.[6] This example illustrates that time aggregation creates

[4] Furher and Rudebusch (2004) provide also a more general version of eq. (11). We choose the simple version for analytical tractability. Moreover, we focus in a model where the dynamics involve variables not available at the frequency we set up the model.

[5] The description of the different matrices is in Appendix 7.2.

[6] See Appendix 7.2 for more details on this point.

non-linear combinations of the parameters which describe the monthly process. These non-linear combinations make recovering the original parameters impossible. Moreover, if we consider the zero restrictions, we see that now they vary in the two approaches. While in the naive model we have

$$A_0^N = \begin{bmatrix} X & X & X \\ 0 & X & X \\ X & X & X \end{bmatrix}, A_1^N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{bmatrix}, \Sigma^N = \begin{bmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{bmatrix},$$

in the quarterly aggregated model the matrices are of the form:

$$A_0^Q = \begin{bmatrix} X & X & X \\ 0 & X & X \\ X & X & X \end{bmatrix}, A_1^i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & X & X \\ 0 & X & X \end{bmatrix}, \Sigma^Q = \begin{bmatrix} X & 0 & 0 \\ 0 & X & X \\ 0 & X & X \end{bmatrix}.$$

Comparing the restrictions of this example emphasizes the second problem related to time aggregation. The naive econometrician, who estimates the same structural model setting the agents' decision interval equal to the sampling interval at which all the data are available, imposes zero restrictions even where in the proper aggregated model there are not, namely, she assumes by mistake that $A_1^N(2,3) = 0$ and $A_1^N(3,2) = 0$. The naive econometrician imposes no effects between output and the lag of the interest rate, while in the properly aggregated model there is a dynamic relation between the same two variables. Hence, it is not surprising that the two models will provide different estimates of the structural parameters.

2.3 Exploiting mixed frequency data to deal with identification issues

In this section we show how the parameters of the model at monthly frequency can be identified when we exploit mixed frequency data. Taking into account the information which is available at monthly frequency (i.e. data on inflation rate and interest rate), we solve the identification issue, which we face when we aggregate all the series at quarterly level.

We first write the three equations represented in a compact form by (13) as:

$$\frac{1}{\sigma_s}\pi_t + Fy_t^* + GR_t = \epsilon_{st} \tag{15}$$

$$Hy_t^* + LR_t = py_{t-1}^* + \epsilon_{dt} \tag{16}$$

$$\frac{\phi_\pi}{\sigma_r}(\rho_r - 1)\pi_t + \frac{\phi_y}{\sigma_r}(\rho_r - 1)y_t^* + \frac{1}{\sigma_r}R_t = \frac{1}{\sigma_r}\rho_r R_{t-1} + \epsilon_{rt}, \tag{17}$$

where the matrices F, G, H, L are defined in the Appendix 10.6. We do not have any problems in estimating eq. (15) and (17) at $t = 3, 6, 9...$, since all the data are available

at this frequency. However, we cannot estimate eq. (16) since y^*_{t-1} is not observable. Therefore, we need to modify eq. (16) in such a way that it contains only variables which are available at the time of estimation. If we substitute y^*_{t-1} with its own expression $y^*_{t-1} = \frac{p}{H} y^*_{t-2} - \frac{L}{H} R_{t-1} + \frac{1}{H}\epsilon_{dt-1}$, and then we repeat it again for y^*_{t-2}, we obtain:

$$y^*_t = \left(\frac{p}{H}\right)^3 y^*_{t-3} - \frac{L}{H} R_t - \frac{L}{H}\left(\frac{p}{H}\right) R_{t-1} - \frac{L}{H}\left(\frac{p}{H}\right)^2 R_{t-2} + \xi_t. \tag{18}$$

From eq. (15), 18) and (17), we can now identify all the parameters. In particular, from eq. (15), we identify σ_s and obtain F and G; from eq. (17), we identify σ_r, ρ_r, ϕ_y and ϕ_π, and from eq. (18), we obtain $\frac{L}{H}$ and $\frac{p}{H}$. These ratios, together with F, G and the definition of F, G, H, L, allow us to identify all the remaining parameters, β, k, τ, p.

From this example, we can see how the use of mixed frequency data can solve, or at least alleviate, the identification issue. However, in general, it is not always possible to recover all the parameters, even with mixed frequency data. Consider for example the case when R_t is also not available on a monthly basis but only at the quarterly frequency, so that π_t is the only observable monthly variable. Now, the model for R_t and y^*_t can be written as

$$\begin{aligned}\begin{pmatrix} R_t \\ y^*_t \end{pmatrix} &= A_1 \begin{pmatrix} R_{t-1} \\ y^*_{t-1} \end{pmatrix} - \begin{pmatrix} H\phi_\pi \frac{\rho_r - 1}{H + L\phi_y - L\rho_r\phi_y} & 0 \\ -L\phi_\pi \frac{\rho_r - 1}{H + L\phi_y - L\rho_r\phi_y} & 0 \end{pmatrix} \pi_t \\ &\quad + \begin{pmatrix} \frac{H}{H + L\phi_y - L\rho_r\phi_y} & \frac{H\phi_y - H\rho_r\phi_y}{H + L\phi_y - L\rho_r\phi_y} \\ -\frac{L}{H + L\phi_y - L\rho_r\phi_y} & \frac{H}{H + L\phi_y - L\rho_r\phi_y} \end{pmatrix} \begin{pmatrix} \sigma_r \epsilon_{rt} \\ \frac{1}{H}\epsilon_{dt} \end{pmatrix}, \end{aligned} \tag{19}$$

where

$$A_1 = \begin{pmatrix} H\frac{\rho_r}{H + L\phi_y - L\rho_r\phi_y} & \frac{1}{H}p\frac{H\phi_y - H\rho_r\phi_y}{H + L\phi_y - L\rho_r\phi_y} \\ -L\frac{\rho_r}{H + L\phi_y - L\rho_r\phi_y} & \frac{p}{H + L\phi_y - L\rho_r\phi_y} \end{pmatrix},$$

which is a VAR(1) with π_t as exogenous variable (observable at the monthly level). Since R_t and y^*_t are only observable quarterly, this VAR(1) cannot be estimated. We can only estimate the corresponding model aggregated at quarterly frequency, which is

$$\begin{pmatrix} R_t \\ y^*_t \end{pmatrix} = A_1^3 \begin{pmatrix} R_{t-3} \\ y^*_{t-3} \end{pmatrix} - (1 + A_1 L + A_1^2 L^2) \begin{pmatrix} H\phi_\pi \frac{\rho_r - 1}{H + L\phi_y - L\rho_r\phi_y} & 0 \\ -L\phi_\pi \frac{\rho_r - 1}{H + L\phi_y - L\rho_r\phi_y} & 0 \end{pmatrix} \pi_t + \begin{pmatrix} \nu_{rt} \\ \nu_{dt} \end{pmatrix}, \tag{20}$$

where $t = , 3, 6, 9, \ldots$, L is the lag operator at the monthly frequency (since π_t is observable monthly) and the error process

$$\begin{pmatrix} \nu_{rt} \\ \nu_{dt} \end{pmatrix} = (1 + A_1 L + A_1^2 L^2) \begin{pmatrix} \frac{H}{H + L\phi_y - L\rho_r\phi_y} & \frac{H\phi_y - H\rho_r\phi_y}{H + L\phi_y - L\rho_r\phi_y} \\ -\frac{L}{H + L\phi_y - L\rho_r\phi_y} & \frac{H}{H + L\phi_y - L\rho_r\phi_y} \end{pmatrix} \begin{matrix} \sigma_r \epsilon_{rt} \\ \frac{1}{H}\epsilon_{dt} \end{matrix},$$

remains uncorrelated at the quarterly frequency. From the system in (20) it is no longer

possible to recover the structural parameters $\sigma_r, \rho_r, \phi_y, \phi_\pi, \frac{L}{H}$ and $\frac{p}{H}$, and therefore we loose identifiability even when using the available higher frequency information on π_t.

If instead π_t and y_t are only available quarterly but R_t monthly, it can be easily shown that the model in (15), 18) and (17) can be still estimated.

Therefore, for the structural model in (15) and (17), when the output variable can be only observed at the quarterly frequency, it is necessary and sufficient to have monthly information on R_t to achieve identification.

Unfortunately, it is not possible to provide a general rule on when and to what extent the mixed frequency information helps, since this depends on the specific structure of the structural model and amount and type of available mixed frequency information. But, as the examples illustrate, the use of the available mixed frequency information can only improve the identifiability of the system.

2.4 Estimation

We now present the general method for the estimation of a structural model with data released at different frequencies, which we later implement in our Monte Carlo and empirical exercises. We follow and generalize the analysis of Mariano and Murasawa (2010), and we provide the state-space representation of the models to be estimated in a maximum-likelihood framework. We focus on log-linearized DSGE models, and more generally on all the models whose solution can be cast in state-space form, where the low frequency series are then considered as high frequency series with missing observations. [7] The framework can be easily generalized to more than two frequencies, at the cost of increasing the notational complexity and computational efforts required. Therefore, we focus the exposition on the case of two frequencies only.

In general, the solution of a log-linearized DSGE model can be written in the form:

$$y_t = A(\theta) s_t + u_t, \tag{21}$$

$$s_t = B(\theta) s_{t-1} + C(\theta) \varepsilon_t, \tag{22}$$

where s_t is a $k \times 1$ state vector, y_t is a $N \times 1$ vector of observables, ε_t is a $p \times 1$ vector of shocks, and u_t is a $N \times 1$ vector of possible measurement errors. All the elements depend on θ, the structural parameters of the model. Eq. (22) characterizes the DSGE model solution, while eq. (21) maps the model variables into the observable variables.

In the case we consider here, not all variables are observable at frequency t. We define $\{y_{1t}\}$ as the N_1-variate low frequency series observable every m^{th} period, and $\{y_{2t}\}$ as

[7] The method of Mariano and Murasawa (2010) relies on the use of the Kalman filter in the presence of missing values and mixed-frequency data, and it has become commonly employed in the mixed-frequency literature which does not rely on Bayesian techniques. The use of the Kalman filter to deal with missing data is not new, in particular it has been extensively analyzed by Andrew Harvey in many of his studies already in the 80s (see, e.g., Harvey and Pierce (1984) and Harvey (1985)). However, the Mariano and Murasawa (2010) approach is simpler and particularly well suited for economic applications.

the N_2-variate high frequency series observable every period. $\{y_{1t}^*\}$ represents the latent unobservable high frequency series underlying $\{y_{1t}\}$, such that $y_{1t} = \omega(L)\, y_{1t}^*$ for each t, where l is the lag order of the polynomial $\omega(L)$. Finally, we define the $N \times 1$ vectors y_t and y_t^* respectively as $\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix}$ and $\begin{pmatrix} y_{1t}^* \\ y_{2t} \end{pmatrix}$ for all t, where $N = N_1 + N_2$.

Following this notation, we want to estimate the following system:

$$y_t^* = A(\theta)\, s_t + u_t, \tag{23}$$
$$s_t = B(\theta)\, s_{t-1} + C(\theta)\, \varepsilon_t. \tag{24}$$

Both u_t and ε_t are normally distributed, with $v_t = C(\theta)\, \varepsilon_t$, $E(v_t v_t') = Q(\theta)$, $E(u_t u_t') = H(\theta)$.[8] Hereafter, for simplicity, we write A, B, C, Q and H taking their dependence on θ for given.

