Gradualism and Liquidity Traps*

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Abstract

Modifying the objective function of a discretionary central bank to include an interest-rate smoothing objective increases the welfare of an economy in which large contractionary shocks occasionally force the central bank to lower the policy rate to its effective lower bound. The central bank with an interest-rate smoothing objective credibly keeps the policy rate low for longer than the central bank with the standard objective function. Through expectations, the temporary overheating of the economy associated with such a low-for-long interest rate policy mitigates the declines in inflation and output when the lower bound constraint is binding. In a calibrated quantitative model, we find that the introduction of an interest-rate smoothing objective can reduce the welfare costs associated with the lower bound constraint by about one-half.

Keywords: Gradualism, Inflation Targeting, Interest-Rate Smoothing, Liquidity Traps, Zero Lower Bound

JEL-Codes: E52, E61

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1 Introduction

As a general rule, the Federal Reserve tends to adjust interest rates incrementally, in a series of small or moderate steps in the same direction.

Ben S. Bernanke, on May 20, 2004[1]

Gradual adjustment in the federal funds rate has been a key feature of monetary policy in the United States. Over the two decades prior to December 2008—the beginning of the most recent lower-bound episode—the Federal Open Market Committee (FOMC) changed its target for the federal funds rate at 89 out of 191 meetings. At these 89 meetings, the FOMC adjusted the federal funds target rate, on average, just 33 basis points in absolute terms. More recently, when announcing the first increase in its target range for the federal funds rate in December 2015 after seven years of zero-interest rate policy, the FOMC emphasized that it expected the policy rate to increase only gradually (Federal Open Market Committee (2015)). Indeed, as of July 2017, the federal funds target range has been raised only four times, in steps of 25 basis points, since December 2015.

While there are likely myriad factors behind this gradual adjustment in the policy rate, some evidence suggests that the observed inertia in the policy rate reflects the central bank's deliberate desire to smooth the interest rate path beyond what the intrinsic inertia in economic conditions calls for (Coibion and Gorodnichenko (2012); Givens (2012)). As we will review, several studies suggest that interest-rate smoothing can improve society's welfare in various environments.

In this paper, we revisit the desirability of interest-rate smoothing in an economy in which large contractionary shocks occasionally force the central bank to lower the policy rate to the zero lower bound (ZLB). We conduct our analysis in the framework of policy delegation in which society designs the central bank's objective function and the central bank, in turn, acts under discretion and sets the policy rate in accordance with the objective.[2] In so doing, we stick to the optimal delegation literature's focus on simple non-state-contingent objective functions that involve only a small number of target variables.[3] Using a stochastic New Keynesian model, we ask how modifying the central bank's objective function to include an interest-rate smoothing (IRS) objective affects stabilization policy and society's welfare, as measured by the expected lifetime

[1]Bernanke (2004), "Gradualism," speech delivered at an economics luncheon co-sponsored by the Federal Reserve Bank of San Francisco (Seattle Branch) and the University of Washington, Seattle, Washington, May 20, https://www.federalreserve.gov/boarddocs/speeches/2004/200405202/default.htm.

[2]Prominent examples of adopting the policy delegation approach to the design of the central bank's objective include Rogoff (1985), Persson and Tabellini (1993), Walsh (1995, 2003), and Svensson (1997). For a literature review, see Persson and Tabellini (1999).

[3]In principle, it is often possible to design more complex state-contingent objective functions that better approximate the optimal commitment solution within a particular model than simple objective functions do. However, the literature on policy delegation has found that such more elaborated objective functions are typically too complicated to be of practical interest. See, e.g., Walsh (1995). Our goal, shared with much of the literature, is to analyze the design of objective functions that can be rendered implementable and communicable in practice. Interestingly, Yellen (2012), then Vice Chair of the Board of Governors of the Federal Reserve System, in a speech presents some model-based counterfactual policy simulations in which the central bank's objective function is formalized in a way that is very similar to our specification.

utility of the representative household. We first use a stylized version of the model to transparently describe the key trade-off involved in adopting a gradualist policy. We then move on to the analysis of a quantitative model to understand the quantitative relevance of gradualism.

Our main finding is that adding an IRS objective to central banks' standard inflation and output gap stabilization objectives can go a long way in mitigating the adverse consequences of the ZLB constraint. In the aftermath of a deep recession involving a binding ZLB constraint, a gradualist central bank increases the policy rate more slowly than a central bank with the standard objective. Such a slow increase of the policy rate generates a temporary overheating of the economy, which mitigates the declines in inflation and output while the ZLB constraint is binding, by raising expectations of future inflation and real activity. A smaller contraction at the ZLB, in turn, alleviates the deflationary bias—the systematic undershooting of the inflation target—away from the ZLB via expectations. In equilibrium, interest-rate smoothing increases society's welfare by improving stabilization outcomes not only when the policy rate is at the ZLB but also when the policy rate is away from it.

Interest-rate smoothing, however, does not provide a free lunch. In particular, interest-rate smoothing prevents the central bank from responding sufficiently to less severe shocks that could be neutralized by an appropriate policy rate adjustment without hitting the ZLB. From a normative perspective, when the policy rate is away from the ZLB, the central bank should reduce the policy rate one-for-one to a downward shift in the natural real rate of interest—the real interest rate prevailing in an economy with flexible prices—to offset completely the effect of the shock to the natural real rate. A gradualist central bank will reduce the policy rate by less on impact, thus failing to keep inflation and the output gap fully stabilized.[4] The optimal degree of interest-rate gradualism balances this cost against the aforementioned benefits. We find that the welfare gains from interest-rate smoothing are quantitatively important. In our quantitative model calibrated to match key features of the U.S. economy, a central bank with an optimized weight on its IRS objective improves society's welfare by about one-half.

We also explore a refinement to our baseline IRS objective function that enhances the welfare gains from interest-rate gradualism. Instead of a smoothing objective for the actual policy rate, the refinement requires the central bank to be concerned with smoothing of the shadow policy rate—the policy rate that it would like to set given the current state of the economy if the ZLB were not a constraint for nominal interest rates. If the policymaker aims to smooth the shadow rate, the lagged shadow rate becomes an endogenous state variable that remembers the history of inflation rates and output gaps. In particular, the larger the economic downturn in a liquidity trap, the lower the shadow rate and the longer the actual policy rate remains low. The resulting history dependence is akin to that observed under the optimal commitment policy, and increases

[4] Interest-rate gradualism also prevents the central bank from neutralizing shocks that lead to an increase in the natural real rate, thereby allowing for above-target inflation rates and output gaps. As described in section 3.3, while such transitory overshootings are by themselves associated with lower welfare, they can improve welfare in an economy with an occasionally binding ZLB constraint, as they raise inflation and output gap expectations in states in which the natural rate is low and the policy rate is close to or at the ZLB.

the welfare gains from interest-rate smoothing.

Our paper is related to a body of work that has examined various motives for gradualist monetary policy.[5] The strand of the literature closest to our paper emphasizes the benefits of interest-rate smoothing arising from its ability to steer private-sector expectations by inducing history dependence in the policy rate (Woodford (2003b); Giannoni and Woodford (2003)).[6] Another strand of the literature emphasizes the benefit of interest-rate smoothing arising from its ability to better manage uncertainties about data, parameter values, or the structure of the economy facing the central bank (Sack (1998); Orphanides and Williams (2002); Levin, Wieland, and Williams (2003); Orphanides and Williams (2007)). Some studies emphasize the costs and benefits of interest-rate smoothing arising from its effects on financial stability (Cukierman (1991); Stein and Sunderam (2015)). None of these studies, however, accounts for the ZLB on nominal interest rates. Our contribution is to show that the presence of the ZLB provides a novel rationale for guiding monetary policy by gradualist principles.

Our work is also closely related to a set of papers that explores ways to mitigate the adverse consequences of the ZLB constraint while preserving time consistency. In particular, several approaches try to mimic the prescription of the optimal commitment policy for liquidity traps to keep the policy rate low for long, thus generating a temporary overheating of the economy. Eggertsson (2006) and Burgert and Schmidt (2014) show that in models with non-Ricardian fiscal policy and nominal government debt, discretionary policymakers can provide incentives to future policymakers to keep policy rates low for long periods of time by means of expansionary fiscal policy that raises the nominal level of government debt. Jeanne and Svensson (2007), Berriel and Mendes (2015), and Bhattarai, Eggertsson, and Gafarov (2015) find that central banks' balance sheet policies can, under certain conditions, operate as a commitment device for discretionary policymakers that facilitates the use of "low-for-long" policies. Finally, Billi (2016) explores policy delegation schemes in which the discretionary central bank's standard inflation and output gap stabilization objectives are replaced by either a price-level or a nominal-income stabilization objective. He finds that these delegation schemes can generate low-for-long policies and thereby improve welfare.[7] Compared with these approaches, the relative appeal of our approach is that it neither requires an additional policy instrument nor does it represent a fundamental departure from the inflation-targeting framework currently embraced by many central banks.[8]

The paper is organized as follows. Section 2 describes the baseline model. Section 3 presents the main results on the effect of interest-rate smoothing in the baseline model. Section 4 presents additional results for the baseline model. The first part considers a refinement of the interest-rate

[5] For an early literature overview, see Sack and Wieland (2000).

[6] For the analyses of other monetary policy regimes that induce history dependence, see, for instance, Vestin (2006) and Bilbiie (2014).

[7] Nakata and Schmidt (2014) show that the appointment of an inflation-conservative central banker improves welfare by mitigating the deflationary bias associated with discretionary policy in the presence of the ZLB. However, an inflation-conservative central banker does not follow low-for-long policies.

[8] See also Nakata (2014) for a reputational approach to make the temporary overheating of the economy in the aftermath of the crisis time-consistent.

smoothing objective that helps to further mitigate the welfare costs associated with the ZLB. The second part explores the role of cost-push shocks for the welfare results. Section 5 extends the analysis to a more elaborate quantitative model of the U.S. economy. A final section concludes.

2 The model

This section presents the model, lays down the policy problem of the central bank, and defines the equilibrium.

2.1 Private sector

The private sector of the economy is given by the standard New Keynesian structure formulated in discrete time with an infinite horizon as developed in detail in Woodford (2003a) and Gali (2008). A continuum of identical infinitely living households consumes a basket of differentiated goods and supplies labor in a perfectly competitive labor market. The consumption goods are produced by firms using (industry-specific) labor. Firms maximize profits subject to staggered price setting as in Calvo (1983). Following the majority of the literature on the ZLB, we put all model equations except for the ZLB constraint in semi-loglinear form.

The equilibrium conditions of the private sector are given by the following two equations:

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} \tag{1}$$

$$y_t = E_t y_{t+1} - \sigma \left(i_t - E_t \pi_{t+1} - r_t^n \right), \tag{2}$$

where π_t is the inflation rate between periods $t-1$ and t, y_t denotes the output gap, i_t is the level of the nominal interest rate between periods t and $t+1$, and r_t^n is the exogenous natural real rate of interest. Equation (1) is a standard New Keynesian Phillips curve, and equation (2) is the consumption Euler equation. The parameters are defined as follows: $\beta \in (0,1)$ denotes the representative household's subjective discount factor, $\sigma > 0$ is the intertemporal elasticity of substitution in consumption, and κ represents the slope of the New Keynesian Phillips curve.[9]

In the baseline model, the only source of uncertainty is the natural real interest rate shock r_t^n. In section 4.2 we consider a version of the model augmented with cost-push shocks. The natural real rate is assumed to follow a stationary autoregressive process of order one:

$$r_t^n = (1-\rho_r) r^n + \rho_r r_{t-1}^n + \epsilon_t^r, \tag{3}$$

where $r^n \equiv \frac{1}{\beta} - 1$ is the steady state level of the natural rate, $\rho_r \in [0,1)$ is the persistence parameter and ϵ_t^r is a *i.i.d.* $N(0, \sigma_r^2)$ innovation.

[9] κ is related to the structural parameters of the economy as follows. $\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta(1+\eta\epsilon)} \left(\sigma^{-1} + \eta\right)$, where $\theta \in (0,1)$ denotes the share of firms that cannot reoptimize their price in a given period, $\eta > 0$ is the inverse of the elasticity of labor supply, and $\epsilon > 1$ denotes the price elasticity of demand for differentiated goods.

2.2 Society's welfare and the central bank's problem

Society's welfare is represented by a second-order approximation to the representative household's expected lifetime utility:

$$V_t = u(\pi_t, y_t) + \beta E_t V_{t+1}, \tag{4}$$

where

$$u(\pi, y) = -\frac{1}{2}\left(\pi^2 + \lambda y^2\right). \tag{5}$$

Society's relative weight on output gap stabilization, λ, is a function of the structural parameters and is given by $\lambda = \frac{\kappa}{\epsilon}$.[10] In the remainder of the paper, we will often refer to society's welfare simply as welfare.

