Endogenous Regime Switching Near the Zero Lower Bound

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Abstract

This paper develops a New Keynesian model with a time-varying natural rate of interest (r-star) and a zero lower bound (ZLB) on the nominal interest rate. The representative agent contemplates the possibility of an occasionally binding ZLB that is driven by switching between two local rational expectations equilibria, labeled the "targeted" and "deflation" solutions, respectively. Sustained periods when the real interest rate remains below the central bank's estimate of r-star can induce the agent to place a substantially higher weight on the deflation equilibrium, causing it to occasionally become self-fulfilling. I solve for the time series of stochastic shocks and endogenous forecast rule weights that allow the model to exactly replicate the observed time paths of the U.S. output gap and quarterly inflation since 1988. In model simulations, raising the central bank's inflation target to 4% from 2% can reduce, but not eliminate, the endogenous switches to the deflation equilibrium.

Keywords: *Natural rate of interest, New Keynesian, Liquidity trap, Zero lower bound, Taylor rule, Deflation.*

JEL Classification: E31, E43, E52.

1 Introduction

The sample period from 1988 onwards is generally viewed as an example of consistent U.S. monetary policy aimed at keeping inflation low while promoting sustainable growth and full employment. The nature of this policy is typically described in standard New Keynesian models by a Taylor-type rule in which movements in the federal funds rate are driven by fluctuations in recent inflation and a measure of real activity. Amazingly, the U.S. federal funds rate has been pinned close to zero for about one-fourth of the elapsed time since 1988. The U.S. economy is not alone in experiencing an extended period of zero or mildly negative nominal interest rates in recent decades.

Figure 1 plots three-month nominal Treasury bill yields in the United States, Japan, Switzerland, and the United Kingdom. Nominal interest rates in the United States encountered the zero lower bound during the 1930s and from 2008.Q4 though 2015.Q4. Nominal interest rates in Japan have remained near zero since 1998.Q3, except for the relatively brief period from 2006.Q4 to 2008.Q3. Nominal interest rates in Switzerland have been zero or slightly negative since 2008.Q4. Nominal interest rates in the United Kingdom have been approximately zero since 2009.Q1. Outside of these episodes, all four countries exhibit a strong positive correlation between nominal interest rates and inflation, consistent with the Fisher relationship.

Benhabib, Schmitt-Grohé and Uribe (2001a,b) show that imposing a zero lower bound (ZLB) on the nominal interest rate in a standard New Keynesian model gives rise to two long-run endpoints (steady states).[1] The basic idea is illustrated in Figure 2, which is adapted from Bullard (2010). The two intersections of the ZLB-augmented monetary policy rule (solid red line) with the Fisher relationship (dashed black line) define two long-run endpoints. I refer to these as the "targeted equilibrium" and "deflation equilibrium," respectively. Data since 2008.Q4 lie closer to the deflation equilibrium than the targeted equilibrium.

This paper develops a New Keynesian model with a time-varying natural rate of interest (r-star), i.e., the real short-term interest rate that is consistent with full utilization of economic resources and steady inflation at the central bank's target rate. R-star is an important benchmark for monetary policy because it determines the real interest rate that policymakers should aim for once shocks to the economy have dissipated and the central bank's macroeco-

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[1] I use the terminology "long-run endpoints" rather than "steady states" because the model developed here allows for permanent shifts in the natural rate of interest which, in turn, can shift the long-run values of some macroeconomic variables.

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nomic goals have been achieved.[2] The times series process for r-star in the model is calibrated to closely approximate the path of the U.S. natural rate series estimated by Laubach and Williams (2016).[3]

As is well known, the New Keynesian deflation equilibrium is locally indeterminate. I therefore consider a minimum state variable (MSV) solution that rules out sunspot variables and extra lags of fundamental state variables. The decision rules associated with the deflation equilibrium induce more volatility in the output gap and inflation in response to real interest rate shocks. Model variables in the deflation equilibrium have distributions with lower means and higher variances than those in the targeted equilibrium. But the significant overlap in the various distributions creates a dilemma for an agent who seeks to determine the likelihood that a string of recent data observations are drawn from one equilibrium or the other.

The representative agent in the model contemplates the possibility of an occasionally binding ZLB that is driven by switching between the two local equilibria. This view turns out to be true in the simulations, validating the agent's beliefs. The agent constructs forecasts using a form of model averaging, where the time-varying forecast weights are determined by recent performance, as measured by the root mean squared forecast errors for the output gap and inflation. Sustained periods when the real interest rate remains below the central bank's estimate of r-star can induce the agent to place a substantially higher weight on the deflation equilibrium, causing it to occasionally become self-fulfilling. These episodes are accompanied by highly negative output gaps and a binding ZLB, reminiscent of the U.S. Great Recession. But even outside of recessions or when the ZLB is not binding, the agent may continue to assign a nontrivial weight to the deflation equilibrium, causing the central bank to persistently undershoot its inflation target, similar to the U.S. economy since mid-2012.

In one exercise, I solve for the time series of stochastic shocks and endogenous forecast rule weights that allow the switching model to exactly replicate the observed time paths of the CBO output gap and quarterly PCE inflation since 1988. The model-implied weight on the targeted equilibrium starts to decline in 2008.Q4, eventually reaching a minimum in 2011.Q3. The weight subsequently increases as the U.S. economy recovers from the Great Recession. But even towards the end of the data sample, the weight on the targeted equilibrium remains well below 1.0, helping the model to account for the persistent undershooting of the Fed's inflation target since mid-2012. The path of expected inflation from the switching model

starts to decline after 2008.Q4 and remains below the Fed's 2% inflation target at the end of the data sample. This pattern is similar to some measures of expected inflation in U.S. data.

The framework developed here is similar to that of Aruoba and Schorfheide (2016) and Aruoba, Cuba-Borda, and Schorfheide (2017). These authors construct a stochastic two-regime model in which the economy can switch between a targeted-inflation regime and a deflation regime, depending on the realization of a sunspot variable. The probability of transitioning from one regime to the other is exogenous. In contrast, the regime switching here is driven by the recent performance of forecast rules that employ observed data on macroeconomic variables. Hence, the transition probabilities that govern the regime switches are endogenous and can be influenced by a change in the monetary policy rule. Moreover, the probability assigned by the agent to being in one regime or the other is not restricted to be zero or one, but rather can take on intermediate values, depending on recent data.