We need to modify the state-space form in (23) and (24) to include also the aggregation rule $y_t = \omega(L)\, y_t^*$. Let us define the new state vector as:

$$\underset{(k+N)(l+1)\times 1}{f_t} = \begin{pmatrix} s_t & s_{t-1} & ... & s_{t-l} & u_t & u_{t-1} & ... & u_{t-l} \end{pmatrix}'.$$

The state-space representation is now

$$y_t = G f_t, \tag{25}$$
$$f_t = M f_{t-1} + P z_t, \tag{26}$$

where z_t is defined as:

$$\underset{(p+N)\times 1}{z_t} = \begin{pmatrix} \varepsilon_t \\ u_t \end{pmatrix},$$

and the matrices G, M, P are the following:

$$\underset{N\times(k+N)(l+1)}{G} = [\ H(0)A \ \ ... \ \ H(l)A \ \ H(0) \ \ ... \ \ H(l)\ ],$$

$$\underset{(k+N)(l+1)\times(k+N)(l+1)}{M} = \begin{bmatrix} B & 0 & ... & 0 & ... & & 0 \\ I & ... & 0 & 0 & ... & & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ 0 & ... & I & 0 & ... & & 0 \\ 0 & ... & 0 & 0 & ... & & 0 \\ 0 & & ... & I & ... & 0 & 0 \\ \vdots & & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & & ... & 0 & ... & I & 0 \end{bmatrix}, \quad \underset{(k+N)(l+1)\times(p+N)}{P} = \begin{bmatrix} C & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

[8] For simplicity we consider H diagonal, i.e. the measurement errors are serially uncorrelated, but the method can be extended to the case of serially correlated measurement errors.

We can now estimate the state-space form in (25) and (26) following the procedure described in Mariano and Murasawa (2010), to whom we refer for additional details.

3 A Monte Carlo exercise within a DSGE framework

In this section we provide an illustration of the time aggregation issues in the context of a small DSGE model and assess the finite sample performance of the estimation method introduced in the previous section. The aim is to estimate the model described in Section 2.1, with monthly, mixed frequency and quarterly data only, and compare the different estimates of the structural parameters. We use the standard solution methods for linear rational expectations models, and since the solution of the model has a state-space representation, we obtain maximum likelihood estimates of the structural parameters by making use of the Kalman filter. The vector of structural parameters that describes the model in eq. (1), (2) and (3) is $\Theta = \left(\beta, \theta, \tau, \rho_r, \phi_\pi, \phi_y, \sigma_s, \sigma_d, \sigma_r\right)'$. Even though all the nine structural parameters can be identified, for the purposes of our analysis, we calibrate the values of β and τ, to increase the precision of the estimates of the other parameters.

3.1 Simulation design and results

The simulated data are generated from the reduced form of the model described in (1) - (3). The calibrated values are: $\beta = 0.99$, $\phi_y = 0.5$, $\phi_\pi = 1.5$, $\tau = 1$, $\rho_r = 0.9$, k such that the average duration of price stickiness is equal to 10 months, which implies the Calvo parameter to be $\theta = 0.9$. The standard deviation of the three shocks, σ_s, σ_d and σ_r is fixed at 0.1. Moreover, $\tau = 1$ is consistent with the choice of a consumer's logarithmic utility function. The sample size is equal to 300 monthly observations and 100 quarters to mimic the ensuing application to the US economy in section 5. The number of replications in the Monte Carlo experiment is 1000.

We want to compare the results obtained by estimating the model with mixed frequency data to those obtained by the naive econometrician who simply disregards the aggregation issue and uses quarterly data. As a benchmark, we estimate the model also at monthly frequency, as if all the three series were available at a monthly level.

As mentioned in the Introduction, Kim (2010) runs a similar experiment, with a modified version of our DSGE model and conducting the estimation in a Bayesian framework. Instead, we use a classical method, that of Mariano and Murasawa (2003), so far used for reduced form and forecasting analyses only. Our main goal is to provide an empirical counterpart for the theoretical discussion in the previous Section about the distortionary effects of temporal aggregation.

To estimate the model at monthly frequency, we follow the standard maximum-likelihood technique. We then repeat the same estimation, using only quarterly data,

as a naive econometrician would do. To follow the approach of Section 2.1, we consider a point-in-time aggregation scheme. Therefore, when aggregating from the monthly to the quarterly level, we simply skip-sample the series, keeping one observation every third available. Finally, to use mixed frequency data we cast the model into the modified state-space form described in 2.4, and estimate it by means of the Kalman filter.

Table 1 reports the median value across replications of parameter estimates[9]. In italics, we also report the 10th and 90th percentiles. The results show that with mixed frequency data we approximate very well the monthly structure of the economy. The estimates of the parameters are very similar to those obtained by estimating the benchmark model at monthly frequency. The estimation of the monthly process is of course possible only because we are using simulated data. Using quarterly variables, we notice that for some parameters we obtain quite different estimates and wider confidence intervals. Moreover, even for the parameters whose estimated value is similar, their interpretation can be quite different. In particular, an estimate of the Calvo parameter θ close to 0.9 implies an average price duration of 10 months with the monthly model, but of almost 10 quarters with quarterly data. To obtain the same implied average price duration at quarterly frequency, θ should be equal to 0.7. This evidence is also consistent with the findings of Kim (2010).

4 Robustness analysis

To extend our analysis and assess the robustness of the reported findings of the Monte Carlo experiments, we allow for some modifications of the experimental design.

As a first robustness check, we consider whether the usefulness of mixed frequency data is confirmed with frequency mismatches other than the monthly-quarterly case analyzed before. Specifically, we extend the analysis to the weekly-quarterly (or monthly-annual) case, assuming 12 weeks per quarter so that $m = 12$.[10]

An even sampling frequency typically generates more identification problems than an odd one. Consider as an example an AR(1) process with negative root. If the process is aggregated with m odd, it is still possible to recover the original negative root. If instead m is even, the aggregated process is compatible with both a positive and a negative disaggregate root, so that the latter cannot be uniquely determined.

In the context of the DSGE model, we simulate the data as in Section 3.1, but in this case we generate 3600 weekly observations which correspond to 300 quarterly values.[11]

[9] In computing the median and the percentiles, we excluded the replications for which we didn't obtain convergence in the estimation process.

[10] We consider 12 weeks in a quarter instead of 13 for two reasons. First a frequency mismatch equal to 12 can be also interpreted as a monthly-yearly case, second because in what follows we consider also the case of weekly-monthly-quarterly case, and for this we need a multiple of two frequencies. 12 easily absolves this role, since we can consider 12 weeks in a quarter, 4 weeks in a month, and 3 months in a quarter.

[11] Due to the higher number of missing values when $m = 12$, we increase the size to 300 quarterly

As a second robustness check, we assess what happens when also the mixed frequency process is misspecified, in the sense that the true generating frequency is higher than what assumed in the mixed frequency model. As an example, which we then use in the Monte Carlo simulation, the true DGP is a weekly process and we compare the structural estimates obtained with quarterly data to those resulting from mixed frequency monthly-quarterly data. This situation can arise because either the higher frequency data are not available (e.g., weekly data on inflation) or the decision time is different from what assumed.

Once again, we focus on parameter estimates in the case of the DSGE model. The simulation design follows closely the one described before. The only difference is that the DGP is now a weekly process, our y is obtained by skip-sampling every 12 periods, and our x (inflation and interest rate in the case of our DSGE model) is also obtained by skip-sampling but every 4 periods.

We anticipate that the results, discussed in the details in the next two subsections, confirm the usefulness of the mixed frequency approach, also in finite samples, with a small or medium frequency mismatch ($m = 3$ or $m = 12$), and even when the assumed temporal frequency is misspecified ($m = 3$ vs true $m = 12$). The incorrect choice of mixed frequency does not allow to perfectly capture the dynamic of the high frequency DGP (as instead it was possible in the cases analyzed before). However, the results we obtain exploiting monthly and quarterly information are generally closer to the true weekly DGP than those obtained with quarterly values only. Hence, adding higher frequency information still mitigates the identification problem and the extent of the consequent estimation bias.

4.1 The case of weekly and quarterly data

In this subsection, we repeat the same analysis conducted in Section 3, for a sampling frequency $m = 12$. As mentioned, this frequency mismatch can be interpreted as the case of weekly and quarterly or monthly and annual data.

The simulated data are generated from the reduced form of the model described in (1) - (3). The calibrated values are: $\beta = 0.99$, $\phi_y = 0.5$, $\phi_\pi = 1.5$, $\tau = 1$, $\rho_r = 0.9$, $\theta = 0.9$, σ_s, σ_d and σ_r fixed at 0.1, $\tau = 1$ (as in the main experiment) . The sample size is equal to 3600 high frequency observations which correspond to 300 low frequency ones[12]. The number of replications in the Monte Carlo experiment is 500.

To estimate the model at high frequency, we follow the standard maximum-likelihood technique. To estimate a naive low frequency process we consider a point-in-time aggregation scheme. Therefore, when aggregating from the high to the low frequency, we simply skip-sample the series, keeping one observation every 12th available. Again, to use mixed

observations to obtain more stable results when running the Kalman filter.

[12]We increased the number of low-frequency observations to obtain more stable results even in this case with a high-number of missing observations. This allows to avoid computational issues related only to the short sample size.

frequency data we cast the model into the modified state-space form described in Section 2.4, and estimate it by means of the Kalman filter.

Table 2 reports the median value across replications of parameter estimates[13]. The 10th and 90th percentiles are in italics. The results generally confirm the findings in Section 3: with mixed frequency data we approximate very well the high frequency structure of the economy, finding estimates of the parameters which are very similar to those obtained by estimating the benchmark model at high frequency. Using low frequency variables, some of the estimated parameters (such as ϕ_y and σ_r) turn out to be quite different and with wider confidence bands, in line with the results for the monthly-quarterly case. Moreover, once again, even for the parameters whose estimated value is similar, their interpretation can be quite different when based on the low frequency.

4.2 The mixed frequency process is also misspecified

We now address the case where the assumed mixed frequency is incorrect. To be precise, we consider a weekly DGP while the model is estimated with monthly-quarterly or quarterly only data. Our goal is to check whether the mixed frequency approach still mitigates the problems arising with time aggregation[14].

We run the same experiment as in the previous section, with the high frequency and low frequency cases being exactly the same as in Section 4.1. What changes is the mixed frequency case. Here, rather than using weekly and quarterly data, we have monthly and quarterly observations, despite the DGP being weekly.

As it appears from Table 3, when the mixed frequency process is misspecified in the frequency, it sometimes provides estimates of the parameters which are quite different from the true ones (and are instead more similar to the ones obtained with quarterly data only). However, the confidence intervals generally remain smaller than the ones obtained with quarterly data only. Therefore, we can conclude that exploiting more information at least mitigates the temporal aggregation issues.

5 Two applications with US data

We now conduct an empirical analysis using data for the US economy in a DSGE framework. Our main goal is to compare the shock responses obtained with a mixed frequency approach to those from a standard quarterly model. In the first example we stay close

[13] In computing the median and the percentiles, we excluded the replications for which we didn't obtain convergence in the estimation process.

[14] There is a growing literature which conducts analysis in continuous time. In particular, the need to include financial variables in DSGE models fosters research in the time-continuous framework (see e.g. Christensen et al. (2011)). The main claim in using a continuous-time framework is the usefulness in solving the model and a better link to financial models. However, the goal of this paper is to show identification issues in structural models which are described in a discrete-time framework. Therefore, we leave the analysis in continuous time for future research.

to the small scale model used to obtain the analytical results. We first estimate the parameters of the economy described in eq. (1), (2) and (3), and then analyze the impulse responses to the shocks. As a second example, we conduct a similar analysis for a state-of-the-art DSGE model, the Smets and Wouters (2007) model.

5.1 Small-scale DSGE model

We estimate the model using data for the real GDP growth rate, the inflation rate, measured as the growth rate of the consumer price index, and the Federal Fund rates (FFR). The sample covers the period 1965 - 2007.[15] In estimating the model with data at different frequencies, we consider the quarter-on-quarter GDP growth and the monthly inflation rate and monthly interest rate. Moving to the quarterly DSGE, we aggregate the monthly series at the quarterly level. More specifically, we construct the quarterly inflation rate and the quarterly interest rate as the sum of the three monthly observations over the quarter.