The value for the central bank with an IRS objective generically differs from society's welfare and is given by

$$V_t^{CB} = u^{CB}(\pi_t, y_t, \Delta i_t) + \beta E_t V_{t+1}^{CB}, \tag{6}$$

where $\Delta i_t = i_t - i_{t-1}$ denotes the change in the one-period nominal interest rate between periods $t-1$ and t. The central bank's contemporaneous objective function, $u^{CB}(\cdot, \cdot)$, is given by

$$u^{CB}(\pi, y, \Delta i) = -\frac{1}{2}\left[(1-\alpha)\left(\pi^2 + \lambda y^2\right) + \alpha \Delta i^2\right]. \tag{7}$$

The last term, $\alpha \Delta i^2$, captures the IRS objective, and the parameter $\alpha \in [0, 1]$ determines how the smoothing objective weighs against the central bank's inflation and output gap objectives. When $\alpha = 0$, then $u^{CB}(\cdot) = u(\cdot)$.

We assume that the central bank does not have a commitment technology. Each period t, the central bank chooses the inflation rate, the output gap, and the nominal interest rate to maximize its objective function subject to the behavioral constraints of the private sector, with the policy functions at time t+1 taken as given

$$V_t^{CB}(r_t^n, i_{t-1}) = \max_{\pi_t, y_t, i_t} \quad u^{CB}(\pi_t, y_t, \Delta i_t) + \beta E_t V_{t+1}^{CB}(r_{t+1}^n, i_t), \tag{8}$$

subject to the ZLB constraint

$$i_t \geq 0 \tag{9}$$

and the private-sector equilibrium conditions (1) and (2) previously described. *A Markov-Perfect equilibrium with an IRS objective* is defined as a set of time-invariant value and policy functions $\{V^{CB}(\cdot), \pi(\cdot), y(\cdot), i(\cdot)\}$ that solves the central bank's problem above together with society's value function $V(\cdot)$ that is consistent with $\pi(\cdot)$ and $y(\cdot)$.

Because units of welfare are not particularly meaningful, we express the social welfare of an economy in terms of the perpetual consumption transfer (as a share of its steady state) that

[10] See Woodford (2003a) and Gali (2008).

would make the household in the artificial economy without any shocks indifferent to living in the stochastic economy:

$$W := (1-\beta)\,\frac{\epsilon}{\kappa}\left(\sigma^{-1}+\eta\right)E[V], \tag{10}$$

where the mathematical expectation is taken with respect to the unconditional distribution of r_t^n.[11]

2.3 Calibration and model solution

The values of the structural parameters are listed in Table 1. The interest rate elasticity is set to 2, consistent with the value used in Christiano, Eichenbaum, and Rebelo (2011). Inverse labor supply elasticity, price elasticity of demand, and the share of firms keeping the price unchanged are from Eggertsson and Woodford (2003). The parameters ρ_r and σ_r of the natural real rate shock process are estimated using U.S. data for the period 1984-Q1 to 2016-Q4, following the approach by Adam and Billi (2006). The details of the estimation procedure are described in Appendix B. Under this baseline calibration, the probability of being at the ZLB is about 20 percent when the central bank puts no weight on the IRS objective ($\alpha = 0$).

Table 1: **Parameter values for the baseline model**

Parameter	Value	Economic interpretation
β	0.99	Subjective discount factor
σ	2	Intertemporal elasticity of substitution in consumption
η	0.47	Inverse labor supply elasticity
ϵ	10	Price elasticity of demand
θ	0.8106	Share of firms per period keeping prices unchanged
ρ_r	0.85	AR coefficient natural real rate
σ_r	$\frac{0.4}{100}$	Standard deviation natural real rate shock

To solve the model, we approximate the policy functions using a projection method. The details of the solution algorithm and an assessment of the solution accuracy are described in Appendix C.

3 Results

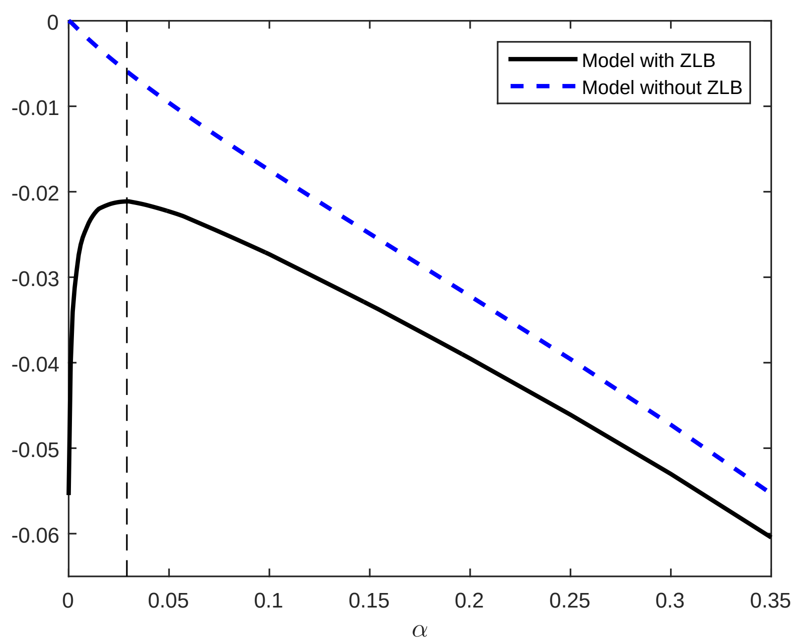
This section analyzes how the introduction of the IRS objective affects the dynamics of the economy and welfare. We first describe how society's welfare depends on the degree of interest-rate gradualism, captured by α. We then analyze how the IRS objective affects the dynamics of the economy to understand the key forces behind the welfare result.

[11] For a derivation of the expression for the welfare-equivalent consumption transfer, see, for instance, Billi (2016).

3.1 Welfare effects of policy gradualism

Figure 1 plots the social welfare measure as defined in equation (10) for alternative values of α over $\alpha \in [0, 0.35]$.[12] The black solid line indicates welfare outcomes when accounting for the ZLB, and the blue dashed line indicates welfare when ignoring the ZLB. In the model without the ZLB, welfare declines monotonically with the degree of interest-rate smoothing α, and it is optimal for society if the central bank focuses only on inflation and output gap stabilization. The reason why welfare declines with interest-rate gradualism is straightforward: The central bank can completely absorb any shock to the natural real rate of interest by setting the policy rate such that in equilibrium, the actual real interest rate equals the natural real rate at each point in time. Indeed, if the central bank is not concerned with interest-rate smoothing, the central bank can completely stabilize output and inflation—in other words, the so-called divine coincidence holds—and welfare is at its maximum value.

Figure 1: Welfare effects of interest-rate smoothing



Note: The figure shows how welfare as defined in equation (10) varies with the relative weight α on the IRS objective. The vertical dashed black line indicates the optimal relative weight on the IRS objective in the model with ZLB.

The welfare effects of interest-rate gradualism change markedly once we account for the ZLB constraint. In the model with the ZLB, welfare depends on the degree of interest-rate smoothing in a nonmonotonic way—it initially increases with the degree of policy gradualism α before starting to decrease. Under our baseline calibration, the optimal weight on the IRS term is $\alpha = 0.029$, as indicated by the vertical dashed line. Welfare can be lower than under the standard objective

[12] For each candidate, we conduct 2,000 simulations, each consisting of 1,100 periods, with the first 100 periods discarded as burn-in periods.

function ($\alpha = 0$) when the degree of interest-rate smoothing is sufficiently high, which happens in our model for values of α larger than 0.3.

The welfare effects of interest-rate smoothing are quantitatively important. According to Table 2, modifying the objective function of a central bank acting under discretion to include an IRS objective with a relative weight of 0.029 reduces the welfare costs associated with the presence of the ZLB constraint by more than one-half (negative 2.11 in the first row versus negative 5.55 in the second row).

Table 2: **Results for the baseline model**

Regime	Optimal α	Welfare ($W \times 100$)	ZLB frequency (in %)
Interest-rate smoothing	0.029	-2.11	5
Standard discretion	-	-5.55	20
Commitment	-	-0.32	11
Shadow-rate smoothing	0.014	-1.19	7

Note: The welfare measure is defined in equation (10).

While the stabilization performance of optimized interest-rate smoothing falls short of the optimal plan under commitment—shown by the third row in Table 2—this welfare improvement due to interest-rate gradualism is significant. In section 4.1, we consider a refinement of the IRS objective function that brings the optimal discretionary policy closer to the optimal commitment policy.

3.2 Why some degree of gradualism is desirable

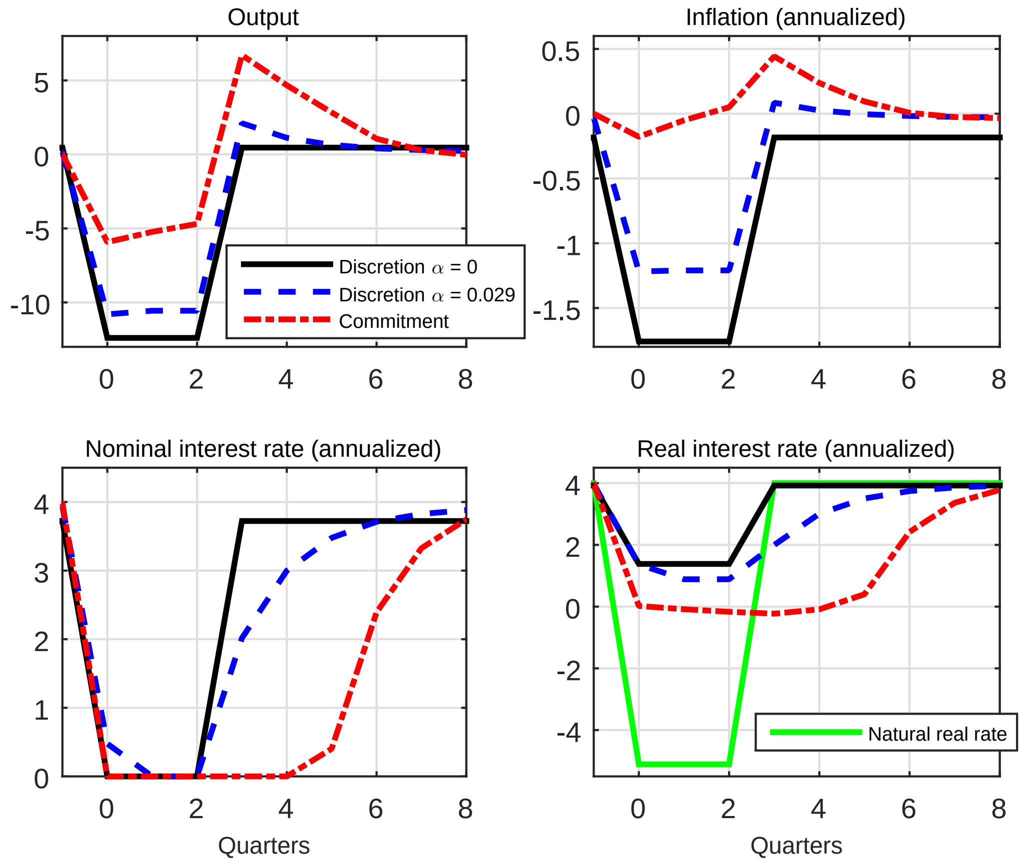
To understand the benefits of interest-rate smoothing in the model with the ZLB, we consider the following liquidity trap scenario. The economy is initially in the risky steady state.[13] In period 0, the natural real rate of interest falls into negative territory and stays at the new level for three quarters before jumping back to its steady state level. At each point in time, households and firms account for the uncertainty regarding the future path of the natural real rate in making their decisions. The considered scenario is arguably rather extreme given the assumed autoregressive process for the natural real rate, but it is useful in cleanly illustrating the implications of the IRS objective for monetary policy and stabilization outcomes.

Figure 2 plots the dynamics of the economy in this experiment for three regimes: the standard discretionary regime without an IRS objective (solid black lines), the augmented discretionary regime with an optimally weighted objective for policy gradualism of $\alpha = 0.029$ (dashed blue lines), and the optimal commitment policy (dash-dotted red lines). The exogenous path of the natural real rate is shown in the lower-right chart (solid green line).