Another related paper is one by Dordal-i-Carrera et al. (2016). These authors develop a New Keynesian model with volatile and persistent "risk shocks" (i.e., shocks that drive a wedge between the nominal policy rate and the short-term bond rate) to account for infrequent but long-lived ZLB episodes. A risk shock in their model is isomorphic to a real interest rate shock here. Large adverse risk shocks are themselves infrequent and long-lived. As the binding ZLB episode becomes more frequent or more long-lived, the optimal inflation target increases. Unlike here, their analysis does not consider model solutions near the deflation equilibrium, but rather focuses on scenarios in which fundamental shocks are large enough to push the targeted equilibrium to a point where ZLB becomes binding.[4] In contrast, the model developed here accounts for infrequent but long-lived ZLB episodes via endogenous switching between two local equilibria, i.e., the shock process itself is not the sole driving force for the infrequent and long-lived ZLB episodes.

As part of the quantitative analysis, I examine how raising the central bank's inflation target can influence the ZLB binding frequency and the volatility of macro variables in the switching model. I find that even with an inflation target of 4%, the ZLB binding frequency remains elevated at 9.9%, the average duration of a ZLB episode is 11.2 quarters, and the maximum duration of a ZLB episode is 132 quarters, or 33 years. Once the deflation equilibrium is taken into account, raising the inflation target is a less effective solution for avoiding ZLB episodes. Reducing the degree of interest rate smoothing in the monetary policy rule serves

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[4]This is also the methodology pursued by Reifschneider and Williams (2000), Schmitt-Grohé and Uribe (2010), Chung et al. (2012), Coibion, Gorodnichenko, and Wieland (2012), Dennis (2016), and Kiley and Roberts (2017).

to increase the ZLB binding frequency, but the episodes exhibit shorter duration on average.

Lastly, I introduce an adaptive learning algorithm into a simplified version of the model. When the agent estimates correctly specified decision rules, the algorithm quickly converges to the vicinity of the targeted equilibrium and remains there. But when the agent estimates misspecified decision rules that fail to control for some white noise shocks, the model exhibits low frequency oscillations between the two local equilibria that are qualitatively similar to those observed in the original switching model with full-knowledge.

1.1   Related literature

A number of papers introduce adaptive learning type mechanisms to examine the dynamics of convergence to either the targeted or the deflation equilibrium. A typical conclusion is that the targeted equilibrium is locally (but not globally) stable under least squares learning (Evans and Honkapohja 2005, Eusepi 2007, Evans, Guse, and Honkapohja 2008, Benhabib, Evans and Honkapohja 2014, Christiano, Eichenbaum, and Johanssen 2016). Arifovic, Schmitt-Grohé, and Uribe (2017) demonstrate that both equilibria can be locally stable under a form of social learning. Hursey and Wolman (2010) examine the global perfect-foresight dynamics of the ZLB-augmented New Keynesian model. They conclude that "the model only tells us what equilibria exist, not how likely they are to occur" (p. 335).

Alstadheim and Henderson (2006) and Sugo and Ueda (2008) describe interest rate rules that can preclude the deflation equilibrium. Armenter (2014) considers an extension of Benhabib, Schmitt-Grohé and Uribe (2001b) in which monetary policy is governed not by a Taylor-type rule, but rather by the optimal time-consistent rule that minimizes the central bank's loss function. He shows that it may not be possible to achieve the targeted equilibrium if agents' initial inflation expectations are below the central bank's inflation target.

Numerous papers consider optimal monetary policy in response to a time-varying natural rate of interest. The models typically impose the ZLB (or effective lower bound), but the deflation equilibrium is ignored, i.e., the analysis is local to the targeted equilibrium. Examples include Eggertsson and Woodford (2003), Adam and Billi (2007), Nakov (2008), Nakata (2013), Hamilton, et al. (2016), Basu and Bundick (2015), Evans, et al. (2015), and Gust, Johannsen, López-Salido (2017). One finding of this literature is that more uncertainty about the future natural rate implies looser monetary policy today or more policy inertia.

The model developed here shares some similarities with the work of Sargent (1999) in which the model economy can endogenously switch between regimes of high versus low inflation,

depending on monetary policymakers' perceptions about the slope of the long-run Phillips curve in light of recent data. Here, the endogenous regime switching depends on private-sector agents' perceptions about whether recent data are more likely to have been generated by the targeted equilibrium or the deflation equilibrium.

2 Model

The framework for the analysis is a standard New Keynesian model, augmented by a zero lower bound on the short-term nominal interest rate. The log-linear version of the standard New Keynesian model is taken to represent a set of global equilibrium conditions, with the only nonlinearity coming from the ZLB.[5] Private-sector behavior is governed by the following equilibrium conditions:

$$y_t = E_t\, y_{t+1} - \alpha[i_t - E_t\, \pi_{t+1} - r_t] + \nu_t, \quad \nu_t \sim N\left(0,\, \sigma_\nu^2\right), \tag{1}$$

$$\pi_t = \beta E_t\, \pi_{t+1} + \kappa y_t + u_t, \quad u_t \sim N\left(0,\, \sigma_u^2\right) \tag{2}$$

where equation (1) is the representative household's consumption Euler equation and equation (2) is the Phillips curve that is derived from the representative firm's optimal pricing decision. The variable y_t is the output gap (the log deviation of real output from potential output), π_t is the quarterly inflation rate (log difference of the price level), i_t is the short-term nominal interest rate, r_t is the exogenous real interest rate, and E_t is the rational expectations operator. Fluctuations in r_t can be interpreted as arising from changes in the representative agent's rate of time preference or changes in the expected growth rate of potential output.[6] The terms ν_t and u_t represent an aggregate demand shock and a cost-push shock, respectively. None of the results in the paper are sensitive to the introduction of a discount factor applied to the term $E_t\, y_{t+1}$ in equation (1), along the lines of McKay, Nakamura, and Steinsson (2016).