We estimate the DSGE model within a maximum-likelihood framework, first at a standard quarterly frequency, and then with the mixed frequency approach, rewriting the model as described in Section 2.4. We calibrate the value of the discount factor β at 0.99, the most common value in the literature.

Figure 1 reports the estimated impulse responses obtained with the two approaches, mixed frequency (solid line) or quarterly data only (dashed line). It is worth to point out that the mixed-frequency model allows to obtain monthly responses. However, in order to compare the results with those obtained from the quarterly model, we focus on the corresponding quarterly aggregates, despite the possibility to analyze also the intra-quarterly dynamics in the mixed-frequency case. The aggregation of the impulse responses is a delicate issue, since the aggregation method depends on the nature of the analyzed series. In our example, we sum the impulse responses over the quarter, since the variables under analysis represent rates.

The figure reveals some discrepancies between the two approaches. The patterns remain similar and in line with economic theory, since a supply shock increases inflation and reduces output, with the former effect dominating and leading to higher interest rates; a demand shock increases all the three variables; and a restrictive monetary policy shock lowers both output and inflation. However, the magnitude of the responses and their persistence can be influenced by the data frequency. For example, the monetary policy shock has a stronger effect on output and weaker effect on inflation when using the mixed frequency data.

Finally, to further analyze the role of temporal aggregation in shaping the results, and how the use of mixed-frequency data can play a role in this, we consider a monthly version

[15] The sample starts at the same point as in Smets and Wouters (2007), but it is updated to 2007. We exclude the crisis period since a proper treatment of this episode is beyond the scope of the current analysis.

of our small New Keynesian model, using the growth rate in the industrial production (IP) as a proxi for output.

With the resulting monthly estimates, we compute the impulse responses and then aggregate them at the quarterly frequency, to make them comparable with those obtained from the mixed-frequency and quarterly models. The impulse responses are shown in Figure 2. The monthly and mixed-frequency responses exhibit some differences, but generally smaller than those obtained comparing the monthly and quarterly responses.

It is not obvious that monthly IP is a good proxi for output, since the share of IP in value added is rather limited and decreasing over time. Given this, we think the mixed frequency results remain more reliable, but the similarity with the monthly responses (even in the presence of different proxies for output) is reassuring.

5.2 Smets and Wouters (2007) model

The Smets and Wouters model (SW2007 henceforth) is a medium-scale DSGE model which incorporates different types of real and nominal frictions and seven structural shocks. It is considered by now a workhorse and benchmark model for analyzing various types of demand, supply and monetary policy shocks.

The model considers sticky nominal prices and wages, habit formation in the consumption, investment adjustment costs, variable capital utilization and fixed costs in production. We briefly report here only the main features of the model, the Appendix 8.1 provides the key equations while for a full derivation we refer to Smets and Wouters (2007, SW).

The model variables for the sticky wage and price economy are: output (y_t), consumption (c_t), investment (i_t), Tobin's q (q_t), utilized capital (k_t^s), installed capital (k_t), capacity utilization (z_t), rental rate of capital (r_t^k), price markup (μ_t^p), inflation rate (π_t), wage markup (μ_t^w), real wage (w_t), total hours worked (l_t), and nominal interest rate (r_t). For the corresponding flexible economy: output (y_t^*), consumption (c_t^*), investment (i_t^*), Tobin's q (q_t^*), utilized capital (k_t^{s*}), installed capital (k_t^*), capacity utilization (z_t^*), rental rate of capital (r_t^{k*}), price markup (μ_t^{p*}), wage markup (μ_t^{w*}), real wage (w_t^*), and total hours worked (l_t^*).

The shocks are: total factor productivity (ε_t^a), investment-specific technology (ε_t^i), government purchases (ε_t^g), risk premium (ε_t^b), monetary policy (ε_t^r), wage markup (ε_t^w) and price markup (ε_t^p). While the total factor productivity, investment-specific technology, government purchase, risk premium and monetary policy shocks follow AR(1) processes, the wage markup and price markup shocks follow ARMA(1,1) processes.

The model is estimated using seven macroeconomic variables: the log difference of real GDP, real consumption, real investment and real wage, log hours worked, the log difference of the GDP deflator, and the federal fund rate (FFR). The equations describing the model are reported in Appendix 8.1.

In our exercise we estimate the SW model by Maximum Likelihood techniques with

either quarterly or mixed frequency data, following the method described in Section 2.4. In order to fully exploit monthly information whenever available, we slightly change the data used for estimation with respect to SW. More in detail, while the log difference of real GDP, real consumption, real investment and real wage stay the same, we use monthly values of FFR, the log difference of the consumer price index and (average weekly) hours. The quarterly aggregates of the monthly series are very similar (or even the same, as in the case of the FFR) to the ones originally used in SW. We also extend the sample to 2007 (skipping however the recent financial crisis since a proper treatment of this episode is beyond the scope of the current analysis.).

Despite the different estimation method, the different data used for some variables and the longer sample, we show in Appendix 8.2 that we can replicate fairly well the quarterly impulse response functions of SW (for a comparison of Bayesian and MLE estimates of SW see also Iskrev (2008)). Hence, we can move to assessing whether and to what extent the responses to shocks differ when mixed frequency or quarterly only data are used for estimation.

Before discussing the results, let us comment on some of the features of the mixed-frequency model. As already discussed in the small-scale DSGE case, the mixed-frequency model allows to obtain monthly responses. We therefore would have the possibility to analyse also the intra-quarterly dynamics of the shock propagation. However, in order to compare the results with those obtained from the quarterly model, we present the corresponding quarterly aggregates only. Specifically, we skip sample the impulse responses for the variables in (log-)levels, and sum the responses over the quarter for the variables which represent rates. We focus on the impulse responses of output and hours (following the skip sample scheme) and of inflation and the FFR (summing over the quarter). These are the same four variables analyzed by Smets and Wouters (2007).

We group the main shocks under three categories. First, investment-specific technology, government purchase, and risk premium shocks, which can be all considered as demand shocks since output and inflation move in the same direction. According to SW, these demand shocks are the main drivers of output fluctuations in the short run.

Second, the wage markup and productivity shocks, which can be considered as supply shocks since output and inflation move in opposite directions. Supply shocks are the main determinants of output movements in the long run.

Third, the monetary policy shock that, according to SW, has only a very small role in driving output fluctuations, but is nonetheless typically evaluated in DSGE analyses.

The impulse responses are graphed in Figures 3-8 where, as before, the solid lines represent the mixed frequency responses and the dashed lines the quarterly responses. Overall, the patterns are qualitatively similar to those in SW, but there are some interesting differences that we now comment upon.

Starting with the demand shocks, a first feature emerging from Figures 3, 4 and 5 is that for all the three shocks the response of output is stronger with mixed frequency

than with quarterly data. On the other hand, in all cases hours react less, at least in the short run, with the mixed frequency data. The responses of inflation, in line with those of output, are generally stronger with the mixed frequency data. Actually, in the case of the investment specific technology shock there is even a negative reaction of inflation when using quarterly data. Finally, the reaction of the interest rate is more heterogeneous across shocks, it is stronger with quarterly data for the investment and spending shocks, weaker for the risk premium shock.

For the two supply shocks, Figures 6 and 7, an increase in the wage mark-up has stronger effects on inflation (positive) and output and hours (negative) with the mixed frequency data. The effects are instead more limited with the mixed frequency data in the case of the productivity shock, which persistently increases output and lowers hours, while the effects on inflation and the interest rate are rather limited (in line with the SW results). Therefore, we confirm the SW finding of a negative persistent effect on hours of productivity increases.

Finally, about the monetary policy shock, the responses in Figure 8 are again qualitatively similar to those in SW. However, with respect to the mixed frequency case, when using quarterly data there is only a very small decrease in inflation, and a more limited effect on output but a stronger reaction of hours.

We can therefore conclude that the use of mixed frequency information can also affect the results on the propagation of different types of shocks in realistic DSGE models. It is also noticeable that our estimation method provides meaningful results even for rather large models, despite the additional computational difficulties when dealing with the mixed-frequency data.

6 Conclusions

In the recent econometric literature, unbalanced datasets have attracted a substantial attention. Different methods have been proposed to deal with mixed frequency data, but the focus has only been on improving the forecasts of key series such as GDP growth, which are usually available at a lower frequency only. In this paper, we shift the attention to the use of mixed frequency data in the context of structural models.

The common approach in the literature is to estimate the deep parameters of the economy at a frequency such that data for all the variables are available, independently of the fact that the agents take decisions within a different time framework. Hence, structural models are typically estimated with quarterly data even if monthly or even higher frequency information on some variables is available.

We show that this practice can have important consequences when trying to give an economic interpretation to the estimated parameters and model dynamics. Using examples from the New Keynesian DSGE literature, we derive the analytical mapping from a monthly specification to a quarterly specification of the same model, showing that

it is in general impossible to identify the parameters which describe the monthly process when using quarterly data only. Even when identification is possible, the naive approach which overlooks aggregation issues can bring to misleading results, since a monthly model aggregated at quarterly level is clearly different from a quarterly model which replicates the structural relations of the monthly model and just changes the agents' time decision interval.

We also show that the identification issue arising from aggregation can be mitigated, and in some cases even solved, by the use of mixed frequency data. Using data at different frequencies allows us to exploit the information included in the intra-quarter lags of the monthly variables to identify more parameters than in the case we just use quarterly data.

We then provide a general classical estimation method to deal with mixed frequency data in a structural context, based on a modified state-space framework combined with the use of the Kalman filter to deal with the missing observations in the low frequency series.

Finally, our Monte Carlo analysis and empirical examples, based on the estimation of DSGE models using simulated and US data, confirm the practical importance of the aggregation issue, and that it can be alleviated by the use of mixed frequency data.

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Table 1: Estimates of structural DSGE parameters with monthly, mixed-frequency and quarterly simulated data

	DGP	monthy estimates		quarterly estimates		mixed frequency estimates	
ϑ	0.9	0.899		0.893		0.898	
		0.876	*0.926*	*0.857*	*0.935*	*0.876*	*0.925*
φ_y	0.5	0.584		0.375		0.633	
		0.170	*1.589*	*0.061*	*2.348*	*0.121*	*1.744*
φ_π	1.5	1.404		1.403		1.492	
		0.856	*2.593*	*0.511*	*2.784*	*0.764*	*3.308*
ρ_r	0.9	0.911		0.887		0.917	
		0.839	*0.956*	*0.803*	*0.967*	*0.818*	*0.960*
σ_s	0.1	0.100		0.100		0.100	
		0.095	*0.105*	*0.091*	*0.110*	*0.095*	*0.105*
σ_d	0.1	0.100		0.099		0.099	
		0.094	*0.105*	*0.089*	*0.108*	*0.090*	*0.108*
σ_r	0.1	0.100		0.141		0.100	
		0.092	*0.108*	*0.124*	*0.162*	*0.090*	*0.109*

Notes: The estimates are obtained for a sample of 300 monthly observations or, equivalently 100 quarterly observations. The DGP is represented by the reduced form of the model in eq. (1) - (3). Column 2 reports the true parameters, from which we generated the data. Column 3 reports median, the 10th and 90th percentile across replications of the parameters estimated with monthly data, Column 4 with quarterly data and Column 5 with mixed-frequency data. The number of replications is fixed at 1000.
* A Calvo parameter θ equal to 0.9 at monthly frequency implies an average price duration of 10 months. To obtain the same implied average price duration at quarterly frequency θ should be equal to 0.7.