Under the standard discretionary regime, the central bank immediately lowers the nominal

[13] The risky steady state describes the point at which the economy eventually settles as existing economic disturbances dissipate, while taking into account the uncertainty associated with future disturbances. In our model, the presence of ZLB risk introduces a wedge between the risky steady state and the deterministic steady state. See also Hills, Nakata, and Schmidt (2016).

Figure 2: Liquidity trap scenario



Note: In the considered liquidity trap scenario, the economy is initially in the risky steady state. In period 0, the natural real rate falls into negative territory and stays at the new level for three quarters before jumping back to its steady-state level.

interest rate to zero. The real interest rate stays strictly positive, leading to large declines in output and inflation, which drop by 12.4 and 1.8 percent, respectively. When the economy exits the liquidity trap in period 3, the nominal interest rate is raised immediately to its risky steady-state level, and the real interest rate closely tracks the natural rate.

Now, consider the IRS regime. Due to its desire for a gradual adjustment in the policy rate, the central bank refrains from immediately lowering the policy rate all the way to zero in period 0. Nevertheless, the declines in output and inflation are smaller (10.8 and 1.2 percent, respectively) than under the standard discretionary regime. In period 1, the policy rate reaches the ZLB and the real interest rate declines further. At the same time, output and inflation slightly rise beyond their previous period's troughs. Upon exiting the liquidity trap in period 3, the policy rate is raised only gradually, resulting in a temporarily negative real rate gap—that is, a real interest rate that is below its natural rate counterpart. This negative real rate gap boosts output and inflation above their longer-run targets. In period 4, output and inflation are 2.1 and 0.1 percent, respectively.

Because households and firms are forward-looking, the anticipated temporary overheating of the economy leads to less deflation and smaller output losses at the outset of the liquidity trap event compared with the standard discretionary regime.

The history dependence just described manifests itself in one of the optimality conditions of the gradualist central bank's maximization problem:

$$\begin{aligned}0 =& \alpha(1+\beta)i_t - \alpha i_{t-1} - \beta\alpha E_t i(r^n_{t+1}, i_t) \\ &+ \beta(1-\alpha)\frac{\partial E_t \pi(r^n_{t+1}, i_t)}{\partial i_t}\pi_t + (1-\alpha)\left(\frac{\partial E_t y(r^n_{t+1}, i_t)}{\partial i_t} + \sigma \frac{\partial E_t \pi(r^n_{t+1}, i_t)}{\partial i_t}\right)(\lambda y_t + \kappa \pi_t) \\ &- (1-\alpha)\sigma(\lambda y_t + \kappa\pi_t) - \phi^{ZLB}_t. \end{aligned} \tag{11}$$

The optimality condition shows that for given economic conditions, a gradualist central bank aims to set the contemporaneous policy rate such that the deviations from the lagged policy rate as well as from the expected future policy rate are small in equilibrium. Notice that if $\alpha = 0$—that is, if the central bank has no smoothing objective—then the right-hand side terms in the first two rows of equation (11) vanish and the equation is reduced to the familiar static target criterion (accounting for the ZLB) of the standard discretionary regime.[14]

The policy of keeping the interest rate low for long under gradualism is shared by the optimal commitment policy. Under the commitment policy, the central bank lowers the policy rate immediately all the way to zero and keeps the policy rate at the ZLB even after the natural rate becomes positive. The promise of an extended period of holding the policy rate at the ZLB leads to an even larger overshooting of inflation and the output gap than observed under the gradualist central bank, which in turn results in smaller deflation and output losses during the crisis period.

The benefit of interest-rate gradualism—smaller declines in inflation and output at the ZLB—spills over to the stabilization outcomes when the policy rate is away from the ZLB through expectations. As described in detail in Nakata and Schmidt (2014) and Hills, Nakata, and Schmidt (2016), the standard discretionary regime fails to fully stabilize inflation and output even at the risky steady state—in which the policy rate is comfortably above the ZLB—due to the asymmetry in the distribution of future inflation and output induced by the possibility of returning to the ZLB. For our calibration, under the standard discretionary regime, the inflation rate is negative 0.18 and the output gap is 0.46 at the risky steady state.[15] Because the decline in inflation at the ZLB is smaller under the IRS regime than under the standard discretionary regime, the distribution is less asymmetric and inflation and output away from the ZLB are better stabilized under interest-rate gradualism. With the optimized IRS weight, the inflation rate and the output gap are negative 0.03 and 0.19, respectively, at the risky steady state. Thus, interest-rate smoothing improves stabilization outcomes not only at the ZLB but also at the risky steady state.

[14] The second row on the right-hand side of equation (11) vanishes if $\alpha = 0$ because the nominal interest rate ceases to be a state variable and hence the partial derivative terms become zero.

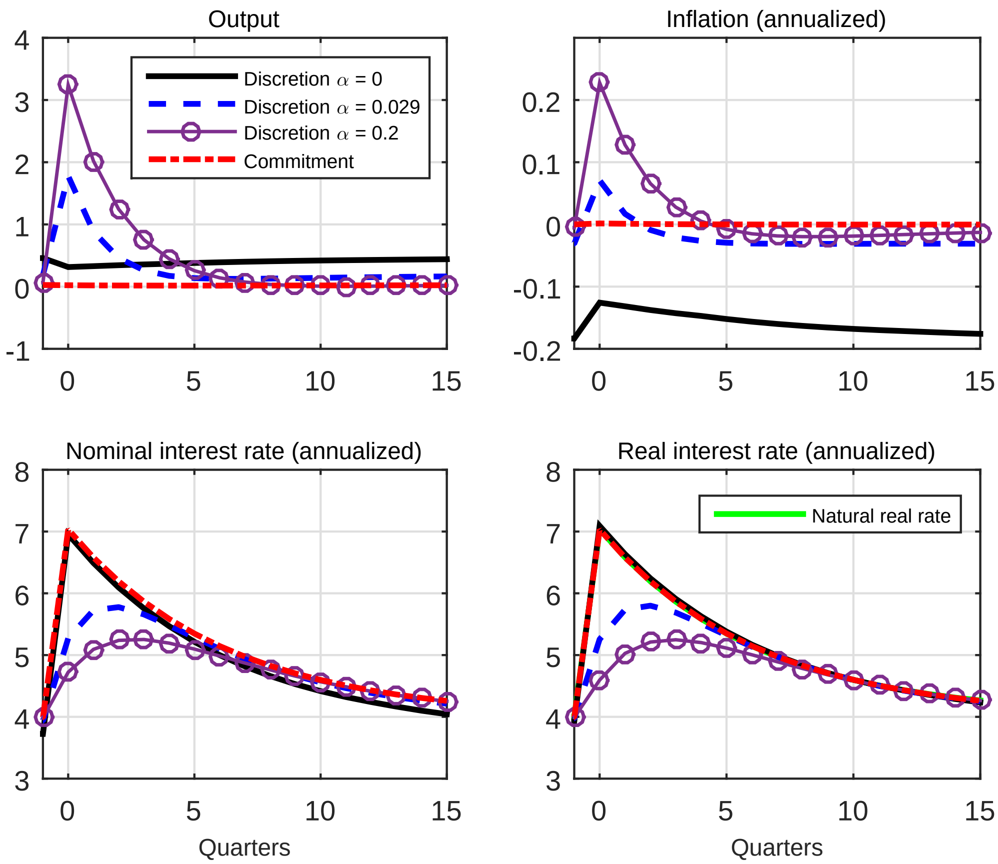
[15] The welfare costs associated with this stabilization shortfall are non-negligible. If we take the welfare loss of an economy that stays permanently in the risky steady state associated with the standard discretionary regime as a proxy, they make up 25 percent of the overall welfare costs.

3.3 Why too much gradualism is undesirable

While the introduction of an IRS objective improves welfare for a wide range of weights α, we have seen that putting too much weight on the smoothing objective delivers lower welfare than the central bank with the standard objective function ($\alpha = 0$) (see Figure 1). This section takes a closer look at the costs associated with excessive interest-rate gradualism.

Figure 3 shows impulse responses to a natural real rate shock of one unconditional standard deviation when the economy is initially at the risky steady state for the three regimes previously considered as well as for an IRS regime with a higher-than-optimal weight on the gradualism objective, $\alpha = 0.2$ (thin purple solid line with circles).

Figure 3: Impulse responses to a positive natural rate shock



Note: In the considered scenario, the economy is initially in the risky steady state. In period 0, the natural real rate increases by one unconditional standard deviation. The shock recedes in subsequent periods according to its law of motion.

Under the standard discretionary regime, the central bank raises the policy rate such that the real interest rate closely tracks the path of the natural real rate, making the latter hardly visible in the lower-right chart. The larger buffer against hitting the ZLB slightly mitigates the downward bias in expected output and inflation, which attenuates the stabilization trade-off for the central bank. Output and inflation move closer to their target levels so long as the shock prevails, albeit by a small amount.

Under the two IRS regimes—one with the optimal weight and the other with a higher-than-optimal weight—the central bank raises the nominal interest rate only sluggishly so that the path of the real interest rate is temporarily below that of the natural rate. This more accommodative monetary policy stance stimulates output and inflation, and both variables overshoot their targets for a few quarters. The larger the weight on the smoothing objective, the more gradually interest rates respond and the larger the positive deviations of output and inflation from target. Such overshooting, while costly in terms of contemporaneous utility flows, has the desirable effect of increasing inflation expectations in states in which the ZLB constraint is binding, as rational agents take into account how the central bank responds to shocks in the future when forming expectations. However, in the case of too much gradualism, the welfare costs of these target overshootings outweigh the gains from improved expectations. The discretionary regime with the optimized weight on the smoothing objective optimally trades off the gains from gradual policy rate adjustments against these costs.

Before closing this section, it is interesting to observe that in this experiment, away from the ZLB, the interest rate response under the optimal commitment policy is very similar to the one under the standard discretionary regime. Thus, contrary to the casual impression one might get from the liquidity trap scenario, policy inertia is not a generic feature of the optimal commitment policy. Under both, the standard discretionary policy and the optimal commitment policy, the central bank wants to adjust the policy rate to neutralize the effects of shocks to the natural real rate of interest. If there is a sudden change in the natural real rate, both types of policy regimes will adjust the policy rate instantaneously.

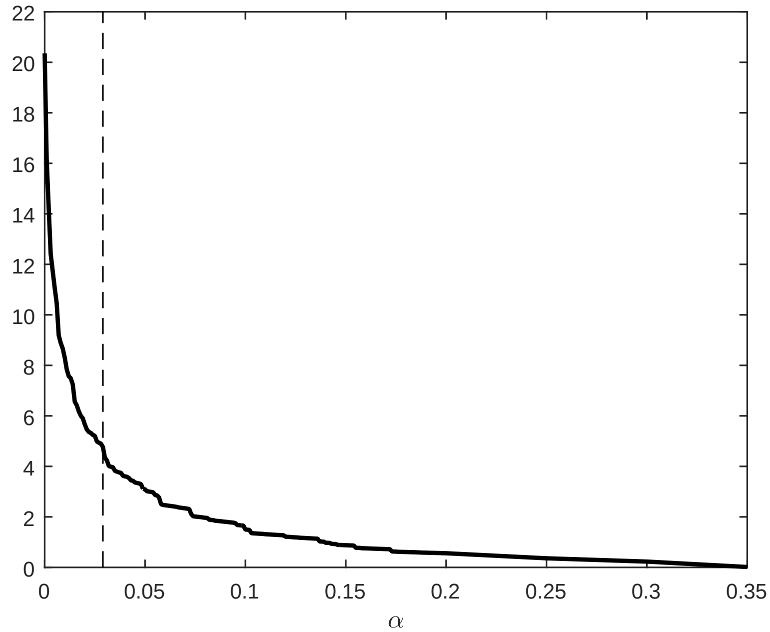
3.4 Gradualism and the frequency of hitting the zero lower bound

As shown in Table 2, under optimal IRS the ZLB constraint is binding less often than under the standard discretionary regime and the optimal commitment regime. To put more light on how IRS affects the frequency of hitting the ZLB, Figure 4 plot the average frequency of ZLB events as a function of α. The vertical dashed black line indicates the frequency of ZLB events for the optimal relative weight on the IRS objective.

The frequency of zero interest rates is declining in α. Two factors explain this result. First, as shown in Figure 2, a discretionary central bank with an IRS objective lowers the policy rate more gradually towards zero in response to a large contractionary natural real rate shock than a discretionary central bank without an IRS objective does.[16] Second, as explained in section 3.2, under the standard discretionary regime, the possibility that the ZLB constraint might be binding in the future puts downward pressure on inflation expectations, and thereby on actual inflation, in all states of nature. This implies that in equilibrium the ZLB constraint is not only binding

[16]This feature of IRS regimes is not shared by the optimal commitment policy, and it is undesirable from a welfare perspective.