The time series process for the real rate of interest is given by

$$r_t = \rho_r\, r_{t-1} + (1 - \rho_r)\, r_t^* + \varepsilon_t, \quad \varepsilon_t \sim N\left(0,\, \sigma_\varepsilon^2\right), \tag{3}$$

$$r_t^* = r_{t-1}^* + \eta_t, \quad \eta_t \sim N\left(0,\, \sigma_\eta^2\right). \tag{4}$$

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[5]Armenter (2016) adopts a similar approach in computing the optimal monetary policy in the presence of two steady states. Eggertsson and Sing (2016) show that the log-linear New Keynesian model behaves very similar to the true nonlinear model in the vicinity of the targeted equilibrium.

[6]Specifically, we have $r_t \equiv -\log\left[\beta \exp\left(\zeta_t\right)\right] + \gamma E_t \Delta \bar{y}_{t+1}$, where ζ_t is a shock to the agent's time discount factor β, $\bar{y}_t$ is the logarithm of real potential output, and $\gamma = \alpha^{-1}$ is the coefficient of relative risk aversion. For the derivation, see Hamilton, et al. (2016) or Gust, Johannsen, and Lopez-Salido (2017).

Equations (3) and (4) summarize a "shifting endpoint" time series process since the long-run endpoint r_t^* can vary over time due to the permanent shock η_t. In any given period, r_t can deviate from r_t^* due to the temporary shock ε_t. The persistence of the "real interest rate gap" $r_t - r_t^*$ is governed by the parameter ρ_r, where $|\rho_r| < 1$. Kozicki and Tinsely (2012) employ this type of time series process to describe U.S. inflation. When $\rho_r = 1$, we recover the random walk plus noise specification employed by Stock and Watson (2007) to describe U.S. inflation.[7]

Using equation (3) to substitute out r_t from equation (1) yields the following alternative version of the consumption Euler equation:

$$y_t = E_t\, y_{t+1} - \alpha[i_t - E_t\, \pi_{t+1} - r_t^*] + u_t + \alpha\varepsilon_t + \alpha\rho_r\left(r_{t-1} - r_{t-1}^* - \eta_t\right), \qquad (5)$$

where the last three terms could be consolidated into a single aggregate demand shock. From this version, we can interpret r_t^* as the unobservable "natural rate of interest," i.e., the real interest rate that is consistent with full utilization of economic resources and steady inflation at the central bank's target rate. This interpretation is consistent with the empirical strategies of Laubach and Williams (2016), Lubik and Matthes (2015), and Kiley (2015) which view the natural rate of interest as a longer-term economic concept. In contrast, empirical strategies that employ micro-founded New Keynesian models typically view the natural (or equilibrium) rate of interest as a short-term concept, more along the lines of the variable r_t in equation (1).[8] The real interest rate gap $r_t - r_t^*$ captures a concept that has been emphasized by Fed policymakers in recent speeches, namely, a distinction between estimates of the "short-term natural of interest" and its longer-term counterpart (Yellen 2015, Dudley 2015, and Fischer 2016). Here I will refer to r_t^* as the natural rate of interest.

In the model, the agent's rational forecast for the real interest rate gap at any horizon $h \geq 1$ is given by

$$E_t\left(r_{t+h} - r_{t+h}^*\right) = \left(\rho_r\right)^h \left(r_t - E_t r_t^*\right), \qquad (6)$$

where $E_t r_t^*$ represents the agent's current estimate of the natural rate computed using the Kalman filter so as to minimize the mean squared forecast error. When $|\rho_r| < 1$ as assumed here, the real interest rate gap is expected to shrink to zero as the forecast horizon h increases.

[7]But unlike here, Stock and Watson (2007) allow for stochastic volatility in the permanent and temporary shocks.

[8]See, for example, Barsky, Justiniano, and Melosi (2014), Cúrdia, et al. (2015), and Del Negro, et al. (2017).

In Appendix A, I show that the Kalman filter expression for $E_t r_t^*$ is

$$E_t r_t^* = \lambda \left[\frac{r_t - \rho_r r_{t-1}}{1 - \rho_r} \right] + (1 - \lambda) E_{t-1} r_{t-1}^* \qquad (7)$$

$$\lambda = \frac{-(1 - \rho_r)^2 \phi + (1 - \rho_r) \sqrt{(1 - \rho_r)^2 \phi^2 + 4\phi}}{2}, \qquad (8)$$

where λ is the Kalman gain parameter and $\phi \equiv \sigma_\eta^2 / \sigma_\varepsilon^2$. For the quantitative analysis, the values of ρ_r, σ_η^2, and σ_ε^2 are chosen so that the time path of $E_t r_t^*$ from equation (7) approximates the path of the U.S. natural rate series estimated by Laubach and Williams (2016, updated) for the sample period 1988.Q1 to 2017.Q2. Their estimation strategy assumes that the natural rate exhibits a unit root, consistent with equation (4). Hamilton, et al. (2016) present evidence that the ex-ante real rate of interest $i_t - E_t \pi_{t+1}$ in U.S. data is nonstationary, but they find that the gap between the ex-ante real rate and their estimate of the world long-run real rate appears to be stationary. This evidence is also consistent with equations (3) and (4) which imply that real rate gap $r_t - r_t^*$ is stationary.

The central bank's monetary policy rule is given by

$$i_t^* = \rho i_{t-1}^* + (1 - \rho) [E_t r_t^* + \pi^* + g_\pi (\overline{\pi}_t - \pi^*) + g_y (y_t - y^*)], \qquad (9)$$

$$\overline{\pi}_t = \omega \pi_t + (1 - \omega) \overline{\pi}_{t-1}, \qquad (10)$$

$$i_t = \max \{0, i_t^*\}, \qquad (11)$$

where i_t^* is the desired nominal interest rate that responds to deviations of recent inflation $\overline{\pi}_t$ from the central bank's target rate π^* and to deviations of the output gap from its targeted long-run endpoint y^*. Recent inflation $\overline{\pi}_t$ is an exponentially-weighted moving average of past quarterly inflation rates so as to approximate the compound average inflation rate over the past 4 quarters—a typical central bank target variable.[9] The parameter ρ governs the degree of interest rate smoothing as i_t^* adjusts partially each period toward the value implied by the terms in square brackets.