Table 2: Estimates of structural DSGE parameters with high-, mixed- and low-frequency simulated data (frequency mismatch: m =12)

	DGP	HF estimates		LF estimates		MF estimates	
ϑ	0.9	0.900		0.885		0.900	
		0.893	0.908	0.865	0.917	0.893	0.908
φ_y	0.5	0.559		0.184		0.521	
		0.334	0.779	0.035	1.667	0.184	1.234
φ_π	1.5	1.580		1.572		1.596	
		1.179	2.152	0.804	2.093	0.856	2.841
ρ_r	0.9	0.907		0.845		0.903	
		0.876	0.926	0.794	0.958	0.838	0.947
σ_s	0.1	0.100		0.100		0.100	
		0.098	0.101	0.095	0.106	0.098	0.101
σ_d	0.1	0.100		0.099		0.099	
		0.099	0.102	0.094	0.105	0.094	0.105
σ_r	0.1	0.101		0.169		0.101	
		0.097	0.103	0.156	0.192	0.093	0.105

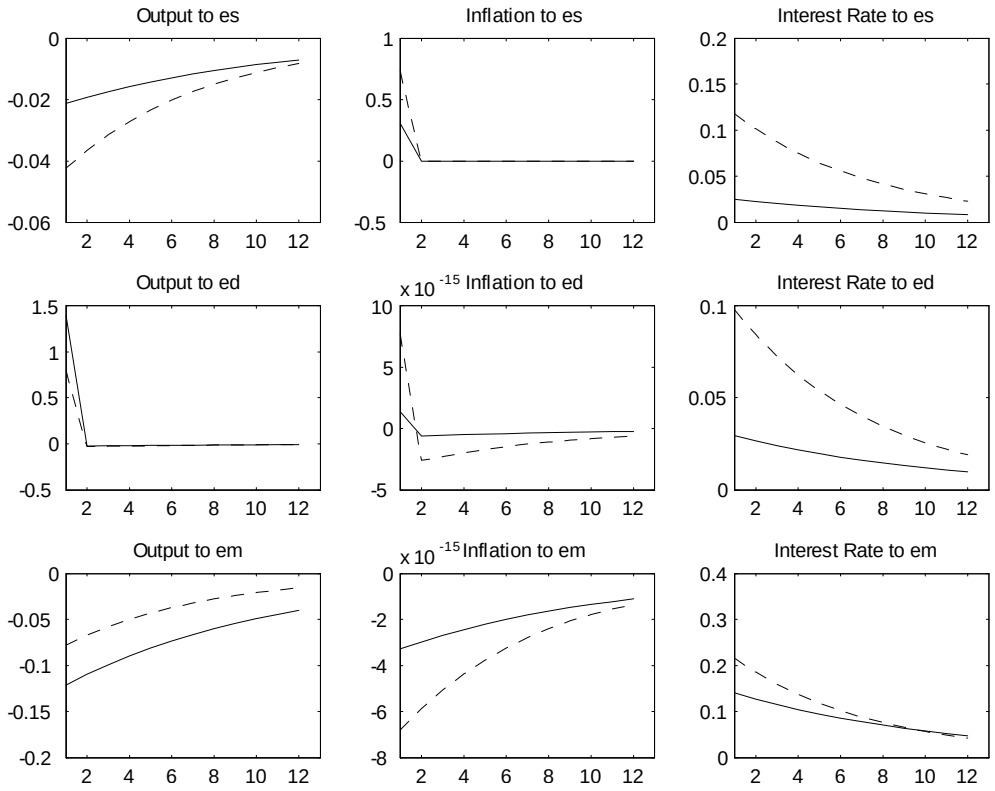
Notes: The estimates are obtained for a sample of 3600 high-frequency observations or, equivalently 300 low-frequency observations. The DGP is represented by the reduced form of the model in eq. (1) - (3). Column 2 reports the true parameters, from which we generated the data. Column 3 reports median, the 10th and 90th percentile across replications of the parameters estimated with high-frequency data, Column 4 with low-frequency data and Column 5 with mixed-frequency data. The number of replications is fixed at 500.

Table 3: Estimates of structural DSGE parameters with high-, mixed- and low-frequency simulated data (true DGP: weekly - data availability: monthly and quarterly.)

	DGP	HF estimates		LF estimates		MF estimates	
ϑ	0.9	0.900		0.885		0.885	
		0.893	0.908	0.865	0.917	0.870	0.900
φ_y	0.5	0.559		0.184		0.162	
		0.334	0.779	0.035	1.667	0.040	0.622
φ_π	1.5	1.580		1.572		1.118	
		1.179	2.152	0.804	2.093	0.916	1.783
ρ_r	0.9	0.907		0.845		0.832	
		0.876	0.926	0.794	0.958	0.782	0.909
σ_s	0.1	0.100		0.100		0.101	
		0.098	0.101	0.095	0.106	0.097	0.104
σ_d	0.1	0.100		0.099		0.099	
		0.099	0.102	0.094	0.105	0.094	0.105
σ_r	0.1	0.101		0.169		0.145	
		0.097	0.103	0.156	0.192	0.135	0.160

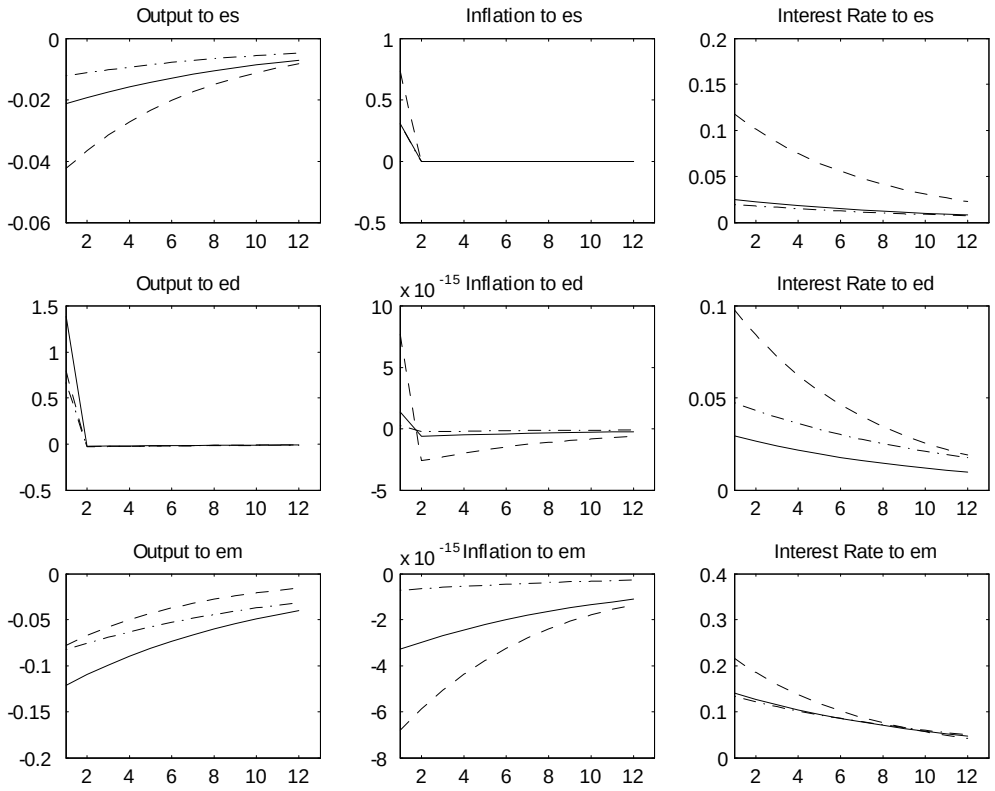
Notes: The estimates are obtained for a sample of 3600 weekly observations or, equivalently 900 monthly and 300 quarterly observations. The DGP is represented by the reduced form of the model in eq. (1) - (3). Column 2 reports the true parameters, from which we generated the data. Column 3 reports median, the 10th and 90th percentile across replications of the parameters estimated with weekly data, Column 4 with quarterly data and Column 5 with monthly and quarterly data. The number of replications is fixed at 500.

Figure 1: Impulse Responses for a small-scale New Keynesian DSGE model



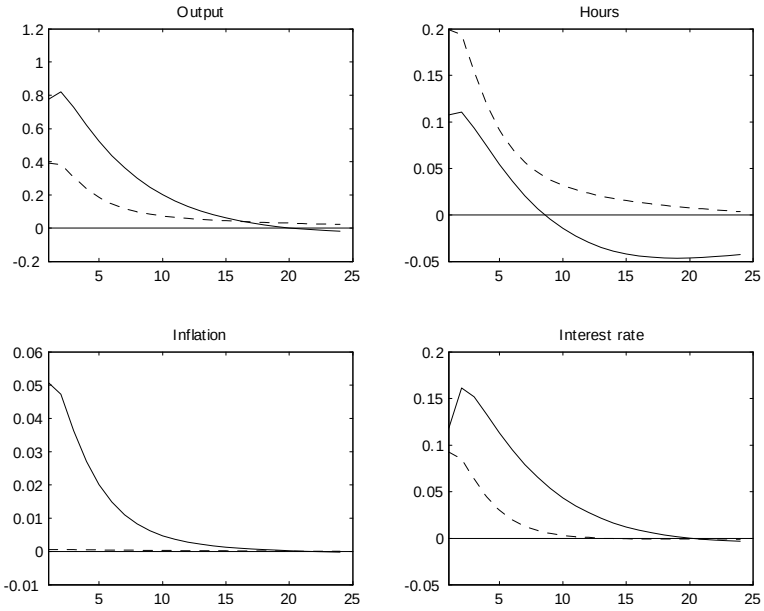
Notes: The impulse responses in solid lines are those obtained with mixed-frequency data at a monthly frequency and then aggregated to a quarterly level. The dashed ones are those obtained with the quarterly data. The shocks are indicated as follows: es the supply shock, ed the demand shock, em the monetary policy shock. The sample considered for the estimation spans 1965-2007.

Figure 2: Impulse Responses for a small-scale New Keynesian DSGE model: a monthly benchmark



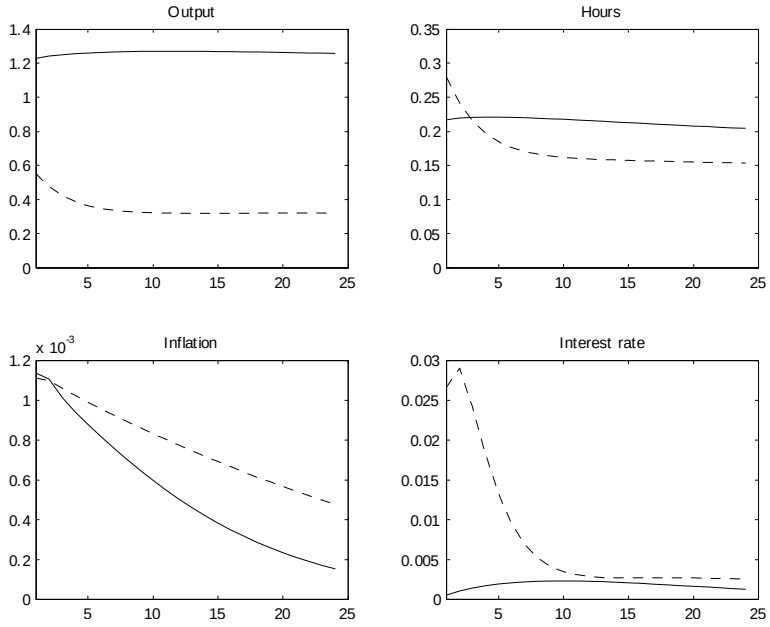
Notes: The impulse responses in solid lines are those obtained with mixed-frequency data at a monthly frequency and then aggregated to a quarterly level. The dashed ones are those obtained with the quarterly data. The dashed-dotted ones are those obtained with the monthly data, considering industrial production as measure of real activity, and then aggregated to a quarterly level. The shocks are indicated as follows: es the supply shock, ed the demand shock, em the monetary policy shock. The sample considered for the estimation spans 1965-2007.