Figure 4: Frequency of a binding ZLB constraint



Note: The figure shows how the average frequency of a binding ZLB constraint (in %) varies with the relative weight α on the IRS objective. The vertical dashed black line indicates the optimal relative weight on the IRS objective in the model with ZLB.

in states where the natural real rate is negative but also in states where the natural real rate is strictly positive but close to zero. Since IRS improves stabilization outcomes at the ZLB, it also mitigates the deflationary expectations in states where the natural real rate is above but close to zero, with effect that the gradualist policymaker can implement a higher policy rate than the standard discretionary policymaker in these states.

One might expect that IRS regimes also entail a channel that should increase the frequency of periods in which the policy rate is zero. Specifically, after a liquidity trap event, the discretionary policymaker with the standard objective function raises the policy rate approximately one-for-one with the natural real rate of interest, whereas the policymaker with the IRS objective raises the policy rate more gradually. However, even so the policy rate path after a liquidity trap event is temporarily lower under IRS than under the standard objective, that path is still strictly positive. This is because the policy rate set by the policymaker with an IRS objective is a function of three terms: the lagged policy rate, the expected future policy rate and the weighted sum of current inflation and output gap that prescribes the target criterion for optimal discretionary monetary policy in the absence of IRS.[17] While the first term is zero in the immediate aftermath of a liquidity trap event, the other two terms are strictly positive.

[17] See optimality condition (11).

4 Additional results

In the first part of this section, we consider a refinement of the IRS regime that further increases the welfare gains from interest-rate gradualism by smoothing the path of the actual policy rate with respect to the lagged shadow policy rate—the policy rate that the central bank would like to set given current economic conditions if it had not been constrained by the ZLB—as opposed to the actual lagged policy rate. In the second part, we assess the desirability of interest-rate smoothing in an economy that is buffeted by both natural real rate shocks and cost-push shocks.

4.1 Shadow interest-rate smoothing

Shadow interest-rate smoothing (SIRS) aims to enhance the ability of the discretionary policymaker to keep the policy rate low for long in the aftermath of a recession. The shadow interest rate keeps track of the severity of the recession and makes the period for which the policy rate is kept at the ZLB depend on the severity of the recession. The value of the central bank with a SIRS objective is given by

$$V_t^{CB,SIRS} = u^{CB,SIRS}(\pi_t, y_t, i_t, i_{t-1}^*) + \beta E_t V_{t+1}^{CB,SIRS}, \tag{12}$$

where the central bank's contemporaneous objective function, $u^{CB,SIRS}(\cdot,\cdot,\cdot,\cdot)$, is given by

$$u^{CB,SIRS}(\pi_t, y_t, i_t, i_{t-1}^*) = -\frac{1}{2}\left[(1-\alpha)\left(\pi_t^2 + \lambda y_t^2\right) + \alpha(i_t - i_{t-1}^*)^2\right]. \tag{13}$$

Each period t, the central bank with a SIRS objective first chooses the shadow nominal interest rate in order to maximize the value today subject to the behavioral constraints of the private sector, with the value and policy functions at time $t+1$—$V_{t+1}^{CB,SIRS}(\cdot,\cdot)$, $y_{t+1}(\cdot,\cdot)$, $\pi_{t+1}(\cdot,\cdot)$—taken as given:

$$i_t^* = \text{argmax}_x \qquad u^{CB,SIRS}(\pi(x), y(x), x, i_{t-1}^*) + \beta E_t V_{t+1}^{CB,SIRS}(r_{t+1}^n, x), \tag{14}$$

with

$$\begin{aligned} y(x) =& E_t y_{t+1}(r_{t+1}^n, x) - \sigma(x - E_t \pi_{t+1}(r_{t+1}^n, x) - r_t^n) \\ \pi(x) =& \kappa y(x) + \beta E_t \pi_{t+1}(r_{t+1}^n, x). \end{aligned} \tag{15}$$

The actual policy rate i_t is given by

$$i_t = \max(i_t^*, 0). \tag{16}$$

That is, the actual policy rate today is zero when $i_t^* < 0$, and it is equal to i_t^* when $i_t^* \geq 0$.

The central bank's value today is given by

$$V_t^{CB,SIRS}(r_t^n, i_{t-1}^*) = u^{CB,SIRS}(\pi_t, y_t, i_t, i_{t-1}^*) + \beta E_t V_{t+1}^{CB,SIRS}(r_{t+1}^n, i_t^*), \tag{17}$$

where inflation and the output gap are given by

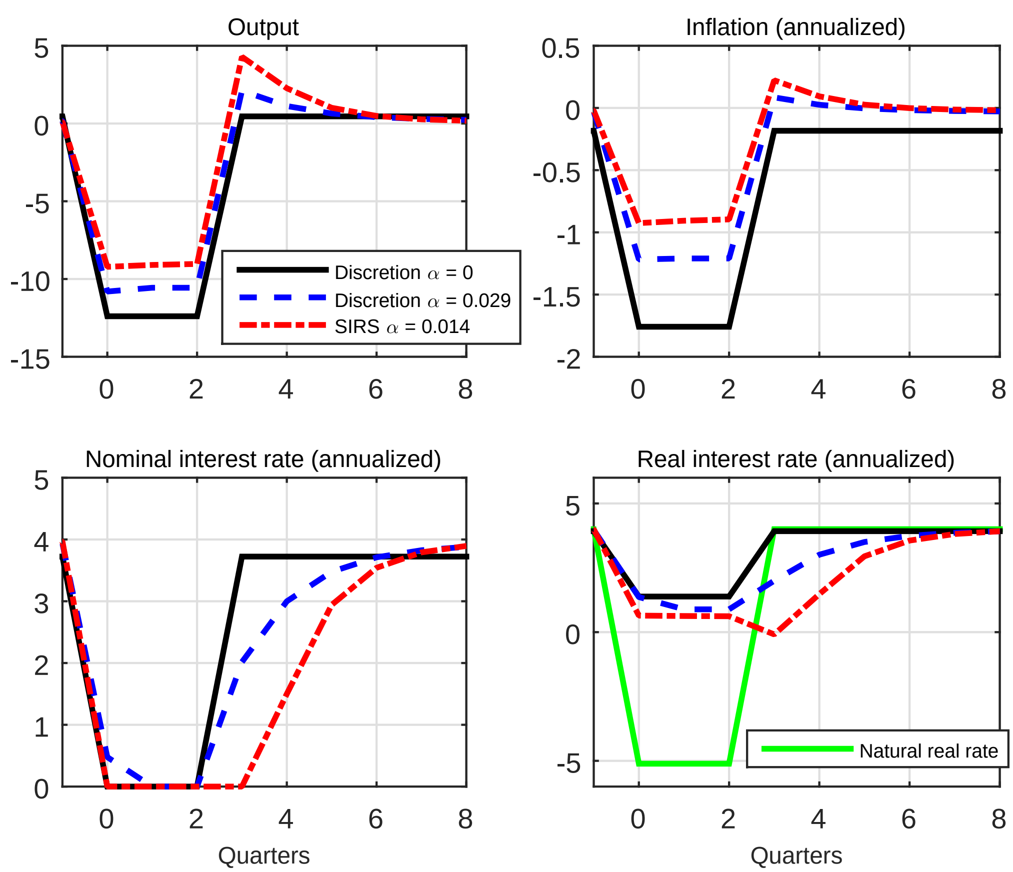
$$
\begin{aligned}
y_t =& E_t y_{t+1}(r^n_{t+1}, i^*_t) - \sigma(i_t - E_t \pi_{t+1}(r^n_{t+1}, i^*_t) - r^n_t) \\
\pi_t =& \kappa y_t + \beta E_t \pi_{t+1}(r^n_{t+1}, i^*_t).
\end{aligned}
$$

The definition of the Markov-Perfect equilibrium with the shadow interest-rate smoothing is similar to that with the standard IRS objective and is relegated to Appendix A.

The fourth row of Table 2 reports the optimal weight, welfare, and ZLB frequency for the SIRS regime. The optimal relative weight on the SIRS objective in the central bank's objective function is considerably smaller than under the standard IRS regime, while welfare is higher than under the standard IRS regime.

Figure 5 compares the dynamics of the economy under the SIRS regime with those under the standard IRS regime and the discretionary regime with zero weight on the IRS objective in the context of the liquidity trap scenario of Section 3.2. As a result of the lower optimized weights

Figure 5: Liquidity trap scenario: Shadow interest-rate smoothing



Note: In the considered liquidity trap scenario, the economy is initially in the risky steady state. In period 0, the natural real rate falls into negative territory and stays at the new level for three quarters before jumping back to its steady-state level.

on the SIRS objective, and in contrast to standard IRS, under SIRS the policy rate is lowered immediately to its lower bound when the shock buffets the economy. The SIRS regime also raises the policy rate more slowly when the shock has receded, leading to a more accommodative real interest rate path. The economic boom upon exiting the liquidity trap is therefore larger under the SIRS regime than under the IRS regime, and as a result, the drop in the inflation rate and the output gap during the liquidity trap is smaller.

A key difference between the SIRS framework and the standard IRS framework lies in the endogenous state variable. Under the IRS regime, the endogenous state variable is the actual policy rate i_t, while it is the shadow interest rate i_t^* under the SIRS regime. Unlike the actual policy rate, the shadow interest rate can go below zero. This unconstrained nature of the shadow rate has two important implications for interest rate policy. The first implication is that, in the face of large contractionary shocks, the policy rate is lowered more aggressively than under standard IRS. This more aggressive lowering reflects the fact that the shadow rate is anticipated to enter negative territory, while the policy rate is anticipated not to fall below zero under the standard IRS regime. Because the SIRS regime smoothes the shadow rate path, the shadow rate declines faster than the policy rate in the standard IRS regime. The policy rate path under the SIRS regime simply mimics the shadow rate path subject to the ZLB constraint.

The second implication is that, as large contractionary shocks dissipate, the policy rate is kept at the ZLB for a longer period under the SIRS regime than under the standard IRS regime. The shadow rate remembers the severity of the recession: The larger the downturn, the lower the shadow rate. As the policy rate follows the shadow rate path subject to the ZLB constraint, a larger downturn thus leads to a lower path of interest rates under the SIRS regime, akin to the optimal commitment policy.[18] In contrast, under conventional interest-rate smoothing, history dependence operates via the nominal interest rate, which has a lower bound of zero. Thus, once the ZLB is reached, a further decline in the natural rate has no direct implications for the size of the subsequent monetary stimulus.

4.2 A simple model with natural real rate and cost-push shocks

In our baseline model, the only exogenous disturbance is a natural real rate shock. We now extend the analysis to an economy that is subject to both natural real rate shocks and cost-push shocks. The New Keynesian Phillips curve augmented with a cost-push shock then becomes:

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t, \tag{18}$$

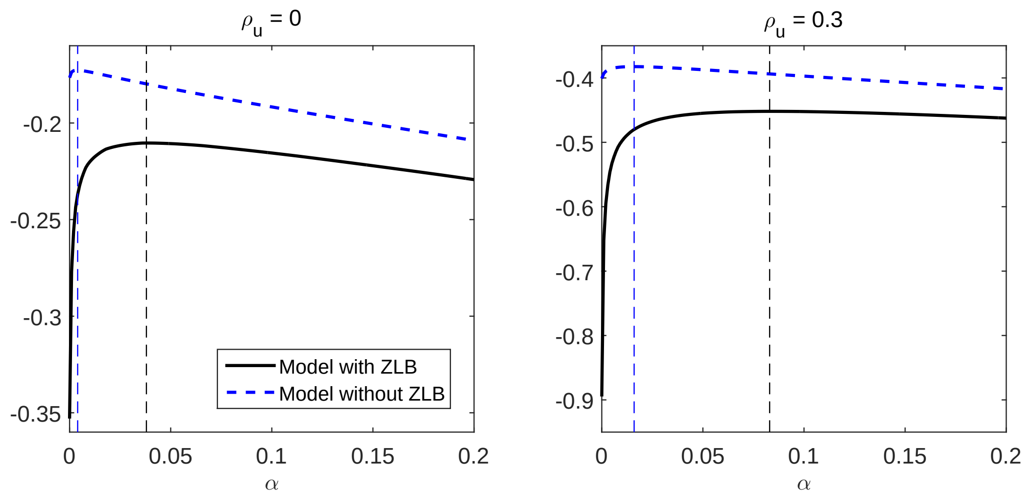
where u_t follows a stationary autoregressive process of order one, $u_t = \rho_u u_{t-1} + \epsilon_t^u$. Parameter

[18] The conduct of interest-rate policy under SIRS also has similarities with that observed when the interest rate is set according to a truncated inertial Taylor rule with a lagged shadow policy rate (considered in Hills and Nakata (2014), Hills, Nakata, and Schmidt (2016), Gust, Lopez-Salido, and Smith (2012)) or a Reifschneider-Williams (2000) rule (Reifschneider and Williams (2000)). Under these policy rules, how long the central bank keeps the policy rate at the ZLB also depends on the severity of the recession.