The quantity $E_t r_t^* + \pi^*$ represents the targeted long-run endpoint of i_t^*. Including $E_t r_t^*$ in the policy rule implies that monetary policymakers continually update their estimate of the unobservable r_t^*. Support for this idea can be found in the Federal Open Market Committee's Summary of Economic Projections (SEP). Meeting participants provide their views on the projected paths of macroeconomic variables over the next three calendar years and in the

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[9]Specifically, the value of ω is set to achieve $\overline{\pi}_t \simeq [\Pi_{j=0}^3 (1 + \pi_{t-j})]^{0.25} - 1$

longer run. Since the natural rate of interest is a longer-run concept, we can infer the median SEP projection for r_t^* by subtracting the median longer-run projection for inflation from the median longer-run projection for the nominal federal funds rate. The median SEP projection for r_t^* computed in this way has ratcheted down over time, as documented by Lansing (2016), and currently stands at about 1%.[10]

Equation (11) is the ZLB that constrains the nominal policy interest rate i_t to be non-negative. In the model simulations, I implement the occasionally binding ZLB by making the substitution $i_t = 0.5\,i_t^* + 0.5\sqrt{(i_t^*)^2}$ in the global equilibrium condition (1). Details are contained in the appendix.

2.1 Long-run endpoints

The Fisher relationship $i_t = r_t + E_t\,\pi_{t+1}$ is embedded in the non-stochastic version of equation (1).[11] Consequently, when $g_\pi > 1$, the model has two long-run endpoints (steady states) as shown originally by Benhabib, Schmitt-Grohé, and Uribe (2001a,b). The novelty here is that the long-run endpoints can shift due to shifts in r_t^*. Straightforward computations using the model equations yield the following long-run endpoints that characterize the "targeted equilibrium" and the "deflation equilibrium," respectively.

Table 1. Long-run Endpoints

Targeted equilibrium	Deflation equilibrium
$\pi_t = \pi^*$	$\pi_t = -r_t^*$
$y_t = y^* = \pi^*\,(1-\beta)/\kappa$	$y_t = -r_t^*\,(1-\beta)/\kappa$
$i_t^* = r_t^* + \pi^*$	$i_t^* = (r_t^* + \pi^*)\,[1 - g_\pi - g_y\,(1-\beta)/\kappa]$
$i_t = r_t^* + \pi^*$	$i_t = 0$

In the targeted equilibrium, long-run inflation is at the central bank's target rate π^* and the long-run output gap y^* is slightly positive for typical calibrations with $0.99 < \beta < 1$. The long-run desired nominal policy rate i_t^* conforms to the Fisher relationship and the ZLB is not binding such that $i_t = i_t^* > 0$, provided that $r_t^* > -\pi^*$. In the model simulations, I impose bounds on fluctuations in r_t^* that are based on the range of natural rate estimates obtained by Laubach and Williams (2016) for the sample period since 1988. In the deflation equilibrium,

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[10]Gust, Johannsen, and Lopez-Salido (2017) show that a Taylor-type rule that includes a time-varying intercept that moves with perceived changes in the equilibrium real interest rate can achieve results that are similar to optimal discretionary policy. Carlstrom and Fuerst (2016) compute the optimal response coefficient on the natural rate of interest in a Taylor-type rule.

[11]Cochrane (2016) and Williamson (2017b) show that Fisherian effects can dominate the short-term comovement between the nominal interest rate and inflation in standard New Keynesian models.

the long-run inflation rate, the long-run output gap, and the long-run desired nominal interest rate are all negative when $r_t^* > 0$.[12]

2.2   Local linear forecast rules

Given the linearity of the model aside from the ZLB, it is straightforward to derive the agent's rational decision rules for y_t and π_t in the vicinity of the long-run endpoints associated with each of the two equilibria. For the targeted equilibrium, the local decision rules are unique linear functions of the state variables: r_t, $E_t r_t^*$, $\overline{\pi}_{t-1}$, i_{t-1}^*, ν_t, and u_t. For the deflation equilibrium, I solve for the minimum state variable (MSV) solution which abstracts from extraneous sunspot variables and extra lags of fundamental state variables.[13]

Given the local linear decision rules, we can construct the agent's conditional forecasts for y_{t+1} and π_{t+1} in each of the two local equilibria. In the stochastic simulations, I substitute the local linear forecast rules into the global equilibrium conditions (1) and (2). I allow for an occasionally binding ZLB by making the substitution $i_t = 0.5\, i_t^* + 0.5 \sqrt{(i_t^*)^2}$ in equation (1). Together with the monetary policy rule (9), this procedure yields a system of three equations that can be solved each period to obtain the three realizations y_t, π_t, and i_t^*. Details are contained in Appendices B and C.

The decision rule coefficients applied to the state variable $r_t - E_t r_t^*$ are much larger in magnitude in the deflation equilibrium than in the targeted equilibrium (see Appendices B and C). Consequently, the deflation equilibrium exhibits more volatility and undergoes a more severe recession in response to an adverse shock sequence that causes $r_t - E_t r_t^*$ to be persistently negative. The higher volatility in the deflation equilibrium is due to the binding ZLB which prevents the central bank from taking action to mitigate the consequences of the adverse shock sequence.

The local linear forecast rules for the targeted equilibrium are derived under the assumption that $i_t^* > 0$ and hence do not take into account the possibility that a shock sequence could be large enough to cause the ZLB to become binding in the future. The error induced by this assumption will depend on the frequency and duration of ZLB episodes in the targeted equilibrium. Based on model simulations, the targeted equilibrium experiences a binding ZLB in only 1.5% of the periods, with an average duration of 4.1 quarters. Consequently, the

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[12]Evans, Honkopoja, and Mitra (2016) develop a New Keynesian models that imposes a lower bound on the inflation rate that is more negative than $-r^*$ (which is assumed to be constant in their model). They show that this additional constraint gives rise to a third steady state in which the ZLB binds but the Fisher relationship does not hold.