Figure 3: Impulse Responses to a risk-premium shock



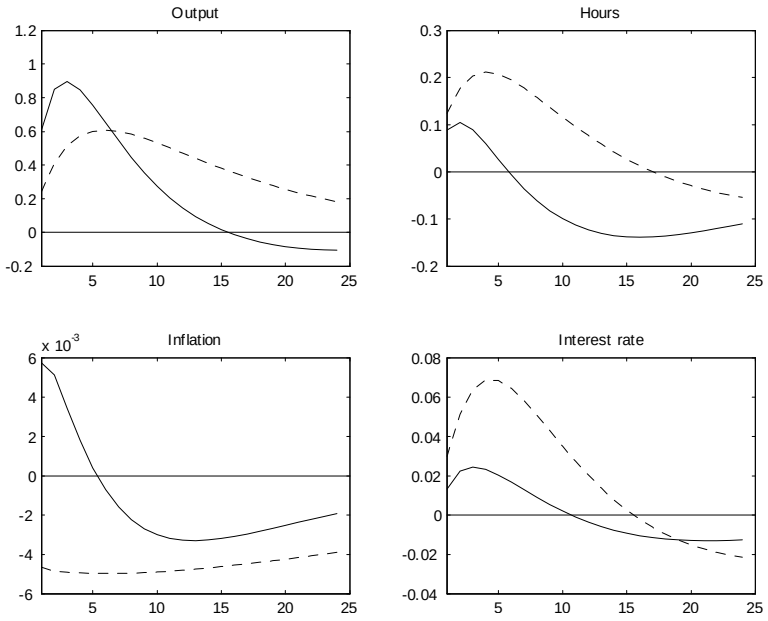
Notes: The impulse responses in solid lines are those obtained with mixed-frequency data at a monthly frequency and then aggregated to a quarterly level. The dashed ones, are those obtained with the quarterly data. The sample considered for the estimation spans 1965-2007.

Figure 4: Impulse Responses to an exogenous spending shock



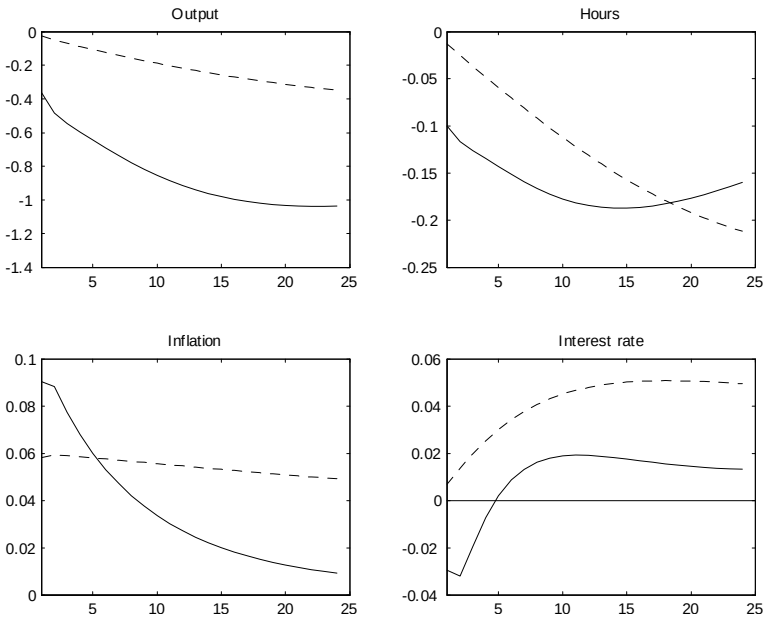
Notes: See Notes at Figure 3.

Figure 5: Impulse Responses to an investment-specific technology shock



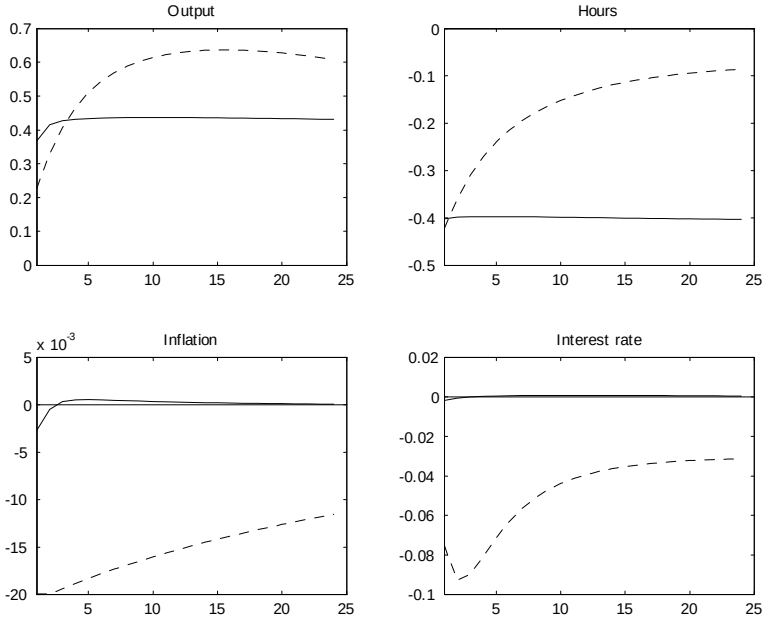
Notes: See Notes at Figure 3.

Figure 6: Impulse Responses to a wage markup shock



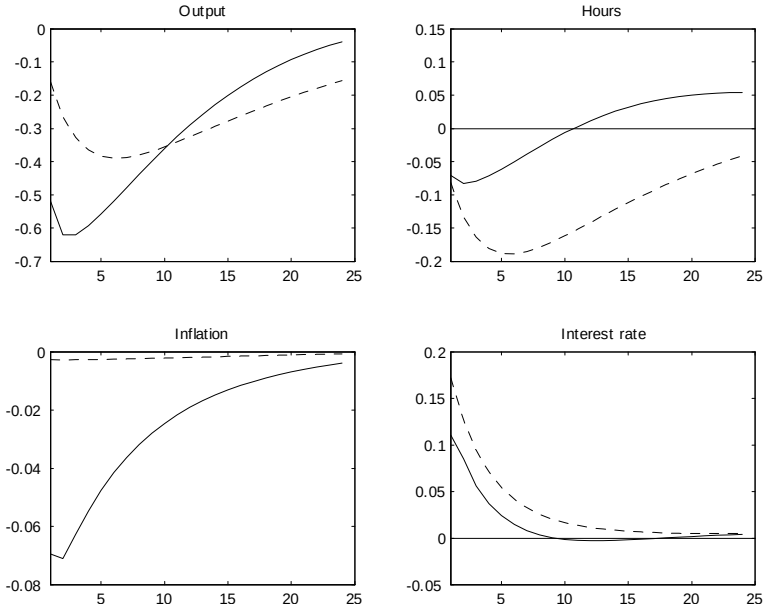
Notes: See Notes at Figure 3.

Figure 7: Impulse Responses to a productivity shock



Notes: See Notes at Figure 3.

Figure 8: Impulse Responses to a monetary policy shock



Notes: See Notes at Figure 3.

7 Appendix A

7.1 A basic New Keynesian model: mapping and identification issues

Our basic New Keynesian model is described by the following three equations:

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + \kappa y_t^* + \varepsilon_{st}, && (27) \\ y_t^* &= E_t y_{t+1}^* - \tau \left(R_t - E_t \pi_{t+1} \right) + \varepsilon_{dt}, && (28) \\ R_t &= \rho_r R_{t-1} + (1-\rho_r) \left(\phi_\pi \pi_t + \phi_y y_t^* \right) + \varepsilon_{rt}. && (29) \end{aligned}$$

Let us rewrite the model in a matrix form:

$$B_0 X_t^* = C X_{t-1}^* + D E_t X_{t+1}^* + \epsilon_t, \tag{30}$$

where $X_t^* = \left[\begin{array}{ccc} \pi_t & y_t^* & R_t \end{array} \right]'$ and $\epsilon_t = \left[\begin{array}{ccc} \epsilon_{st} & \epsilon_{dt} & \epsilon_{rt} \end{array} \right]'$, with $\epsilon_t \sim N(0, I_3)$.

The matrices B_0, C, D have the following form:

$$\begin{aligned} B_0 &= \left[\begin{array}{ccc} \frac{1}{\sigma_s} & -\frac{k}{\sigma_s} & 0 \\ 0 & \frac{1}{\sigma_d} & \frac{\tau}{\sigma_d} \\ \frac{\phi_\pi}{\sigma_r}(\rho_r - 1) & \frac{1}{\sigma_r}\phi_y (\rho_r - 1) & \frac{1}{\sigma_r} \end{array} \right], \\ C &= \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sigma_r}\rho_r \end{array} \right], \\ D &= \left[\begin{array}{ccc} \frac{\beta}{\sigma_s} & 0 & 0 \\ \frac{\tau}{\sigma_d} & \frac{1}{\sigma_d} & 0 \\ 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

The unique stable solution for this model is given by

$$A_0 X_t^* = A_1 X_{t-1}^* + \epsilon_t, \tag{31}$$

with A_0 and A_1 defined as follows:

$$A_0 = \begin{bmatrix} \frac{1}{\sigma_s} & -\frac{k}{\sigma_s} & G \\ 0 & \frac{1}{\sigma_d} & F \\ \frac{\phi_\pi}{\sigma_r}(\rho_r - 1) & \frac{1}{\sigma_r}\phi_y(\rho_r - 1) & \frac{1}{\sigma_r} \end{bmatrix},$$

$$A_1 = C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sigma_r}\rho_r \end{bmatrix}$$

where

$$G = \frac{\beta}{\sigma_r \sigma_s}\rho_r \frac{G\sigma_r\sigma_s + Fk\sigma_d\sigma_r}{G\phi_\pi\sigma_s + F\sigma_d\phi_y + Fk\phi_\pi\sigma_d - G\phi_\pi\sigma_s\rho_r - F\sigma_d\rho_r\phi_y - Fk\phi_\pi\sigma_d\rho_r + 1}$$

and

$$F = \frac{\tau}{\sigma_d} + \frac{1}{\sigma_r}\rho_r \left(\begin{array}{c} F\frac{\sigma_r}{G\phi_\pi\sigma_s + F\sigma_d\phi_y + Fk\phi_\pi\sigma_d - G\phi_\pi\sigma_s\rho_r - F\sigma_d\rho_r\phi_y - Fk\phi_\pi\sigma_d\rho_r + 1} + \\ \frac{\tau}{\sigma_d}\frac{G\sigma_r\sigma_s + Fk\sigma_d\sigma_r}{G\phi_\pi\sigma_s + F\sigma_d\phi_y + Fk\phi_\pi\sigma_d - G\phi_\pi\sigma_s\rho_r - F\sigma_d\rho_r\phi_y - Fk\phi_\pi\sigma_d\rho_r + 1} \end{array} \right).$$

It can be checked that all the parameters of this model can be easily identified.

We now provide the details for the second step of our analysis, i.e. how the monthly process can be aggregated to a quarterly level.