$\rho_u \in [0, 1)$ and ϵ_t^u is a *i.i.d.* $N(0, \sigma_u^2)$ innovation. The remainder of the model structure stays the same as in Section 2. We set $\sigma_u = 0.17/100$ as estimated by Ireland (2011) for the U.S. economy.

Figure 6 plots the social welfare measure as defined in equation (10) for alternative values of $\alpha \in [0, 0.2]$.[19] The left panel shows results for the case when cost-push shocks are purely transitory, $\rho_u = 0$, as estimated by Ireland (2011), and the right panel shows results for the case of persistent cost-push shocks, $\rho_u = 0.3$.[20] In each panel, the black solid line indicates welfare outcomes when accounting for the ZLB, and the blue dashed line indicates welfare when ignoring the ZLB.

Figure 6: Welfare effects of interest-rate smoothing: Model augmented with cost-push shocks



Note: The figure shows how welfare as defined in equation (10) varies with the relative weight α on the IRS objective. The vertical black dashed line indicates the optimal weight on the smoothing objective in the model with ZLB, and the vertical blue dashed line indicates the optimal weight in the model without the ZLB.

First, consider the left panel. In the presence of transitory cost-push shocks, the optimal degree of interest-rate smoothing is no longer zero, even when one ignores the ZLB constraint. This result arises because the optimal (time-inconsistent) response to a cost-push shock entails endogenous persistence in the inflation rate and the output gap. If the economy is buffeted by a transitory inflationary cost-push shock, the optimal commitment policy is to raise the policy rate above the steady state for more than one period in order to undershoot the inflation target in the second period. Such a response improves the trade off between inflation and output gap stabilization in the period when the shock hits the economy through the expectations channel (see, for instance, Gali (2008)). Putting a small positive weight on the IRS objective allows a discretionary central bank to mimic the gradual response of the optimal commitment policy to cost-push shocks.

As in our baseline model that is exposed to natural real rate shocks only, the presence of the ZLB increases the optimal degree of interest-rate smoothing. In the model with the ZLB, the optimal weight is $\alpha = 0.038$, as indicated by the vertical dotted line, versus $\alpha = 0.004$ in the model without

[19] For each candidate, we conduct 2,000 simulations, each consisting of 1,100 periods, with the first 100 periods discarded as burn-in periods.

[20] The latter calibration is in line with estimates of the cost-push shock process in Justiniano, Primiceri, and Tambalotti (2013) for the U.S. economy.

the ZLB. Reflecting the additional benefit of interest-rate smoothing arising from the presence of cost-push shocks in the model with the ZLB, this optimal weight is larger than that in the model with natural real rate shocks only, which is 0.029, as shown in figure 1. As before, the welfare gains from interest-rate smoothing are quantitatively important. At the optimal weight $\alpha = 0.038$, the welfare costs are more than one-third smaller than under the standard discretionary monetary policy regime.

Now, consider the right panel. When the cost-push shocks are persistent, the optimal relative weight on the IRS objective increases relative to the case with purely transitory cost-push shocks, both in the model without the ZLB and in the model with the ZLB. The optimal α continues to be larger in the model with the ZLB ($\alpha = 0.083$) than in the model without the ZLB ($\alpha = 0.016$), and the difference between the optimal relative weights in the two models, i.e. the difference between the vertical dashed black line and the vertical dashed blue line, is larger when the cost-push shocks are persistent than when they are transitory. Hence, the mechanism that makes interest-rate smoothing desirable in the presence of the ZLB remains quantitatively important even if we increase the importance of the conventional mechanism associated with the stabilization bias of discretionary policy. Finally, the relative welfare gains from including an optimally weighted IRS objective in the central bank's objective function are bigger when the cost-push shocks are persistent than when they are transitory. While this is true for the model with ZLB and for the model without the ZLB, the relative welfare gain is much larger when accounting for the ZLB.[21]

5 A quantitative model

In this section, we examine the desirability of gradualism in a more elaborate model. The model provides an empirically more plausible framework to quantify the desirability of interest-rate smoothing.

5.1 Model and calibration

The quantitative model features price and wage rigidities as in Erceg, Henderson, and Levin (2000), and non-reoptimized prices and wages that are partially indexed to past price inflation. Two exogenous shocks—a natural real rate shock and a cost-push shock—buffet the economy.

The aggregate private sector behavior of the quantitative model is summarized by the following

[21]For $\rho_u = 0.3$, the welfare gain from an optimally-weighted IRS regime relative to the baseline regime with no IRS objective is 50% in the model with the ZLB and 5% in the model without the ZLB.

system of equations:

$$\begin{aligned}
\pi_t^p - \iota_p \pi_{t-1}^p &= \kappa_p w_t + \beta \left(E_t \pi_{t+1}^p - \iota_p \pi_t^p \right) + u_t, && (19) \\
\pi_t^w - \iota_w \pi_{t-1}^p &= \kappa_w \left(\left(\frac{1}{\sigma} + \eta \right) y_t - w_t \right) + \beta \left(E_t \pi_{t+1}^w - \iota_w \pi_t^p \right), && (20) \\
\pi_t^w &= w_t - w_{t-1} + \pi_t^p, && (21) \\
y_t &= E_t y_{t+1} - \sigma \left(i_t - E_t \pi_{t+1}^p - r_t^n \right), && (22) \\
i_t &\geq i_{ELB}. && (23)
\end{aligned}$$

Equation (19) captures the price-setting behavior of firms, where w_t is the composite real wage rate and u_t is a cost-push shock. Equation (20) summarizes the nominal wage setting behavior of households, where π_t^w denotes nominal wage inflation between periods $t-1$ and t. Parameters ι_p and ι_w represent the degree of indexation of prices and wages to past price inflation. Equation (21) relates nominal wage inflation to the change in the real wage and the price inflation rate, and equation (22) is the Euler equation and r_t^n is the natural rate shock. Finally, equation (23) represents the effective lower bound (ELB) constraint on the policy rate. Parameters satisfy $\kappa_p = \frac{(1-\theta_p)(1-\theta_p\beta)}{\theta_p}$ and $\kappa_w = \frac{(1-\theta_w)(1-\theta_w\beta)}{\theta_w(1+\eta\epsilon_w)}$, where $\theta_p \in (0,1)$ and $\theta_w \in (0,1)$ denote share of firms and households that cannot reoptimize their price and wage in a given period, respectively. $\epsilon_p > 1$ is the price elasticity of demand for differentiated goods, whereas $\epsilon_w > 1$ is the wage elasticity of demand for differentiated labor services. The notations for η, σ, and β are the same as in the stylized model.

The natural rate shock r_t^n and the price mark-up shock are assumed to follow a stationary autoregressive process of order one:

$$r_t^n = (1-\rho_r) r^n + \rho_r r_{t-1}^n + \epsilon_t^r, \tag{24}$$

$$u_t = \rho_u u_{t-1} + \epsilon_t^u, \tag{25}$$

where $r^n \equiv \frac{1}{\beta} - 1$ is the steady state level of the natural rate. $\rho_r \in [0,1)$ and $\rho_u \in [0,1)$ are the persistence parameter. ϵ_t^r and ϵ_t^u are *i.i.d.* $N(0,\sigma_r^2)$ and $N(0,\sigma_u^2)$ innovations, respectively.

Society's welfare at time t is given by the expected discounted sum of future utility flows.

$$V_t = u(\pi_t^p, y_t, \pi_t^w, \pi_{t-1}^p) + \beta E_t V_{t+1} \tag{26}$$

where society's contemporaneous utility function $u(\cdot)$ is given by the following second-order approximation to the household's utility:[22]

$$u(\pi_t^p, y_t, \pi_t^w, \pi_{t-1}^p) = -\frac{1}{2} \left[\left(\pi_t^p - \iota_p \pi_{t-1}^p \right)^2 + \lambda y_t^2 + \lambda_w \left(\pi_t^w - \iota_w \pi_{t-1}^p \right)^2 \right], \tag{27}$$

[22] We assume that the deterministic steady-state distortions associated with imperfect competition in goods and labor markets are eliminated by appropriate subsidies.

where the relative weights are functions of the structural parameters.[23]

The central bank acts under discretion. The central bank's contemporaneous utility function $u^{CB}(\cdot)$ is given by,

$$u^{CB}(\pi_t^p, y_t, \pi_t^w, \pi_{t-1}^p, i_t, i_{t-1}) = -\frac{1}{2}\Big\{(1-\alpha)\left[\left(\pi_t^p - \iota_p \pi_{t-1}^p\right)^2 + \lambda y_t^2 + \lambda_w\left(\pi_t^w - \iota_w \pi_{t-1}^p\right)^2\right] + \alpha(i_t - i_{t-1})^2\Big\}, \tag{28}$$

where α is the weight on the interest-rate smoothing term. When $\alpha = 0$, the central bank's objective function collapses to society's objective function.

Each period t, the central bank chooses the price and wage inflation rate, the output gap, the real wage, and the nominal interest rate to maximize its objective function subject to the private-sector equilibrium conditions (equation (19) - (23)), with the value and policy functions at time $t+1$ taken as given:

$$V_t^{CB}(u_t, r_t^n, i_{t-1}, \pi_{t-1}^p, w_{t-1}) = \max_{(\pi_t^p, \pi_t^w, y_t, w_t, i_t)} u^{CB}(\pi_t^p, y_t, \pi_t^w, \pi_{t-1}^p, i_t, i_{t-1}) + \beta E_t V_{t+1}^{CB}(u_{t+1}, r_{t+1}^n, i_t, \pi_t^p, w_t). \tag{29}$$

We quantify the effects of gradualism on society's welfare by the perpetual consumption transfer (as a share of its steady state) that would make a household in the artificial economy without any fluctuations indifferent to living in the economy just described. This welfare-equivalent consumption transfer is given by

$$W := (1-\beta)\frac{\epsilon_p}{\kappa_p}E[V]. \tag{30}$$

Parameter values, shown in Table 3, are chosen so that the key moments implied by the model under $\alpha = 0$ are in line with those in the U.S. economy over the last two decades.[24] The model-implied standard deviations of inflation, output, and the policy rate are 0.63 percent (annualized), 2.9 percent, and 2.3 percent. The same moments from the U.S. data are 0.52 percent (annualized), 2.8 percent, and 2.2 percent.[25] The model-implied probability of being at the ELB is about 28 percent, while the federal funds rate was at the ELB constraint 35 percent of the time over the past two decades.

[23] Specifically, $\lambda = \kappa_p\left(\frac{1}{\sigma}+\eta\right)\frac{1}{\epsilon_p}$ and $\lambda_w = \lambda\frac{\epsilon_w}{\kappa_w\left(\frac{1}{\sigma}+\eta\right)}$.

[24] The first order conditions to the policy problem and the numerical algorithm for model solution are described in Appendix D.

[25] Our sample is from 1997:Q3 to 2017Q2. Inflation rate is computed as the annualized quarterly percentage change (log difference) in the personal consumption expenditure core price index. The measure of the output gap is based on the FRB/US model. The quarterly average of the (annualized) federal funds rate is used as the measure for the policy rate.