[13]For background on MSV solutions, see McCallum (1999).

agent's use of forecast rules that assume $i_t^* > 0$ seems quite reasonable.[14] The local linear forecast rules for the deflation equilibrium are derived under the assumption that $i_t^* \leq 0$ and hence do not take into account the possibility that a shock sequence could be large enough to cause the ZLB to become slack in the future. Based on model simulations, the deflation equilibrium experiences a binding ZLB in 77% of the periods, with an average duration of 30 quarters. The higher volatility of the deflation equilibrium causes the assumption of $i_t^* \leq 0$ to be violated in 23% of the periods. Hence, the error induced by the agent's use of local linear forecast rules would appear to be more significant in the deflation equilibrium. Nevertheless, as shown in Section 4, the agent's forecast errors in the deflation equilibrium are close to white noise, giving no clear indication that the linear forecast rules are misspecified.[15]

2.3 Endogenous regime switching

I now consider a more sophisticated agent who contemplates the possibility of an occasionally binding ZLB that is driven by switching between the two local equilibria, implying that one set of linear forecast rules might perform better than the other. The agent constructs forecasts using a form of model averaging—a technique that is often employed to improve forecast performance in situations where the true data generating process is unknown (Timmerman 2006). The agent in the switching model can be viewed as someone thinking along the lines of Bullard (2010), i.e., the agent is aware of the two local equilibria implied by the New Keynesian framework and is concerned about the possibility of getting stuck in a deflation trap. The forecast rules in the switching model are given by

$$\widehat{E}_t\, y_{t+1} = \mu_t E_t^{\text{targ}}\, y_{t+1} \ + \ (1 - \mu_t)\, E_t^{\text{defl}}\, y_{t+1}, \tag{12}$$

$$\widehat{E}_t\, \pi_{t+1} = \mu_t E_t^{\text{targ}}\, \pi_{t+1} \ + \ (1 - \mu_t)\, E_t^{\text{defl}}\, \pi_{t+1}, \tag{13}$$

where μ_t is the value that minimizes the root mean squared forecast error computed over a moving window of recent data. Specifically, μ_t is the value that minimizes:

$$RMSFE_{t-1} \ = \ \sum_{j=1}^{T_w} \left\{ \tfrac{1}{T_w} \left[y_{t-j} - \mu_t E_{t-j-1}^{\text{targ}}\, y_{t-j} - (1-\mu_t)\, E_{t-j-1}^{\text{defl}}\, y_{t-j} \right]^2 \right.$$

$$\left. + \tfrac{1}{T_w} \left[\pi_{t-j} - \mu_t E_{t-j-1}^{\text{targ}}\, \pi_{t-j} - (1-\mu_t)\, E_{t-j-1}^{\text{defl}}\, \pi_{t-j} \right]^2 \right\}^{0.5}, \tag{14}$$

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[14]Richter and Throckmorton (2016) compare linear model solutions for the targeted equilibrium in which agents ignore the possibility of future ZLB episodes to nonlinear model solutions that account for this possibility.

[15]Aruoba, Cuba-Borda, and Schorfheide (2017) solve for piece-wise linear decision rules in both the targeted equilibrium and the deflation equilibrium to account for the occasionally binding nature of the ZLB constraint.

which shows that μ_t is computed using data dated $t-1$ or earlier. In the simulations, I impose the restriction $0 \leq \mu_t \leq 1$. Very similar results are obtained if μ_t is determined by a discrete choice framework along the lines of Brock and Hommes (1998).[16]

Given the representative agent's conditional forecasts from equations (12) and (13), the realizations of the macroeconomic variables are determined by the following global equilibrium conditions:

$$y_t = \widehat{E}_t y_{t+1} - \alpha \left[i_t - \widehat{E}_t \pi_{t+1} - r_t \right] + \nu_t, \tag{15}$$

$$\pi_t = \beta \widehat{E}_t \pi_{t+1} + \kappa y_t + u_t, \tag{16}$$

$$i_t^* = \rho i_{t-1}^* + (1-\rho) \left[E_t r_t^* + \pi^* + g_\pi (\overline{\pi}_t - \pi^*) + g_y (y_t - y^*) \right], \tag{17}$$

$$i_t = 0.5\, i_t^* + 0.5 \sqrt{(i_t^*)^2}, \tag{18}$$

where $\overline{\pi}_t = \omega\, \pi_t + (1-\omega)\, \overline{\pi}_{t-1}$.

As a check, I also compute the time-varying weight μ_t using a form of Bayesian model averaging. In this case, μ_t is the average conditional probability that a given sequence of quarterly inflation observations are drawn from one of two populations with known densities.[17] In this model, the Bayes law computation takes the form

$$\mu_t = \frac{\mu_{t-1} \frac{1}{T_w} \sum_{j=1}^{T_w} \mathrm{f}^{\,\mathrm{targ}}(\pi_{t-j})}{\mu_{t-1} \frac{1}{T_w} \sum_{j=1}^{T_w} \mathrm{f}^{\,\mathrm{targ}}(\pi_{t-j}) + (1 - \mu_{t-1}) \frac{1}{T_w} \sum_{j=1}^{T_w} \mathrm{f}^{\,\mathrm{defl}}(\pi_{t-j})}, \tag{19}$$

where $\mathrm{f}^{\,\mathrm{targ}}(\pi_{t-j})$ and $\mathrm{f}^{\,\mathrm{defl}}(\pi_{t-j})$ are the probability density functions for the quarterly inflation distributions under the targeted equilibrium and the deflation equilibrium, respectively. These distributions are assumed known to the agent.[18] For the quantitative analysis, I run a pre-simulation to compute the moments of the quarterly inflation distributions in each of the two local rational expectations equilibria. I impose bounds on the agent's prior such that $0.01 \leq \mu_{t-1} \leq 0.99$ during the simulation so that the agent never rules out the possibility of switching from one equilibrium to the other.

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[16]In this case, $\mu_t = \{1 + \exp[\psi(RMSFE_{t-1}^{\mathrm{targ}} - RMSFE_{t-1}^{\mathrm{defl}})]\}^{-1}$, where $RMSFE_{t-1}^{\mathrm{targ}}$ and $RMSFE_{t-1}^{\mathrm{defl}}$ are the fitness measures associated with the two sets of local linear forecast rules and ψ is the "intensity of choice" parameter. As ψ becomes larger, the resulting sequence for μ_t takes on values approaching either 1 or 0, with intermediate values less likely.