The process in (31) can be rewritten as

$$A_0 X_t^* = A_1 A_0^{-1} A_0 X_{t-1}^* + \epsilon_t, \tag{32}$$

and then by recursively substituting $A_0 X_{t-i}^*$ with its equivalent $A_1 A_0^{-1} A_0 X_{t-i}^* + \epsilon_{t-i}$, we obtain:

$$A_0 X_t^* = A_1 A_0^{-1} A_1 A_0^{-1} A_1 X_{t-3}^* + A_1 A_0^{-1} A_1 A_0^{-1}\epsilon_{t-2} + A_1 A_0^{-1}\epsilon_{t-1} + \epsilon_t \tag{33}$$

which we can write simply as

$$A_0 X_t^* = A_1^Q X_{t-3}^* + u_t, \tag{34}$$

where $u_t \sim N\left(0, \Sigma^Q\right)$ and

$$A_1^Q = A_1 A_0^{-1} A_1 A_0^{-1} A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\mathbf{1}}{\boldsymbol{\sigma}_r}\frac{\rho_r^3}{\left(G\phi_\pi\sigma_s + F\sigma_d\phi_y + Fk\phi_\pi\sigma_d - G\phi_\pi\sigma_s\rho_r - F\sigma_d\rho_r\phi_y - Fk\phi_\pi\sigma_d\rho_r + 1\right)^2} \end{bmatrix}.$$

We can show that from the elements of the matrices A_0 and A_1^Q defined above, we can recover all the parameters driving the monthly process. Specifically, from the matrix

A_0 we identify $\sigma_s, \sigma_d, \sigma_r$ and k. Moreover, we obtain F and G, and also the values of the combinations of parameters $\phi_\pi (1-\rho_r)$ and $\phi_y (1-\rho_r)$. We can rewrite the element of the matrix A_1^Q, $A_1^Q(3,3) = \frac{\mathbf{1}}{\boldsymbol{\sigma}_r} \frac{\rho_r^3}{\left(G\phi_\pi\sigma_s + F\sigma_d\phi_y + Fk\phi_\pi\sigma_d - G\phi_\pi\sigma_s\rho_r - F\sigma_d\rho_r\phi_y - Fk\phi_\pi\sigma_d\rho_r + 1\right)^2}$ as $A_1^Q(3,3) = \frac{\rho_r^3}{\boldsymbol{\sigma}_r\left(G\phi_\pi(1-\rho_r)\sigma_s + F\sigma_d\phi_y(1-\rho_r) + Fk\phi_\pi(1-\rho_r)\sigma_d + 1\right)^2}$, where all the elements in the denominator are known and therefore we can recover ρ_r from the numerator. Having ρ_r, we also identify ϕ_π and ϕ_y. The last two parameters, β and τ can be obtained from the definition of F and G.

7.2 Introducing more dynamic in the Euler equation

Our New Keynesian model is described by the following three equations:

$$\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + k y_t^* + \varepsilon_{st}, && (35)\\
y_t^* &= E_t y_{t+1}^* - \tau\left(R_t - E_t\pi_{t+1}\right) + p y_{t-1}^* + \varepsilon_{dt}, && (36)\\
R_t &= \rho_r R_{t-1} + (1-\rho_r)\left(\phi_\pi \pi_t + \phi_y y_t^*\right) + \varepsilon_{rt}, && (37)
\end{aligned}$$

which can be written in matrix form as:

$$B_0 X_t^* = C X_{t-1}^* + D E_t X_{t+1}^* + \epsilon_t, \tag{38}$$

where $X_t^* = \left[\begin{array}{ccc} \pi_t & y_t^* & R_t \end{array}\right]'$ and $\epsilon_t = \left[\begin{array}{ccc} \epsilon_{st} & \epsilon_{dt} & \epsilon_{rt} \end{array}\right]'$, with $\epsilon_t \sim N(0, I_3)$.

We normalize σ_d to 1.

The matrices B_0, C, D have the following form:

$$\begin{aligned}
B_0 &= \left[\begin{array}{ccc} \frac{1}{\sigma_s} & -\frac{k}{\sigma_s} & 0 \\ 0 & 1 & \tau \\ \frac{\phi_\pi}{\sigma_r}(\rho_r - 1) & \frac{1}{\sigma_r}\phi_x(\rho_r - 1) & \frac{1}{\sigma_r} \end{array}\right],\\
C &= \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & \frac{1}{\sigma_r}\rho_r \end{array}\right],\\
D &= \left[\begin{array}{ccc} \frac{\beta}{\sigma_s} & 0 & 0 \\ \tau & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].
\end{aligned}$$

The unique stable solution for this model is given by

$$A_0 X_t^* = A_1 X_{t-1}^* + \epsilon_t, \tag{39}$$

with A_0 and A_1 defined as follows:

$$\begin{aligned} A_0 &= \begin{bmatrix} \frac{1}{\sigma_s} & F & G \\ 0 & H & L \\ \frac{\phi_\pi}{\sigma_r}(\rho_r - 1) & \frac{1}{\sigma_r}\phi_x(\rho_r - 1) & \frac{1}{\sigma_r} \end{bmatrix}, \\ A_1 &= C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & \frac{1}{\sigma_r}\rho_r \end{bmatrix}, \end{aligned}$$

where

$$F = p\frac{\beta}{\sigma_s}\frac{F\sigma_s + G\sigma_s\phi_x - G\sigma_s\rho_r\phi_x}{H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r} - \frac{k}{\sigma_s},$$

$$G = \frac{\beta}{\sigma_r\sigma_s}\rho_r\frac{GH\sigma_r\sigma_s - FL\sigma_r\sigma_s}{H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r},$$

$$H = 1 - p\begin{pmatrix} \frac{G\phi_\pi\sigma_s - G\phi_\pi\sigma_s\rho_r + 1}{H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r} \\ -\tau\frac{F\sigma_s + G\sigma_s\phi_x - G\sigma_s\rho_r\phi_x}{H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r} \end{pmatrix},$$

$$L = \tau + \frac{1}{\sigma_r}\rho_r\begin{pmatrix} L\frac{\sigma_r}{H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r} \\ +\tau\frac{GH\sigma_r\sigma_s - FL\sigma_r\sigma_s}{H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r} \end{pmatrix}.$$

It can be checked that all the parameters of the model are identified.

To aggregate the process at a quarterly level, we conduct the same steps as in Appendix 7.1 and obtain

$$A_0 X_t^* = A_1^Q X_{t-3}^* + u_t, \tag{40}$$

where $u_t \sim N\left(0, \Sigma^Q\right)$ and

$$A_1^Q = A_1 A_0^{-1} A_1 A_0^{-1} A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M & N \\ 0 & P & Q \end{bmatrix},$$

with M, N, P, Q highly non-linear functions of all the structural parameters:

$$M = p\begin{pmatrix} p^2\frac{(G\phi_\pi\sigma_s - G\phi_\pi\sigma_s\rho_r + 1)^2}{(H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r)^2} \\ -Lp\rho_r\frac{\phi_x - \rho_r\phi_x - F\phi_\pi\sigma_s + F\phi_\pi\sigma_s\rho_r}{(H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r)^2} \end{pmatrix},$$

$$N = -\frac{1}{\sigma_r}\rho_r\begin{pmatrix} Lp^2\sigma_r\frac{G\phi_\pi\sigma_s - G\phi_\pi\sigma_s\rho_r + 1}{(H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r)^2} \\ +HLp\sigma_r\frac{\rho_r}{(H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r)^2} \end{pmatrix},$$

$$P = p \begin{pmatrix} \frac{H}{\sigma_r}\rho_r^2 \frac{\phi_x-\rho_r\phi_x-F\phi_\pi\sigma_s+F\phi_\pi\sigma_s\rho_r}{(H+L\phi_x-L\rho_r\phi_x+GH\phi_\pi\sigma_s-FL\phi_\pi\sigma_s-GH\phi_\pi\sigma_s\rho_r+FL\phi_\pi\sigma_s\rho_r)^2} \\ +\frac{p}{\sigma_r}\rho_r\left(G\phi_\pi\sigma_s - G\phi_\pi\sigma_s\rho_r + 1\right)\frac{\phi_x-\rho_r\phi_x-F\phi_\pi\sigma_s+F\phi_\pi\sigma_s\rho_r}{(H+L\phi_x-L\rho_r\phi_x+GH\phi_\pi\sigma_s-FL\phi_\pi\sigma_s-GH\phi_\pi\sigma_s\rho_r+FL\phi_\pi\sigma_s\rho_r)^2} \end{pmatrix},$$

$$Q = \frac{1}{\sigma_r}\rho_r \begin{pmatrix} H^2\frac{\rho_r^2}{(H+L\phi_x-L\rho_r\phi_x+GH\phi_\pi\sigma_s-FL\phi_\pi\sigma_s-GH\phi_\pi\sigma_s\rho_r+FL\phi_\pi\sigma_s\rho_r)^2} \\ -Lp\rho_r\frac{\phi_x-\rho_r\phi_x-F\phi_\pi\sigma_s+F\phi_\pi\sigma_s\rho_r}{(H+L\phi_x-L\rho_r\phi_x+GH\phi_\pi\sigma_s-FL\phi_\pi\sigma_s-GH\phi_\pi\sigma_s\rho_r+FL\phi_\pi\sigma_s\rho_r)^2} \end{pmatrix}.$$

While it is still possible to easily identify σ_s and σ_r, we need to solve highly non-linear equations to find the other parameters, which do not give rise to a unique solution.

In general, we can write the solution of a DSGE model as

$$s_t = As_{t-1} + Bu_t. \tag{41}$$

Aggregating the process to a quarterly level, we obtain

$$s_\tau = A^3 s_{\tau-1} + \varepsilon_\tau, \tag{42}$$

with the possibility of an MA component, depending on the aggregation method chosen. Identifying the monthly process from the quarterly one would imply at least to get a unique A from the matrix A^3. Therefore, we want to find the cube roots of a matrix. But, we can show in the easiest case possible that this is not necessarily unique. Let us consider for simplicity A^3 as a 2×2 identity matrix[16]. It is obvious to see that $A = I_2$ is a cube root of A^3. However, it is possible to check that any matrix

$$A = \begin{bmatrix} -d-1 & -\frac{1}{f}\left(d^2+d+1\right) \\ f & d \end{bmatrix}, \quad d, f \in \mathbf{R}, \quad f \neq 0, \tag{43}$$

is also a cube root. This is therefore a very easy example which shows that from A^3 we cannot always uniquely identify A.

7.3 Obtaining identification by exploiting mixed frequency data

The unique stable solution for the model

$$\pi_t = \beta E_t\pi_{t+1} + ky_t^* + \varepsilon_{st}, \tag{44}$$

$$y_t^* = E_t y_{t+1}^* - \tau\left(R_t - E_t\pi_{t+1}\right) + py_{t-1}^* + \varepsilon_{dt}, \tag{45}$$

$$R_t = \rho_r R_{t-1} + (1-\rho_r)\left(\phi_\pi\pi_t + \phi_y y_t^*\right) + \varepsilon_{rt}, \tag{46}$$

[16] The dimension of the matrix is set at $n = 2$ for simplicity, but it can be extended to any other n.

is given by

$$A_0 X_t^* = A_1 X_{t-1}^* + \epsilon_t, \tag{47}$$

with A_0 and A_1 defined as follows:

$$\begin{aligned} A_0 &= \begin{bmatrix} \frac{1}{\sigma_s} & F & G \\ 0 & H & L \\ \frac{\phi_\pi}{\sigma_r}(\rho_r - 1) & \frac{1}{\sigma_r}\phi_x(\rho_r - 1) & \frac{1}{\sigma_r} \end{bmatrix}, \\ A_1 &= C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & \frac{1}{\sigma_r}\rho_r \end{bmatrix}. \end{aligned}$$

We can rewrite (47) as a system of three equations:

$$\frac{1}{\sigma_s}\pi_t + F y_t^* + G R_t = \epsilon_{st} \tag{48}$$

$$H y_t^* + L R_t = p y_{t-1}^* + \epsilon_{dt} \tag{49}$$

$$\frac{\phi_\pi}{\sigma_r}(\rho_r - 1)\pi_t + \frac{\phi_y}{\sigma_r}(\rho_r - 1) y_t^* + \frac{1}{\sigma_r} R_t = \frac{1}{\sigma_r}\rho_r R_{t-1} + \epsilon_{rt}. \tag{50}$$