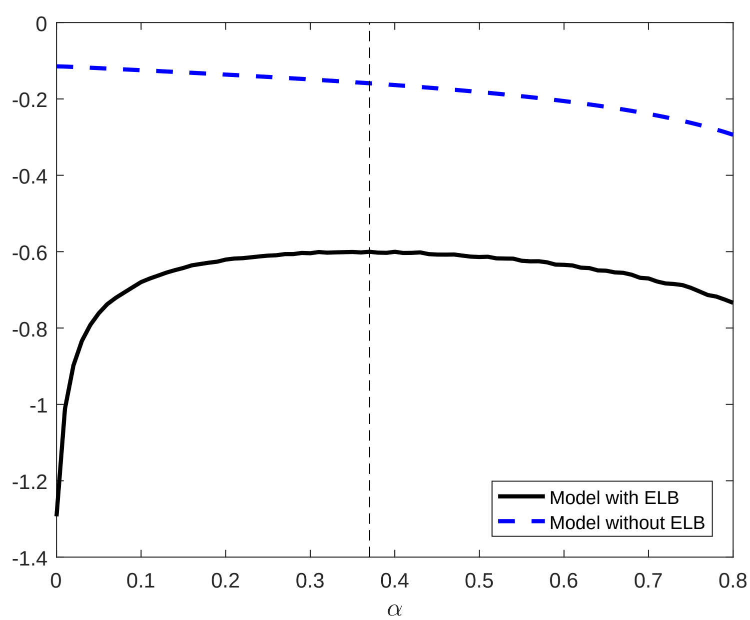
Table 3: **Parameter values for the quantitative model**

Parameter	Description	Parameter Value
β	Discount rate	0.9925
σ	Intertemporal elasticity of substitution in consumption	4
η	Inverse labor supply elasticity	2
ϵ_p	Price elasticity of substitution among intermediate goods	11
ϵ_w	Wage elasticity of substitution among labor services	11
θ_p	Share of firms per period keeping prices unchanged	0.9
θ_w	Share of households per period keeping wages unchanged	0.9
ι_p	Degree of indexation of prices to past price inflation	0.1
ι_w	Degree of indexation of wages to past price inflation	0.1
i_{ELB}	Effective lower bound	$\frac{0.125}{400}$
ρ_r	AR(1) coefficient for natural real rate shock	0.85
σ_r	The standard deviation of natural real rate shock	$\frac{0.31}{100}$
ρ_u	AR(1) coefficient for price markup shock	0
σ_u	The standard deviation of price markup shock	$\frac{0.17}{100}$

5.2 Results

Figure 7 shows how the degree of gradualism (α) affects welfare of the economies with and without the ELB constraint—indicated by black solid and blue dashed lines, respectively.

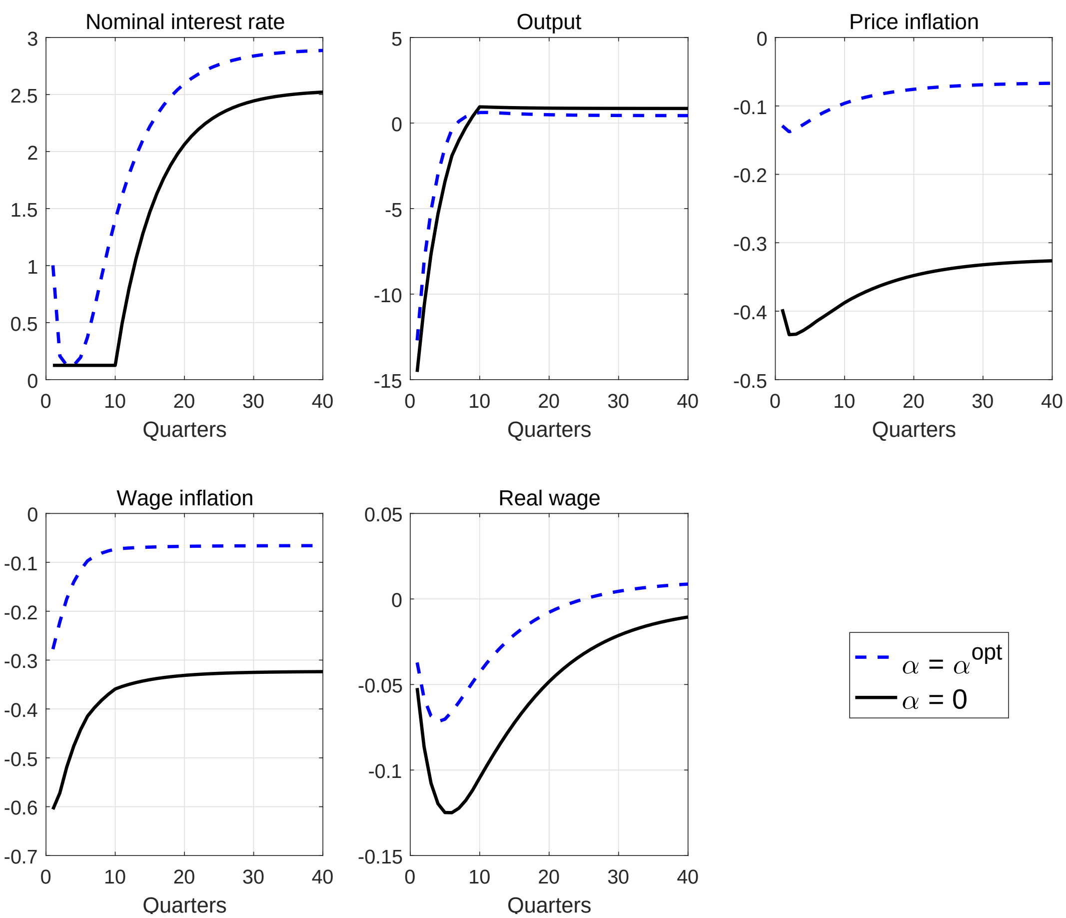
Figure 7: Welfare effects of interest rate smoothing in the quantitative model



Note: The figure shows how welfare as defined in equation (30) varies with the relative weight α on the IRS objective. The vertical dashed black line indicates the optimal relative weight on the IRS objective in the model with ZLB.

Consistent with our earlier analysis of the stylized model, the welfare of the economy without the ELB constraint monotonically decreases as α increases. In principle, the presence of a cost-push shock can make some degree of gradualism desirable. In response to a positive cost-push shock, the central bank with commitment adjusts the interest rates gradually in order to create some history dependence (Woodford (2003a) and Gali (2015)). Such history-dependence in the policy rate can be partially mimicked by the interest-rate smoothing. See the analysis from the stylized model with cost-push shocks in Section 4.2 for more details on this argument. However, in the quantitative model, other factors—such as sticky wages and price/wage indexation—induce the inertia in the policy rate even in the absence of gradualism, making any weight on the interest-rate smoothing term welfare-reducing.

Figure 8: Liquidity trap scenario in the quantitative model



Note: In the considered liquidity trap scenario, the economy is initially in the deterministic steady state. In period 1, the natural rate shock falls to a level that is 2.5 unconditional standard deviations from its steady state level. Thereafter, the natural rate shock returns to its steady state level according to the autoregressive process described in the main text.

In the model with the ELB constraint, the optimal weight on the interest-rate smoothing term is positive, as indicated by the vertical dashed line. This is consistent with what we saw in the stylized model. The optimal α is 0.37. The welfare gain from policy gradualism is quantitatively important. The welfare cost of business cycles is about 50 percent smaller at the optimal α than at $\alpha = 0$.

To understand the effect of gradualism on the dynamics of the economy with the ELB, Figure 8 compares the IRFs under two different values of α when the natural real rate of interest is initially 2.5 unconditional standard deviations below the steady state. Dashed blue and solid black lines are the IRFs under $\alpha = 0.37$ and under $\alpha = 0$, respectively. At the beginning of the recession, gradualism prevents the central bank from reducing the policy rate to the ELB as quickly as in the case with no gradualism. The recession is substantially less severe with $\alpha = \alpha^{opt}$ than with $\alpha = 0$ due to the stabilizing effect of interest-rate smoothing. In equilibrium, because the recession is less severe, the policy rate lifts off from the ELB earlier with gradualism than without gradualism.

Due to the stabilizing effects of gradualism, the probability of being at the ELB is lower with the optimal α than with $\alpha = 0$ (15 percent versus 28 percent). A lower ELB probability manifests itself in better economic outcomes at the risky steady state. In particular, due to a lower possibility of being at the ELB, price and wage inflation are nontrivially higher (and closer to zero), and output and real wages are slightly lower (closer to zero), at the risky steady state with $\alpha = 0.37$ than with $\alpha = 0$. These effects of the ELB risk on the steady-state allocations are consistent with the analysis in 3.2, Hills, Nakata, and Schmidt (2016), and Nakata and Schmidt (2014).

6 Conclusion

Our analysis provides a novel rationale for policy rate gradualism. In a liquidity trap, a gradualist central bank keeps the policy rate low for longer than is warranted by the dynamics of output and inflation alone, mimicking a key feature of the optimal commitment policy. This low-for-long policy creates a transitory boom in future inflation and output, which damps the declines of inflation and real activity during the liquidity trap via expectations.

A discretionary central bank that is only concerned with output and inflation stabilization will find itself unable to credibly commit to keep the policy rate low, for it has an incentive to renege on its past promise and increase the policy rate once the liquidity-trap conditions recede. However, modifying the objective function of a discretionary central bank to include an IRS objective allows society to make low-for-long policies credible. An optimally chosen weight on the IRS objective relative to the central bank's objectives for inflation and output stabilization leads to a significant improvement in society's welfare even though society itself is not intrinsically concerned with the stabilization of changes in the policy rate.

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Technical Appendix for Online Publication

A Interest-rate smoothing regimes

A.1 Interest-rate smoothing

The Lagrange problem of the central bank with an IRS objective at period t is given by

$$\begin{aligned} V_t^{CB}(r_t^n, i_{t-1}) = \max_{\pi_t, y_t, i_t} \Big[&-\frac{1}{2}\left[(1-\alpha)\left(\pi_t^2 + \lambda y_t^2\right) + \alpha(i_t - i_{t-1})^2\right] + \beta E_t V_{t+1}^{CB}(r_{t+1}^n, i_t) \\ &+ \phi_t^{PC}(\pi_t - \beta E_t \pi_{t+1}(r_{t+1}^n, i_t) - \kappa y_t) \\ &+ \phi_t^{IS}(y_t - E_t y_{t+1}(r_{t+1}^n, i_t) + \sigma(i_t - E_t \pi_{t+1}(r_{t+1}^n, i_t) - r_t^n)) \\ &+ \phi_t^{ZLB} i_t \Big] \end{aligned} \tag{A.1}$$

where the central banker takes the value and policy functions next period as given. The FONC are

$$(1-\alpha)\pi_t - \phi_t^{PC} = 0 \tag{A.2}$$

$$(1-\alpha)\lambda y_t + \kappa \phi_t^{PC} - \phi_t^{IS} = 0 \tag{A.3}$$

$$\begin{aligned} &\alpha(i_t - i_{t-1}) - \beta \frac{\partial E_t V_{t+1}^{CB}(r_{t+1}^n, i_t)}{\partial i_t} + \beta \frac{\partial E_t \pi(r_{t+1}^n, i_t)}{\partial i_t} \phi_t^{PC} \\ &+ \left(\frac{\partial E_t y(r_{t+1}^n, i_t)}{\partial i_t} + \sigma \frac{\partial E_t \pi(r_{t+1}^n, i_t)}{\partial i_t} - \sigma\right) \phi_t^{IS} - \phi_t^{ZLB} = 0 \end{aligned} \tag{A.4}$$

as well as the complementary slackness conditions and the NKPC and IS equation. Combining the first two conditions, we get

$$(1-\alpha)(\lambda y_t + \kappa \pi_t) = \phi_t^{IS} \tag{A.5}$$

Furthermore, note that

$$\frac{\partial V_t^{CB}(r_t^n, i_{t-1})}{\partial i_{t-1}} = \alpha(i_t - i_{t-1}) \tag{A.6}$$

We can then consolidate the third optimality condition to obtain an interest-rate target criterion

$$\begin{aligned} 0 =& \alpha(1+\beta)i_t - \alpha i_{t-1} - \beta\alpha E_t i(r_{t+1}^n, i_t) \\ &+ \beta(1-\alpha)\frac{\partial E_t \pi(r_{t+1}^n, i_t)}{\partial i_t}\pi_t + (1-\alpha)\left(\frac{\partial E_t y(r_{t+1}^n, i_t)}{\partial i_t} + \sigma \frac{\partial E_t \pi(r_{t+1}^n, i_t)}{\partial i_t}\right)(\lambda y_t + \kappa \pi_t) \\ &- (1-\alpha)\sigma(\lambda y_t + \kappa \pi_t) - \phi_t^{ZLB} \end{aligned} \tag{A.7}$$

A.2 Shadow interest-rate smoothing

The value of the central bank with a shadow interest-rate smoothing (SIRS) objective is given by

$$V_t^{CB,SIRS} = u^{CB,SIRS}(\pi_t, y_t, i_t, i_{t-1}^*) + \beta E_t V_{t+1}^{CB,SIRS} \tag{A.8}$$

where the central bank's contemporaneous objective function, $u^{CB,SIRS}(\cdot,\cdot,\cdot,\cdot)$, is given by

$$u^{CB,SIRS}(\pi_t, y_t, i_t, i^*_{t-1}) = -\frac{1}{2}\left[(1-\alpha)\left(\pi_t^2 + \lambda y_t^2\right) + \alpha(i_t - i^*_{t-1})^2\right] \tag{A.9}$$

Each period t, the central bank with a SIRS objective first chooses the shadow nominal interest rate in order to maximize the value today subject to the behavioral constraints of the private sector, with the value and policy functions at time $t+1$—$V_{t+1}^{CB,SIRS}(\cdot,\cdot)$, $y_{t+1}(\cdot,\cdot)$, $\pi_{t+1}(\cdot,\cdot)$—taken as given:

$$i^*_t = \text{argmax}_x \qquad u^{CB,SIRS}(\pi(x), y(x), x, i^*_{t-1}) + \beta E_t V_{t+1}^{CB,SIRS}(r^n_{t+1}, x) \tag{A.10}$$

with

$$\begin{aligned} y(x) =& E_t y_{t+1}(r^n_{t+1}, x) - \sigma(x - E_t\pi_{t+1}(r^n_{t+1}, x) - r^n_t) \\ \pi(x) =& \kappa y(x) + \beta E_t \pi_{t+1}(r^n_{t+1}, x) \end{aligned} \tag{A.11}$$

The actual policy rate i_t is given by

$$i_t = \max(i^*_t, 0) \tag{A.12}$$

That is, the actual policy rate today is zero when $i^*_t < 0$, and it is equal to i^*_t when $i^*_t \geq 0$.