[17]See Anderson (1958), Chapter 6.

[18]Huh and Lansing (2000) employ a similar setup in a policy credibility model where the agent uses observed inflation rates to infer whether the central bank's inflation target has truly shifted to a lower mean value.

3 Parameter values

Table 2 shows the baseline parameter values used in the model simulations. The top group of parameters appear in the private-sector equilibrium conditions (1) and (2). The middle group of parameters appear in the monetary policy rule (9). The bottom group of parameters pertain to the exogenous real interest rate process and the forecast evaluation window for the switching model.

Table 2. Baseline Parameter Values

Parameter	Value	Description/Target
α	0.25	Interest rate coefficient in Euler equation.
β	0.995	Discount factor in Phillips curve.
κ	0.025	Output gap coefficient in Phillips curve.
σ_ν	0.01	Std. dev. of aggregate demand shock.
σ_u	0.02	Std. dev. of cost push shock.
π^*	0.02	Central bank inflation target.
ω	0.459	$\bar{\pi}_t \simeq$ 4-quarter inflation rate.
g_π	1.5	Policy rule response to inflation.
g_y	1.0	Policy rule response to output gap.
ρ	0.80	Interest rate smoothing parameter.
ρ_r	0.8564	Persistence parameter for r_t.
σ_ε	0.0099	Std. dev. of temporary shock to r_t.
σ_η	0.0019	Std. dev. of permanent shock to r_t.
λ	0.0257	Optimal Kalman gain for $E_t r_t^*$.
T_w	8	Window length in qtrs. for forecast evaluation.

The value $\alpha = 0.25$ for the interest rate sensitivity coefficient in equation (1) implies a coefficient of relative risk aversion of $1/\alpha = 4$. This value is consistent with the small empirical sensitivity of consumption to changes in the interest rate, as show by Campbell and Mankiw (1989). The values $\beta = 0.995$ and $\kappa = 0.025$ are identical to those employed by Evans et al. (2015) and are typical of values employed in the literature. Given the other parameter values, the standard deviations of the aggregate demand shock in equation (1) and the cost push shock in equation (2) are chosen so that the standard deviations of the output gap and the 4-quarter inflation rate in the switching model are reasonably close to those observed in U.S. data for the period 1988.Q1 to 2017.Q2.

The inflation target of $\pi^* = 0.02$ is based on the Federal Open Market Committee's (FOMC) stated goal of 2% inflation, as measured by the 4-quarter change in the personal consumption expenditures (PCE) price index. I choose $\omega = 0.459$ to minimize the squared deviation between the 4-quarter PCE inflation rate and the exponentially-weighted moving average of quarterly PCE inflation computed using equation (10) for the period 1961.Q1 to

2017.Q2. When $\omega = 0.459$, the cumulative weight on the first four terms π_t through π_{t-3} in the moving average is 0.915. The monetary policy rule coefficients g_π, g_y and ρ are based on the Taylor (1999) rule, augmented to allow for a realistic amount of inertia in the desired nominal policy rate.

The parameter values that govern the evolution of r_t and r_t^* in equations (3) and (4) are calibrated so that the Kalman filter estimate $E_t r_t^*$ computed from equation (7) approximates the one-sided Laubach-Williams estimate of the natural rate for the period 1988.Q1 to 2017.Q2. The time series for r_t in the data is constructed as the nominal federal funds rate minus expected quarterly inflation computed from a rolling 40-quarter, 4-lag vector autoregression that includes the nominal funds rate, quarterly PCE inflation (annualized), and the CBO output gap. Equation (3) implies $E_t r_{t+1} = \rho_r\, r_t + (1 - \rho_r)\, E_t r_t^*$. I choose $\rho_r = 0.8564$ to minimize the squared forecast error $[r_{t+1} - \rho_r\, r_t - (1 - \rho_r)\, E_t r_t^*]^2$ over the period 1988.Q1 to 2016.Q4, where $E_t r_t^*$ is given by the Laubach-Williams estimate. Given the value of ρ_r, I choose $\lambda = 0.0257$ to minimize the squared deviations between the model-implied estimate $E_t r_t^*$ from equation (7) and the Laubach-Williams estimate. Given these values for ρ_r and λ, I solve for the value $\phi \equiv \sigma_\eta^2/\sigma_\varepsilon^2 = 0.033$ to satisfy the optimal Kalman gain formula (8). Given ϕ, I solve for the value of σ_ε that allows the model-predicted standard deviation of Δr_t to match the corresponding value in the data for the period 1988.Q1 to 2017.Q2. Finally, given ϕ and σ_ε, we have $\sigma_\eta = \sigma_\varepsilon\sqrt{\phi}$.

The window length in quarters for computing the agent's forecast fitness measure from equation (14) is set to $T_w = 8$. Each period, the agent chooses the weight μ_t on the targeted forecast rules so as to minimize the root mean squared forecast errors over the past 2 years. In simulations, this choice produces a ZLB binding frequency in the vicinity of 20%—reasonably close to the frequency observed in U.S. data since 1988. I also examine the sensitivity of the results to higher values of T_w. Higher values of T_w serve to reduce the ZLB binding frequency by reducing the likelihood of switches to the deflation equilibrium.

Figure 3 plots the one-sided Laubach-Williams estimate of the natural rate through 2017.Q2. The series (dashed red line) shows a downward-sloping trend. This pattern is consistent with the declines in global real interest rates observed over the same period (International Monetary Fund 2014, Rachel and Smith 2015). The time series process for the natural rate in the model (dotted green line) provides a good approximation of the Laubach-Williams series from 1988 onwards. Table 3 compares the properties of the U.S. real interest rate to those implied by the model.