We then need to modify eq. (16) in such a way that it contains only variables which are available at the time of estimation. If we substitute y_{t-1}^* with its own expression $y_{t-1}^* = \frac{p}{H} y_{t-2}^* - \frac{L}{H} R_{t-1} + \frac{1}{H}\epsilon_{dt-1}$, and then we repeat it again for y_{t-2}^*, we obtain:

$$\begin{aligned} y_t^* &= \frac{p}{H}\left(\frac{p}{H} y_{t-2}^* - \frac{L}{H} R_{t-1} + \frac{1}{H}\epsilon_{dt-1}\right) - \frac{L}{H} R_t + \frac{1}{H}\epsilon_{dt-1} = \\ &= \frac{p}{H}\left(\frac{p}{H}\left(\frac{p}{H} y_{t-3}^* - \frac{L}{H} R_{t-2} + \frac{1}{H}\epsilon_{dt-1}\right) - \frac{L}{H} R_{t-1} + \frac{1}{H}\epsilon_{dt-1}\right) - \frac{L}{H} R_t + \frac{1}{H}\epsilon_{dt-1} = \\ &= \left(\frac{p}{H}\right)^3 y_{t-3}^* - \frac{L}{H} R_t - \frac{L}{H}\left(\frac{p}{H}\right) R_{t-1} - \frac{L}{H}\left(\frac{p}{H}\right)^2 R_{t-2} + \xi_t. \end{aligned} \tag{51}$$

From eq. (48), (51) and (50), we can now identify all the parameters. From eq. (48), we identify σ_s and obtain F and G. From eq. (50), we identify σ_r, ρ_r, ϕ_y and ϕ_π. From eq. (51), we obtain $\frac{L}{H}$ and $\frac{p}{H}$. Moreover, we know that F, G, H, L are defined as:

$$F = p\frac{\beta}{\sigma_s}\frac{F\sigma_s + G\sigma_s\phi_x - G\sigma_s\rho_r\phi_x}{H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r} - \frac{k}{\sigma_s},$$

$$G = \frac{\beta}{\sigma_r\sigma_s}\rho_r\frac{GH\sigma_r\sigma_s - FL\sigma_r\sigma_s}{H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r},$$

$$H = 1 - p\begin{pmatrix} \frac{G\phi_\pi\sigma_s - G\phi_\pi\sigma_s\rho_r + 1}{H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r} \\ -\tau\frac{F\sigma_s + G\sigma_s\phi_x - G\sigma_s\rho_r\phi_x}{H + L\phi_x - L\rho_r\phi_x + GH\phi_\pi\sigma_s - FL\phi_\pi\sigma_s - GH\phi_\pi\sigma_s\rho_r + FL\phi_\pi\sigma_s\rho_r} \end{pmatrix},$$

$$L = \tau + \frac{1}{\sigma_r}\rho_r \left(\begin{array}{c} L\frac{\sigma_r}{H+L\phi_x-L\rho_r\phi_x+GH\phi_\pi\sigma_s-FL\phi_\pi\sigma_s-GH\phi_\pi\sigma_s\rho_r+FL\phi_\pi\sigma_s\rho_r} \\ +\tau\frac{GH\sigma_r\sigma_s-FL\sigma_r\sigma_s}{H+L\phi_x-L\rho_r\phi_x+GH\phi_\pi\sigma_s-FL\phi_\pi\sigma_s-GH\phi_\pi\sigma_s\rho_r+FL\phi_\pi\sigma_s\rho_r} \end{array} \right).$$

After some algebraic manipulations, from the definition of G we obtain β, and from the definition of F we identify k. Combining the definitions of H and L, we obtain τ. Once we have τ, we have all the necessary parameters to disentangle H and L, and as a consequence we identify also p.

8 Appendix B

8.1 Smets and Wouters (2007, SW): the model equations

We list here the equations which describe the dynamics of the SW model.

The model variables for the sticky wage and price economy are: output (y_t), consumption (c_t), investment (i_t), Tobin's q (q_t), utilized capital (k_t^s), installed capital (k_t), capacity utilization (z_t), rental rate of capital (r_t^k), price markup (μ_t^p), inflation rate (π_t), wage markup (μ_t^w), real wage (w_t), total hours worked (l_t), and nominal interest rate (r_t). For the corresponding flexible economy: output (y_t^*), consumption (c_t^*), investment (i_t^*), Tobin's q (q_t^*), utilized capital (k_t^{s*}), installed capital (k_t^*), capacity utilization (z_t^*), rental rate of capital (r_t^{k*}), price markup (μ_t^{p*}), wage markup (μ_t^{w*}), real wage (w_t^*), and total hours worked (l_t^*), for the corresponding flexible economy.

The shocks are: total factor productivity (ε_t^a), investment-specific technology (ε_t^i), government purchases (ε_t^g), risk premium (ε_t^b), monetary policy (ε_t^r), wage markup (ε_t^w) and price markup (ε_t^p).

Flexible economy:

$$\varepsilon_t^a = \alpha r_t^{k*} + (1-\alpha)\, w_t^* \tag{52}$$

$$z_t^* = \frac{1}{\frac{\psi}{1-\psi}} r_t^{k*} \tag{53}$$

$$k_t^{s*} = z_t^* + k_{t-1}^* \tag{54}$$

$$r_t^{k*} = w_t^* + l_t^* - k_t^{s*} \tag{55}$$

$$y_t^* = c_y c_t^* + i_y i_t^* + r^{kss} k_y z_t^* + \varepsilon_t^g \tag{56}$$

$$y_t^* = \Phi\left(\alpha\, k_t^{s*} + (1-\alpha)\, l_t^* + \varepsilon_t^a\right) \tag{57}$$

$$i_t^* = \frac{1}{1+\beta\,\gamma^{(1-\sigma_c)}}\left(i_{t-1}^* + \beta\,\gamma^{(1-\sigma_c)} E_t\, i_{t+1}^* + \frac{1}{\gamma^2\,\varphi}\, q_t^*\right) + \varepsilon_t^i \tag{58}$$

$$q_t^* = -r_t^* + \left(1-\beta\,(1-\delta)\,\gamma^{-\sigma_c}\right) E_t\, r_{t+1}^{k*} + \beta\,(1-\delta)\,\gamma^{-\sigma_c}\, E_t q_{t+1}^* - \varepsilon_t^b \tag{59}$$

$$\begin{aligned} c_t^* = {} & \frac{\frac{h}{\gamma}}{1+\frac{h}{\gamma}}\, c_{t-1}^* + \frac{1}{1+\frac{h}{\gamma}}\, E_t c_{t+1}^* + \frac{(\sigma_c-1)\; w^{ss} l^{ss}}{c^{ss}\sigma_c\left(1+\frac{h}{\gamma}\right)}\left(l_t^* - E_t l_{t+1}^*\right) \\ & - \frac{1-\frac{h}{\gamma}}{\sigma_c\left(1+\frac{h}{\gamma}\right)} r_t^* + \varepsilon_t^b \end{aligned} \tag{60}$$

$$w_t^* = \sigma_l l_t^* + \frac{1}{1-\frac{h}{\gamma}} c_t^* - \frac{\frac{h}{\gamma}}{1-\frac{h}{\gamma}} c_{t-1}^* \tag{61}$$

$$\begin{aligned} k_t^* = {} & \frac{(1-\delta)}{\gamma} k_{t-1}^* + \left(1 - \frac{(1-\delta)}{\gamma}\right) i_t^* \\ & + \gamma^2\,\varphi\left(1 - \frac{(1-\delta)}{\gamma}\right)\left(1+\beta\,\gamma^{(1-\sigma_c)}\right)\varepsilon_t^i \end{aligned} \tag{62}$$

Sticky Wage economy:

$$\mu_t^p = \alpha\, r_t^k + (1-\alpha)\; w_t - \varepsilon_t^a \tag{63}$$

$$z_t = \frac{1}{\frac{\psi}{1-\psi}}\, r_t^k \tag{64}$$

$$r_t^k = w_t + l_t - k_t^s \tag{65}$$

$$k_t^s = z_t + k_{t-1} \tag{66}$$

$$i_t = \frac{1}{1+\beta\gamma^{(1-\sigma_c)}} \left(i_{t-1} + \beta\gamma^{(1-\sigma_c)} E_t\, i_{t+1} + \frac{1}{\gamma^2\,\varphi}\, q_t \right) + \varepsilon_t^i \tag{67}$$

$$\begin{aligned} q_t = {} & -r_t + E_t\pi_{t+1} + \left(1 - \beta\,(1-\delta)\,\gamma^{-\sigma_c}\right)\; E_t r_{t+1}^k \\ & + \beta\,(1-\delta)\,\gamma^{-\sigma_c}\, E_t q_{t+1} - \varepsilon_t^b \end{aligned} \tag{68}$$

$$\begin{aligned} c_t = {} & \frac{\frac{h}{\gamma}}{1+\frac{h}{\gamma}}\, c_{t-1} + \frac{1}{1+\frac{h}{\gamma}} E_t\, c_{t+1} + \frac{(\sigma_c - 1)\; w^{ss} l^{ss}}{c^{ss}\sigma_c\; \left(1+\frac{h}{\gamma}\right)}\; (l_t - E_t l_{t+1}) \\ & - \frac{1-\frac{h}{\gamma}}{\sigma_c\; \left(1+\frac{h}{\gamma}\right)}\; (r_t - E_t\pi_{t+1}) + \varepsilon_t^b \end{aligned} \tag{69}$$

$$y_t = c_y\, c_t + i_i\, i_t + r^{kss} k_y\, z_t + \varepsilon_t^g \tag{70}$$

$$y_t = \Phi\; (\alpha\, k_t^s + (1-\alpha)\; l_t + \varepsilon_t^a) \tag{71}$$

$$\begin{aligned} r_t = {} & r_\pi\; (1-\rho)\,\pi_t + (1-\rho)\; r_y\; (y_t - y_t^*) \\ & + r_{\Delta y}\; \left(y_t - y_t^* - y_{t-1} + y_{t-1}^*\right) + \rho\, r_{t-1} + \varepsilon_t^r \end{aligned} \tag{72}$$

$$\begin{aligned} \pi_t = {} & \frac{1}{1+\beta\gamma^{(1-\sigma_c)}\,\iota_p} * \\ & \left(\beta\gamma^{(1-\sigma_c)} E_t\pi_{t+1} + \iota_p\,\pi_{t-1} + \frac{\frac{(1-\xi_p)\left(1-\beta\gamma^{(1-\sigma_c)}\xi_p\right)}{\xi_p}}{1+(\Phi-1)\;\varepsilon_p} \mu_t^p \right) + \varepsilon_t^p \end{aligned} \tag{73}$$

$$\begin{aligned} w_t = {} & \frac{1}{1+\beta\gamma^{(1-\sigma_c)}}\, w_{t-1} + \frac{\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}}\, E_t w_{t+1} + \frac{\iota_w}{1\beta\gamma^{(1-\sigma_c)}}\pi_{t-1} \\ & - \frac{1+\beta\gamma^{(1-\sigma_c)}\,\iota_w}{1+\beta\gamma^{(1-\sigma_c)}}\pi_t + \frac{\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}} E_t\pi_{t+1} \\ & + \frac{(1-\xi_w)\;\left(1-\beta\gamma^{(1-\sigma_c)}\,\xi_w\right)}{\left(1+\beta\gamma^{(1-\sigma_c)}\right)\;\xi_w}\; \frac{1}{1+(\lambda_w-1)\;\varepsilon_w} * \\ & \left(\sigma_l\, l_t + \frac{1}{1-\frac{h}{\gamma}}\, c_t - \frac{\frac{h}{\gamma}}{1-\frac{h}{\gamma}}\, c_{t-1} - w_t \right) + \varepsilon_t^w \end{aligned} \tag{74}$$

$$\begin{aligned} k_t = {} & \frac{(1-\delta)}{\gamma}\, k_{t-1} + \left(1 - \frac{(1-\delta)}{\gamma}\right)\, i_t \\ & + \varphi\,\gamma^2\, \left(1 - \frac{(1-\delta)}{\gamma}\right) \left(1 + \beta\,\gamma^{(1-\sigma_c)}\right) \varepsilon_t^i \end{aligned} \tag{75}$$