The central bank's value today is given by

$$V_t^{CB,SIRS}(r^n_t, i^*_{t-1}) = u^{CB,SIRS}(\pi_t, y_t, i_t, i^*_{t-1}) + \beta E_t V_{t+1}^{CB,SIRS}(r^n_{t+1}, i^*_t) \tag{A.13}$$

where inflation and the output gap are given by

$$\begin{aligned} y_t =& E_t y_{t+1}(r^n_{t+1}, i^*_t) - \sigma(i_t - E_t\pi_{t+1}(r^n_{t+1}, i^*_t) - r^n_t) \\ \pi_t =& \kappa y_t + \beta E_t \pi_{t+1}(r^n_{t+1}, i^*_t) \\ i_t \geq& 0 \end{aligned} \tag{A.14}$$

A Markov-Perfect equilibrium with a SIRS objective is defined as a set of time-invariant value and policy functions $\{V^{CB,SIRS}(\cdot),\ \pi(\cdot),\ y(\cdot),\ i^*(\cdot),\ i(\cdot)\}$ that solves the problem of the central bank above, together with the value function $V(\cdot)$ that is consistent with $\pi(\cdot)$ and $y(\cdot)$.

B Calibration of the process of the natural real rate shock

To calibrate the process of the natural real rate shock, we follow the procedure used by Adam and Billi (2006). Specifically, we construct conditional output gap and inflation expectations. We then plug these expectations along with actual values for the output gap and inflation into the consumption Euler equation

$$y_t = \tilde{E}_t y_{t+1} - \sigma(i_t - i - \tilde{E}_t \pi_{t+1}) + d_t, \tag{B.1}$$

where $\tilde{E}_t y_{t+1}$ and $\tilde{E}_t \pi_{t+1}$ are conditional expectations, i is the mean of the policy rate and d_t is the equation residual. We then identify the natural real rate shock as

$$r_t^n - r^n = \frac{1}{\sigma} d_t. \tag{B.2}$$

We use quarterly data for the U.S. economy from 1984-Q1 to 2016-Q4. For the inflation rate we use the quarterly percentage change of the GDP implicit price deflator. The output gap is constructed as the log difference between real GDP and real potential GDP (both in billions of chained 2009 U.S. dollars). For the policy rate we use the quarterly average of the effective federal funds rate. All data series are obtained from FRED. We then subtract the respective sample mean from the three constructed data series.

Following Adam and Billi (2006), we use actual future values of the output gap and inflation for the conditional expectations. We then estimate an AR(1) model for $r_t^n - r^n$ using OLS, and obtain $\rho_r = 0.851$ (standard error: 0.045) and $\sigma_r^2 = 0.1588/100^2$ (standard error: $0.0165/100^2$), or $\sigma_r = 0.399/100$.

C Numerical algorithm and solution accuracy for the simple model

We use the policy function iteration algorithm described below to solve the simple model for the various monetary policy regimes.

C.1 Numerical algorithm

We approximate the policy functions for the inflation rate, output and the policy rate with a finite elements method using collocation. For the basis functions we use cubic splines. The algorithm uses fixed-point iteration and proceeds in the following steps (here exemplified for the IRS regime):

1. Construct the collocation nodes. The nodes are chosen such that they coincide with the spline breakpoints. Use a Gaussian quadrature scheme to discretize the normally distributed innovation to the natural real rate shock.

2. Start with a guess for the basis coefficients.

3. Use the current guess for the basis coefficients to approximate the expectation terms.

4. Solve the system of equilibrium conditions for inflation, output and the policy rate at the collocation nodes, assuming that the zero lower bound is not binding. For those nodes where the zero bound constraint is violated solve the system of equilibrium conditions associated with a binding zero bound.

5. Update the guess for the basis coefficients. If the new guess is sufficiently close to the old one, the algorithm has converged. Otherwise, go back to step 3.

C.2 Solution accuracy

We assess the solution accuracy by evaluating the residual functions associated with, the New Keynesian Phillips curve ($R_{PC,t}$), the consumption Euler equation ($R_{EE,t}$) and the target criterion (A.7) ($R_{TC,t}$) along a simulated equilibrium path with a length of 100,000 periods. For each equation, the residual function is defined as the absolute value of the difference between the left-hand side and the right-hand side of the equation. Table 4 reports the average and the maximum of these residuals for the optimized interest-rate smoothing regime.

Table 4: **Solution accuracy: Simple model with** $\alpha = 0.029$

	Mean$\left[\log_{10}(\mathrm{R}_{k,t})\right]$	Max$\left[\log_{10}(\mathrm{R}_{k,t})\right]$
$k = PC$: Sticky-price error	-6.54	-4.50
$k = EE$: Euler equation error	-5.46	-3.08
$k = TC$: Target criterion error	-7.66	-5.16

D Numerical algorithm and solution accuracy for the quantitative model

D.1 First-order necessary conditions for central bank's problem

Including private-sector equilibrium conditions (equation (19) - (23)), first-order necessary conditions for the central bank's maximization problem are enumerated as follows:

$$0 = -(1-\alpha)\lambda y_t + \phi_{1,t} - \kappa_w \left(\frac{1}{\sigma} + \eta\right)\phi_{2,t}, \tag{D.1}$$

$$0 = -(1-\alpha)\lambda_w(\pi_t^w - \iota_w \pi_{t-1}^p) + \phi_{2,t} + \phi_{4,t}, \tag{D.2}$$

$$\begin{aligned}
0 = &-(1-\alpha)(\pi_t^p - \iota_p \pi_{t-1}^p) + \beta(1-\alpha)\iota_p(E_t\pi_{t+1}^p - \iota_p\pi_t^p) + \beta(1-\alpha)\lambda_w\iota_w(E_t\pi_{t+1}^w - \iota_w\pi_t^p) \\
&- \phi_{1,t}\left(\frac{\partial E_t y_{t+1}}{\partial \pi_t^p} + \sigma\frac{\partial E_t\pi_{t+1}^p}{\partial \pi_t^p}\right) - \phi_{2,t}\beta\left(\frac{\partial E_t\pi_{t+1}^w}{\partial \pi_t^p} - \iota_w\right) - \beta\iota_w E_t\phi_{2,t+1} \\
&+ \phi_{3,t}\left(1 - \beta\left(\frac{\partial E_t\pi_{t+1}^p}{\partial \pi_t^p} - \iota_p\right)\right) - \beta\iota_p E_t\phi_{3,t+1} - \phi_{4,t},
\end{aligned} \tag{D.3}$$

$$\begin{aligned}
0 = &-\phi_{1,t}\left(\frac{\partial E_t y_{t+1}}{\partial w_t} + \sigma\frac{\partial E_t\pi_{t+1}^p}{\partial w_t}\right) + \phi_{2,t}\left(\kappa_w - \beta\frac{\partial E_t\pi_{t+1}^w}{\partial w_t}\right) - \phi_{3,t}\left(\kappa_p + \beta\frac{\partial E_t\pi_{t+1}^p}{\partial w_t}\right) \\
&- \phi_{4,t} + \beta E_t\phi_{4,t+1},
\end{aligned} \tag{D.4}$$

$$\begin{aligned}
0 = &-\alpha(i_t - i_{t-1}) + \beta\alpha(E_t i_{t+1} - i_t) + \phi_{1,t}\left(\sigma - \frac{\partial E_t y_{t+1}}{\partial i_t} - \sigma\frac{\partial E_t\pi_{t+1}^p}{\partial i_t}\right) \\
&- \phi_{2,t}\beta\frac{\partial E_t\pi_{t+1}^w}{\partial i_t} - \phi_{3,t}\beta\frac{\partial E_t\pi_{t+1}^p}{\partial i_t} + \phi_{5,t},
\end{aligned} \tag{D.5}$$

where $\phi_{1,t}$ - $\phi_{5,t}$ are Lagrangian multipliers for equation (19) - (23), respectively.

D.2 Solution method

There are total of five state variables, which we denote by $\mathbb{S}_t \ni [u_t, r_t^n, \pi_{t-1}^p, w_{t-1}, i_{t-1}]$. The problem is to find a set of policy functions, $\{\pi^p(\mathbb{S}_t)$, $\pi^w(\mathbb{S}_t)$, $y(\mathbb{S}_t)$, $w(\mathbb{S}_t)$, $i(\mathbb{S}_t)$, $\phi_1(\mathbb{S}_t)$, $\phi_2(\mathbb{S}_t)$, $\phi_3(\mathbb{S}_t)$, $\phi_4(\mathbb{S}_t)$, and $\phi_5(\mathbb{S}_t)\}$ that solves the following system of *functional* equations:

$$
\begin{aligned}
\pi^p(\mathbb{S}_t) - \iota_p \pi_{t-1}^p &= \kappa_p w(\mathbb{S}_t) + \beta \left(E_t \pi^p(\mathbb{S}_{t+1}) - \iota_p \pi^p(\mathbb{S}_t)\right) + u_t, && \text{(D.6)} \\
\pi^w(\mathbb{S}_t) - \iota_w \pi_{t-1}^p &= \kappa_w \left(\left(\frac{1}{\sigma} + \eta \right) y(\mathbb{S}_t) - w(\mathbb{S}_t) \right) + \beta \left(E_t \pi^w(\mathbb{S}_{t+1}) - \iota_w \pi^p(\mathbb{S}_t)\right), && \text{(D.7)} \\
\pi^w(\mathbb{S}_t) &= w(\mathbb{S}_t) - w_{t-1} + \pi^p(\mathbb{S}_t), && \text{(D.8)} \\
y(\mathbb{S}_t) &= E_t y(\mathbb{S}_{t+1}) - \sigma \left(i(\mathbb{S}_t) - E_t \pi^p(\mathbb{S}_{t+1}) - r_t^n\right), && \text{(D.9)} \\
i(\mathbb{S}_t) &\geq i_{ELB}. && \text{(D.10)}
\end{aligned}
$$

$$
\begin{aligned}
0 =& -(1-\alpha)\lambda y(\mathbb{S}_t) + \phi_1(\mathbb{S}_t) - \kappa_w \left(\frac{1}{\sigma} + \eta \right) \phi_2(\mathbb{S}_t), && \text{(D.11)} \\
0 =& -(1-\alpha)\lambda_w (\pi^w(\mathbb{S}_t) - \iota_w \pi_{t-1}^p) + \phi_2(\mathbb{S}_t) + \phi_4(\mathbb{S}_t), && \text{(D.12)} \\
0 =& -(1-\alpha)(\pi^p(\mathbb{S}_t) - \iota_p \pi_{t-1}^p) + \beta(1-\alpha)\iota_p (E_t \pi^p(\mathbb{S}_{t+1}) - \iota_p \pi^p(\mathbb{S}_t)) \\
& + \beta(1-\alpha)\lambda_w \iota_w (E_t \pi^w(\mathbb{S}_{t+1}) - \iota_w \pi^p(\mathbb{S}_t)) \\
& - \phi_1(\mathbb{S}_t) \left(\frac{\partial E_t y(\mathbb{S}_{t+1})}{\partial \pi^p(\mathbb{S}_t)} + \sigma \frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial \pi^p(\mathbb{S}_t)} \right) - \phi_2(\mathbb{S}_t)\beta \left(\frac{\partial E_t \pi^w(\mathbb{S}_{t+1})}{\partial \pi^p(\mathbb{S}_t)} - \iota_w \right) - \beta \iota_w E_t \phi_2(\mathbb{S}_{t+1}) \\
& + \phi_3(\mathbb{S}_t) \left(1 - \beta \left(\frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial \pi^p(\mathbb{S}_t)} - \iota_p \right) \right) - \beta \iota_p E_t \phi_3(\mathbb{S}_{t+1}) - \phi_4(\mathbb{S}_t), && \text{(D.13)} \\
0 =& -\phi_1(\mathbb{S}_t) \left(\frac{\partial E_t y(\mathbb{S}_{t+1})}{\partial w(\mathbb{S}_t)} + \sigma \frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial w(\mathbb{S}_t)} \right) + \phi_2(\mathbb{S}_t) \left(\kappa_w - \beta \frac{\partial E_t \pi^w(\mathbb{S}_{t+1})}{\partial w(\mathbb{S}_t)} \right) \\
& - \phi_3(\mathbb{S}_t) \left(\kappa_p + \beta \frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial w(\mathbb{S}_t)} \right) \\
& - \phi_4(\mathbb{S}_t) + \beta E_t \phi_4(\mathbb{S}_{t+1}), && \text{(D.14)} \\
0 =& -\alpha(i(\mathbb{S}_t) - i_{t-1}) + \beta\alpha(E_t i(\mathbb{S}_{t+1}) - i(\mathbb{S}_t)) + \phi_1(\mathbb{S}_t) \left(\sigma - \frac{\partial E_t y(\mathbb{S}_{t+1})}{\partial i(\mathbb{S}_t)} - \sigma \frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial i(\mathbb{S}_t)} \right) \\
& - \phi_2(\mathbb{S}_t)\beta \frac{\partial E_t \pi^w(\mathbb{S}_{t+1})}{\partial i(\mathbb{S}_t)} - \phi_3(\mathbb{S}_t)\beta \frac{\partial E_t \pi^p(\mathbb{S}_{t+1})}{\partial i(\mathbb{S}_t)} + \phi_5(\mathbb{S}_t), && \text{(D.15)}
\end{aligned}
$$