Table 3. Properties of Real Interest Rate: Data versus Model

Statistic	U.S. Data 1988.Q1 to 2017.Q2	Model
Std. Dev. Δr_t	0.0103	0.0103
Std. Dev. $\Delta^2 r_t$	0.0151	0.0179
Std. Dev. $r_t - E_t r_t^*$	0.0173	0.0160
Corr. Lag 1 Δr_t	-0.063	-0.070
Corr. Lag 2 Δr_t	-0.211	-0.060

Notes: $\Delta r_t \equiv r_t - r_{t-1}$. $\Delta^2 r_t \equiv \Delta r_t - \Delta r_{t-1}$. The real interest rate r_t in U.S. data is defined as the nominal federal funds rate minus expected quarterly inflation computed from a rolling 40-quarter, 4-lag vector autoregression that includes the nominal funds rate, quarterly PCE inflation, and the CBO output gap. The Kalman filter estimate $E_t r_t^*$ in U.S. data corresponds to the Laubach-Williams one-sided estimate. Model statistics are computed analytically from the laws of motion (3), (4), and (7).

For the baseline simulation, I impose the bounds $-0.0042 \leq r_t^* \leq 0.037$, which corresponds to the range of values for the Laubach-Williams one-sided estimate since 1988. I also consider an alternative simulation that imposes the wider bounds $-0.015 \leq r_t^* \leq 0.037$, where the lower bound of -1.5% is the long-run value of the natural rate of interest computed by Eggertsson, Mehrotra, and Robbins (2017) using a life cycle model calibrated to the U.S. economy in 2015. In a representative agent model, the long-run natural rate influences the mean real risk free rate of return. The mean risk free rate can be negative if the product of the coefficient of relative risk aversion and the variance of consumption growth are sufficiently high, implying a very strong precautionary saving motive.[19]

4 Quantitative analysis

4.1 U.S. data around the ZLB episode

The top left panel of Figure 4 shows that the real federal funds rate has remained mostly below the Laubach-Williams estimate of the natural rate of interest since early 2009, implying persistently negative values for the state variable $r_t - E_t r_t^*$. The bottom left panel shows that the nominal federal funds rate was approximately zero from 2008.Q4 through 2015.Q4. In the same panel, I plot the nominal federal funds predicted by a Taylor-type rule of the form (9) using the parameter values in Table 2 with $E_t r_t^*$ given by Laubach-Williams one-sided

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[19]In a representative agent model, $\log(R_{t+1}^f) = -\log(E_t M_{t+1})$, where R_{t+1}^f is the gross real risk free rate and M_{t+1} is the agent's stochastic discount factor. Assuming *iid* consumption growth and power utility, the mean risk free rate is given by $E[\log(R_{t+1}^f)] = -\log(\beta) + \gamma \bar{x} - \gamma^2 \sigma_x^2/2$, where β is the agent's time discount factor, γ is the coefficient of relative risk aversion, $\bar{x}$ is the mean growth rate of real per capita consumption and σ_x^2 is the corresponding variance. Assuming $\beta \simeq 1$ such that $\log(\beta) \simeq 0$, the condition $\gamma \sigma_x^2 > 2\bar{x}$ implies $E[\log(R_{t+1}^f)] < 0$. For details of the derivation, see Lansing and LeRoy (2014).

estimate, $\overline{\pi}_t$ given by the 4-quarter PCE inflation rate, and y_t given by the CBO output gap. The desired nominal funds rate predicted by the Taylor-type rule is negative starting in 2009.Q1 and remains negative through 2016.Q4.[20]

The top right panel of Figure 4 shows that the 4-quarter PCE inflation rate was briefly negative in 2009 and has remained below the Fed's 2% inflation target since 2012.Q2. The bottom right panel shows that the Great Recession was very severe, pushing the CBO output gap down to -6.3% at the business cycle trough in 2009.Q2. The output gap remains negative at -0.2% in 2017.Q2, eight years after the Great Recession ended.

The various endpoints plotted in Figure 4 are computed using the expressions in Table 1, with r_t^* given by the Laubach-Williams one-sided estimate. Although not shown, the wide confidence intervals surrounding empirical estimates of r_t^* would not rule out values for the true natural rate that lie deeper into negative territory.[21] As r_t^* approaches zero or becomes negative, the "deflation" equilibrium is characterized by zero or low inflation, allowing this equilibrium to provide a better fit of recent U.S. inflation data.

Figure 5 plots various measures of expected inflation in U.S. data. The top right panel shows 5-year and 10-year breakeven inflation rates derived from yields on Treasury Inflation Protected Securities (TIPS). Breakeven inflation dropped sharply in 2008.Q4, coinciding with the start of the ZLB episode. In the top right panel, we see a similar pattern for 1-year and 5-year expected inflation rates derived from zero coupon inflation swap contracts that are traded in the over-the-counter market (Haubrich, Pennacchi, and Ritchken 2012). All of the market-based measures of expected inflation remain below the Fed's 2% inflation target at the end of the data sample in 2017.Q2.

The lower left panel in Figure 5 shows the median 1-year and 10-year expected inflation rates from the Survey of Professional Forecasters (SPF). The 1-year survey measure dropped sharply in 2008.Q4 and has recovered slowly to a level that remains below its pre-recession range. The 10-year survey measure does not exhibit a sharp drop in 2008.Q4, but has since trended downward to a level that is below its pre-recession range. The bottom right panel plots the Federal Reserve Bank of St. Louis' Price Pressures Measure (PPM). A set of common factors extracted from 104 separate data series are used to estimate the probability that the 4-quarter PCE inflation rate over the next year will exceed 2.5% (Jackson, Kliesen, and Owyang

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[20] Augmenting the Taylor-type rule to allow for a response to other variables (such as 4-quarter real GDP growth and an index of macroeconomic uncertainty) can produce a path for the desired nominal funds rate that turns positive somewhat earlier. See Lansing (2017).

[21] According to Kiley (2015), "the co-movement of output, inflation, unemployment, and real interest rates is too weak to yield precise estimates of r*" (p. 2).

2015). The PPM dropped sharply in 2008.Q4 and is currently hovering around a probability of 10%.[22]

Although not plotted in Figure 5, the Federal Reserve Bank of Atlanta's Business Inflation Expectation (BIE) survey shows that while most respondents understand that the Fed's inflation target is 2%, about two-fifths of respondents currently believe that the Fed is more likely to accept an inflation rate below target than to accept an inflation rate above target (Altig, Parker, and Meyer, 2017).