Dynamics of the shocks:

$$\varepsilon_t^a = \rho_a \varepsilon_{t-1}^a + \eta_t^a \tag{76}$$
$$\varepsilon_t^b = \rho_b \varepsilon_{t-1}^b + \eta_t^b \tag{77}$$
$$\varepsilon_t^g = \rho_g \varepsilon_{t-1}^g + \eta_t^g + \rho_{ga} \eta_t^a \tag{78}$$
$$\varepsilon_t^i = \rho_I \varepsilon_{t-1}^i + \eta_t^I \tag{79}$$
$$\varepsilon_t^r = \rho_r \varepsilon_{t-1}^r + e_t^r \tag{80}$$
$$\varepsilon_t^p = \rho_\pi \varepsilon_{t-1}^p + \eta_t^p - \mu_p \eta_{t-1}^p \tag{81}$$
$$\varepsilon_t^w = \rho_w \varepsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w \tag{82}$$

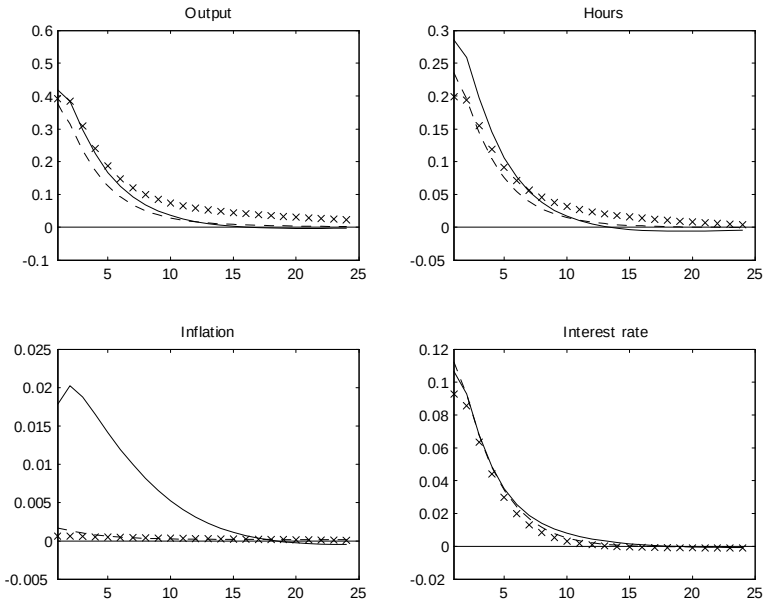
Measurement equations:

$$dlGDP_t = y_t - y_{t-1} + \bar{\gamma} \tag{83}$$
$$dlCONS_t = \bar{\gamma} + c_t - c_{t-1} \tag{84}$$
$$dlINV_t = \bar{\gamma} + i_t - i_{t-1} \tag{85}$$
$$dlWAG_t = \bar{\gamma} + w_t - w_{t-1} \tag{86}$$
$$dlP_t = \pi_t + \bar{\pi} \tag{87}$$
$$FEDFUNDS_t = r_t + \bar{r} \tag{88}$$
$$lHOURS_t = l_t + \bar{l} \tag{89}$$

8.2 Smets and Wouters (2007): comparison between IRFs obtained with Bayesian and Maximum Likelihood estimation

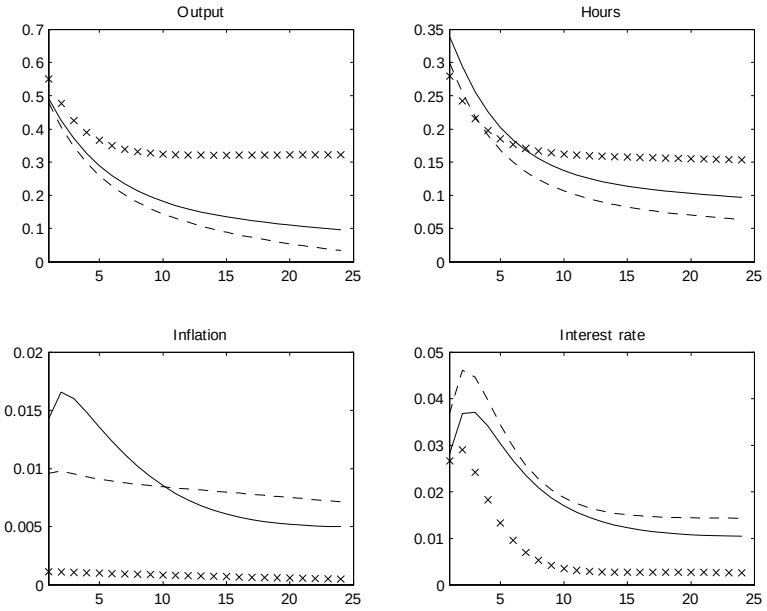
In the following figures we report the impulses responses by Smets and Wouters (in solid line), the ones we obtained by estimating the model with ML on the same sample (in dashed line), and on the sample extended to the end of 2007 (in x-marked line). The graphs are meant to show that the results obtained with ML are fairly similar to the original ones with Bayesian techniques. Moreover, our ML results on the original sample are very similar to the ones obtained by Iskrev (2008) in a similar experiment.

Figure 9: Impulse Responses to a risk-premium shock



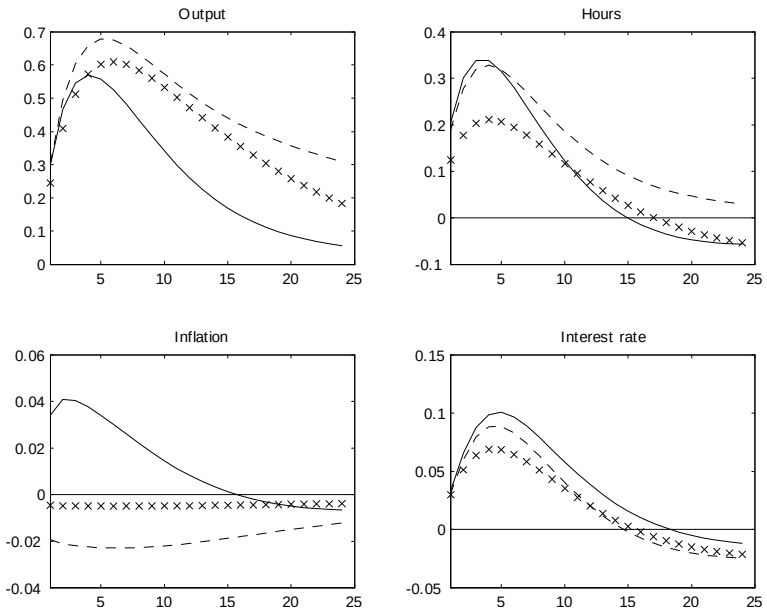
Notes: The impulse responses in solid lines are those obtained by Smets and Wouters (2007). The dashed ones, are those obtained with ML estimation on the same sample. The x-marked ones are those obtained with ML on the sample extended to 2007.

Figure 10: Impulse Responses to an exogenous spending shock



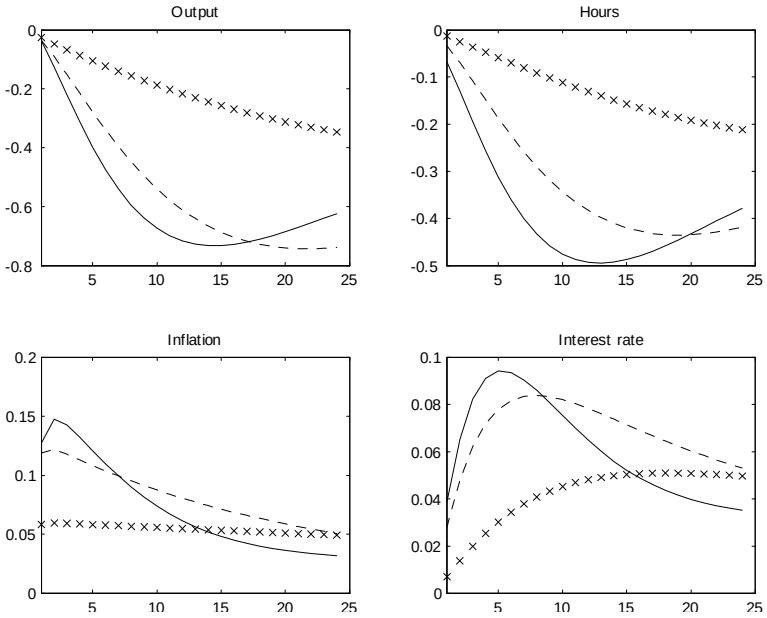
Notes: See Notes at Figure 9.

Figure 11: Impulse Responses to an investment-specific technology shock



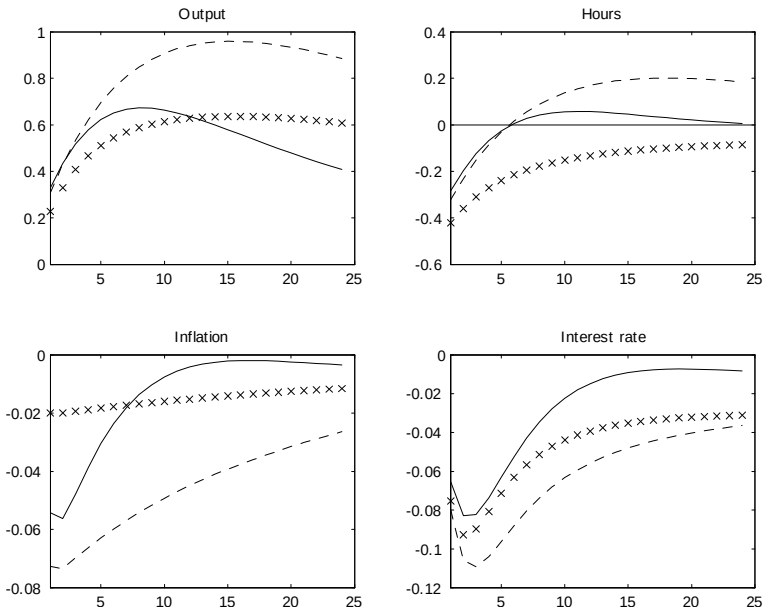
Notes: See Notes at Figure 9.

Figure 12: Impulse Responses to a wage markup shock



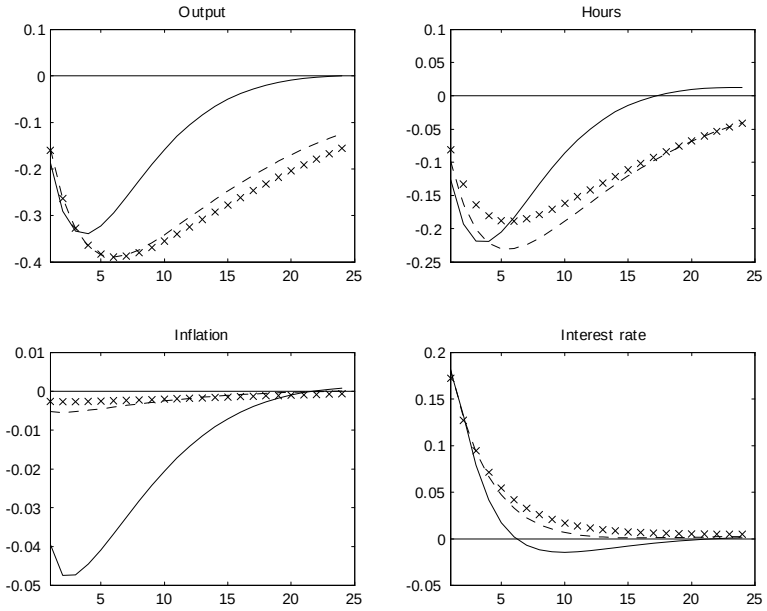
Notes: See Notes at Figure 9.

Figure 13: Impulse Responses to a productivity shock



Notes: See Notes at Figure 9.

Figure 14: Impulse Responses to a monetary policy shock



Notes: See Notes at Figure 9.