Following the idea of Christiano and Fisher (2000), we decompose these policy functions into two parts using an indicator function: one in which the policy rate is allowed to be less than 0, and the other in which the policy rate is assumed to be 0. That is, for any variable Z,

$$
Z(\cdot) = I_{\{R(\cdot) \geq 0\}} Z_{NZLB}(\cdot) + (1 - I_{\{R(\cdot) \geq 0\}}) Z_{ZLB}(\cdot). \tag{D.16}
$$

The problem then becomes finding a set of a *pair* of policy functions, $\{[\pi_{NZLB}^p(\cdot),\ \pi_{ZLB}^p(\cdot)]$,

$\{[\pi^w_{NZLB}(\cdot), \pi^w_{ZLB}(\cdot)], \{[y_{NZLB}(\cdot), y_{ZLB}(\cdot)], [w_{NZLB}(\cdot), w_{ZLB}(\cdot)], [i_{NZLB}(\cdot), i_{ZLB}(\cdot)], [\phi_{1,NZLB}(\cdot), \phi_{1,ZLB}(\cdot)], [\phi_{2,NZLB}(\cdot), \phi_{2,ZLB}(\cdot)], [\phi_{3,NZLB}(\cdot), \phi_{3,ZLB}(\cdot)], [\phi_{4,NZLB}(\cdot), \phi_{4,ZLB}(\cdot)]$, and $[\phi_{5,NZLB}(\cdot)\ \phi_{5,ZLB}(\cdot)]\}$ that solves the system of functional equations above. This approach of Christiano and Fisher (2000) can achieve a given level of accuracy with a considerable less number of grid points relative to the standard approach.

The time-iteration method aims to find the values for the policy and value functions consistent with the equilibrium conditions on a finite number of grid points within the pre-determined grid intervals for the model's state variables. Let $X(\cdot)$ be a vector of policy functions that solves the functional equations above and let $X^{(0)}$ be the initial guess of such policy functions.[26] At the s-th iteration, given the approximated policy function $X^{(s-1)}(\cdot)$, we solve the system of nonlinear equations given by equations (D.6)-(D.15) to find today's $\pi^p_t, \pi^w_t, y_t, w_t, i_t, \phi_{1,t}, \phi_{2,t}, \phi_{3,t}$, $\phi_{4,t}$, and $\phi_{5,t}$ at each grid point. In solving the system of nonlinear equations, we use Gaussian quadrature (with 10 Gauss-Hermite nodes) to discretize and evaluate the expectation terms in the Euler equation, the price and wage Phillips curves, and expectational partial derivative terms.

The values of the policy function that are not on any of the grid points are interpolated or extrapolated linearly. The values of the partial derivatives of the policy functions not on any of the grid points are approximated by the slope of the policy functions evaluated from the adjacent two grid points. That is, for any variable X and Z,

$$\frac{\partial X(\delta_{t+1,t})}{\partial Z_t} = \frac{X(\delta_{t+1}, Z'') - X(\delta_{t+1}, Z')}{Z'' - Z'}. \tag{D.17}$$

where Z' and Z'' are two adjacent grid points to Z_t such that $Z' < Z_t < Z''$. When Z_t is outside the grid interval, the partial derivative is approximated by the slope evaluated at the edge of the grid interval.

The system is solved numerically by using a nonlinear equation solver, `dneqnf`, provided by the IMSL Fortran Numerical Library. If the updated policy functions are sufficiently close to the previously approximated policy functions, then the iteration ends. Otherwise, using the former as the guess for the next period's policy functions, we iterate on this process until the difference between the guessed and updated policy functions is sufficiently small ($\left\| vec(X^s(\delta) - X^{s-1}(\delta)) \right\|_\infty <$ 1E-12 is used as the convergence criteria). The solution method can be extended to models with multiple (non-perfectly correlated) exogenous shocks and with multiple endogenous state variables in a straightforward way.

D.3 Solution accuracy

In this section, we report the accuracy of our numerical solutions for the quantitative model. Following Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015) and Maliar and Maliar (2015), we evaluate the residuals functions along a simulated equilibrium path. The length of the simulation is 100,000.

[26] For all models and all variables, we use flat functions at the deterministic steady-state values as the initial guess.

For the quantitative model, there are six key residual functions of interest. The first three residual functions, denoted by $R_{1,t}$, $R_{2,t}$, and $R_{3,t}$, are associated with the sticky-price equation, the sticky-wage equation, and the Euler equation, respectively (equations (19), (20), and (22)). The last three residual functions, denoted by $R_{4,t}$, $R_{5,t}$, and $R_{6,t}$, are associated with the first-order conditions of the central bank's optimization problem with respect to price inflation, real wage, and the policy rate, respectively (equations (D.3), (D.4), and (D.5)). For each equation, the residual function is defined as the absolute value of the difference between the left-hand side and the right-hand side of the equation. Table 5 shows the average and the maximum of the six residual functions over the 100,000 simulations.

The size of the residuals are comparable to those reported in other numerical works on the New Keynesian model with the ELB constraint, such as Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015), Hills, Nakata, and Schmidt (2016), Hirose and Sunakawa (2015), and Maliar and Maliar (2015).

Table 5: **Solution accuracy: Quantitative model with $\alpha = 0.37$**

	Mean$\left[\log_{10}(\mathrm{R}_{k,t})\right]$	Max$\left[\log_{10}(\mathrm{R}_{k,t})\right]$
$k = 1$: Sticky-price error	-6.46	-4.86
$k = 2$: Sticky-wage error	-6.05	-4.40
$k = 3$: Euler equation error	-4.10	-2.25
$k = 4$: Error in the FONC w.r.t price inflation	-5.07	-4.12
$k = 5$: Error in the FONC w.r.t real wage	-3.90	-3.41
$k = 6$: Error in the FONC w.r.t. policy rate	-2.94	-2.78

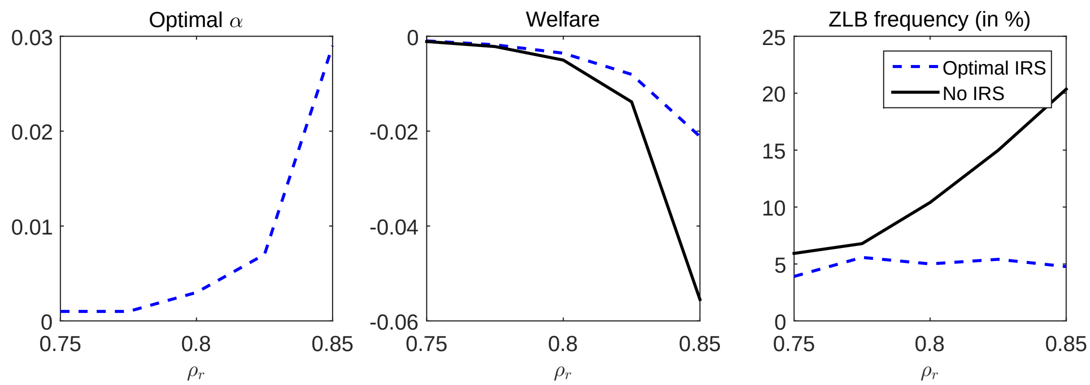
E Sensitivity of results with respect to the calibration of the natural real rate shock process

This section documents how the optimal relative weight on the IRS regime, the welfare gains from IRS, and the frequency of a binding ZLB constraint depend on the calibration of the process of the natural real rate shock r_t^n. The first subsection summarizes results for the simple model of Section 2, and the second subsection for the quantitative model of Section 5.

E.1 Simple model

Figure 9 shows how the calibration of the persistence parameter of the natural real rate shock process, ρ_r, affects the optimal relative weight on the IRS objective (left panel), welfare under the optimal IRS regime and under the standard discretionary regime (middle panel) and the frequency of a binding ZLB constraint under the optimal IRS regime and under the standard discretionary regime (right panel). All other parameters, including the standard deviation of the innovation to the natural real rate shock remain unchanged. Results are only shown for the model with the ZLB, as in the model without the ZLB the efficient equilibrium can be replicated by the standard discretionary regime.

Figure 9: The role of the persistence of the natural rate shock in the simple model with ZLB

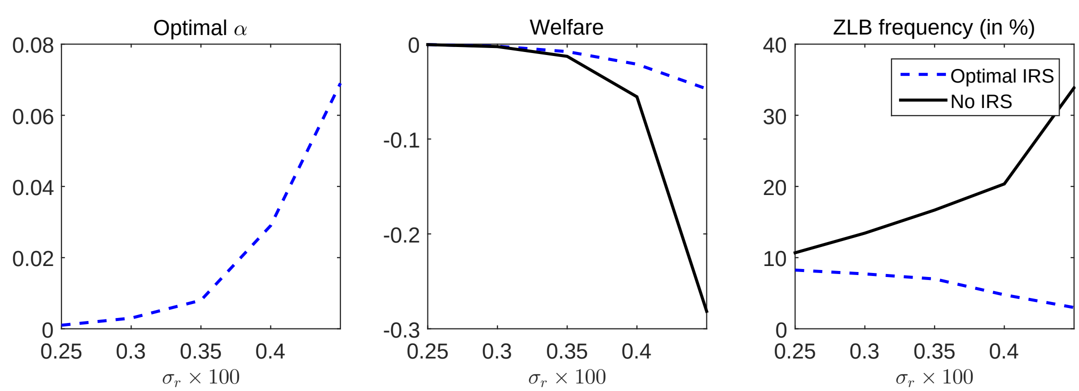


Note: The baseline calibration is $\rho_r = 0.85$. The welfare measure is defined in equation (10).

The optimal relative weight on the IRS objective and the welfare gains from optimal IRS—represented by the difference between the blue dashed line and the solid black line in the middle panel—are both increasing in ρ_r. These two results are intuitive since all else equal, the ZLB is binding more often the higher the persistence of the natural real rate shock. This can be seen from the results for the frequency of ZLB events under the standard discretionary regime (solid black line in the right panel). Finally, we find that in the case of the optimized IRS regime, ρ_r has only small (and non-monotonic) effects on the frequency of ZLB events. This reflects the fact that the optimal relative weight on the IRS objective itself is varying with ρ_r.

Figure 10 shows how results depend on the calibration of the standard deviation of the natural real rate shock innovation, σ_r.

Figure 10: The role of the volatility of the natural rate shock in the simple model with ZLB



Note: The baseline calibration is $\sigma_r = 0.40/100$. The welfare measure is defined in equation (10).

Results are qualitatively similar to those obtained for the persistence parameter ρ_r. In particular, both the optimal relative weight on the IRS objective and the relative welfare gains from optimal IRS are increasing in the standard deviation of the natural real rate shock innovation. The only notable difference is that under the optimal IRS regime, the frequency of a binding ZLB constraint is strictly declining in σ_r.

E.2 Quantitative model

TO BE COMPLETED