4.2 Switching model simulations

Figure 6 plots some key variables from simulations of the switching model. When the exogenous real interest rate gap $r_t - E_t r_t^*$ is negative for a sustained interval (top panel), the resulting downward pressure on y_t and π_t serves to reduce the recent $RMSFE$ of the deflation forecast rules and increase the recent $RMSFE$ of the targeted forecast rules (middle panel). Around period 1725, the shift in relative forecast performance induces the agent to place a substantially lower weight on the targeted equilibrium forecast rules, causing the deflation equilibrium to become temporarily self-fulfilling (bottom panel). Then around period 1800, the real rate gap once again becomes positive, causing the $RMSFE$ of the deflation forecast rules to exceed the $RMSFE$ of the targeted forecast rules. The agent increases the weight on the targeted forecast rules, causing the targeted equilibrium to be restored.

Qualitatively similar results are obtained if the agent employs Bayes law (19) to compute the likelihood that a string of recent π_t observations is drawn from one equilibrium inflation distribution or the other. Interestingly, it is the agent's subjective belief that the deflation equilibrium is possible that allows it to become a reality. If the agent could somehow commit to employing the forecast rule weight $\mu_t = 1$ for all t, then the economy would always remain in the targeted equilibrium.

Figure 7 plots the distributions of macro variables in each of the three model versions. The macro variables in the deflation equilibrium have distributions with lower means but higher variances than those in the targeted equilibrium. But the significant overlap in the various distributions creates a dilemma for an agent who seeks to determine the likelihood that a string of recent data observations are generated by one equilibrium or the other. Variables in the switching model have means that are somewhat lower and variances that are somewhat

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[22]The TIPS breakeven inflation rates and the PPM are from the the Federal Reserve Bank of St. Louis' FRED data base. Expected inflation rates from swap contracts are from the Federal Reserve Bank of Cleveland. Expected inflation rates from the SPF are from the Federal Reserve Bank of Philadelphia.

higher than those in the targeted equilibrium. Consequently, the central bank in the switching model undershoots its inflation target and the volatilities of the output gap and inflation are both higher relative to the targeted equilibrium.

Hills, Nakata, and Schmidt (2016) show that the risk of encountering the ZLB in the future can shift agents' expectations such that the central bank undershoots its inflation target in the present. Something similar is at work here. When the agent increases the weight on the deflation forecast rules, this can cause realized inflation to undershoot the central bank's target for a sustained interval, even when the ZLB is not binding. The switching model allows for low-frequency swings in the level of inflation that are driven solely by expectational feedback, not by changes in the monetary policy rule.[23]

As mentioned above, the U.S. output gap reached -6.3% at the trough of the Great Recession. This was the most severe economic contraction since 1947 as measured by the peak-to-trough decline in real GDP. The bottom right panel of Figure 7 shows that the likelihood of such an event in the targeted equilibrium is essentially zero. In contrast, a Great Recession-type episode is plausible, albeit rare, in the switching model.

Table 4 provides a quantitative comparison between the U.S. data and the results of model simulations. Overall, the statistics generated by the switching model compare favorably to those in U.S. data since 1988. For example, the switching model predicts a ZLB binding frequency of 18.4% versus 24.6% in the data. However, the mean 4-quarter inflation rate in the switching model is only 0.88% versus 2.16% in the data. This particular model prediction is more in line with data from Japan than the United States. But going forward, a continued undershooting of the Fed's 2% inflation target (as has been the case since mid-2012) would push down the mean 4-quarter inflation rate in the data, bringing it closer to the switching model prediction.

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[23]Lansing (2009) achieves a similar result in a model where the representative agent's forecast rule for quarterly inflation is based on a perceived law of motion that follows a Stock and Watson (2007) type time series process.

Table 4. Unconditional Moments: Data versus Model

Statistic	U.S. Data 1988.Q1-2017.Q2	Model Simulations Targeted	Deflation	Switching
Mean y_t	−1.44%	0.40%	−0.43%	0.48%
Std. Dev.	1.75%	1.53%	3.58%	2.20%
Corr. Lag 1	0.95	0.47	0.78	0.67
Mean $\pi_{4,t}$	2.16%	1.98%	−1.69%	0.88%
Std. Dev.	1.09%	0.98%	1.66%	1.58%
Corr. Lag 1	0.89	0.74	0.91	0.91
Mean i_t^*	2.83%	3.67%	−2.68%	2.09%
Std. Dev.	3.42%	1.73%	3.41%	2.86%
Corr. Lag 1	0.99	0.98	0.98	0.99
% periods $i_t = 0$	24.6%	1.53%	77.3%	18.4%
Mean ZLB duration	29 qtrs.	4.1 qtrs.	30.0 qtrs.	11.4 qtrs.
Max. ZLB duration	29 qtrs.	33 qtrs.	295 qtrs.	139 qtrs.

Notes: The ZLB episode in U.S. data is from 2008.Q4 through 2015.Q4. Model results are computed from a 300,000 period simulation. $\pi_{4,t} \equiv [\Pi_{j=0}^{3}(1 + \pi_{t-j})]^{0.25} - 1$.

Using data from all advanced economies since 1950, Dordal-i-Carrera et al. (2016) estimate an average ZLB binding frequency of 7.5% and an average duration for ZLB episodes of 14 quarters. Excluding the high inflation period from 1968 to 1984 serves to raise the average ZLB binding frequency and the average ZLB duration to 10% and 18 quarters, respectively. For the period of consistent U.S. monetary policy since 1988, the single ZLB episode lasted 29 quarters.

Figure 8 plots the distribution of ZLB durations in each model version. Unlike the targeted equilibrium, the switching model can produce infrequent and long-lived ZLB episodes in response to small, normally distributed shocks. The average ZLB duration in the switching model is 11.4 quarters, with a maximum duration of 139 quarters (Table 4). From Figure 8, we see that a 29 quarter ZLB episode is an extremely rare event in the targeted equilibrium but can occur with about a 5% frequency in the switching model. To account for infrequent and long-lived ZLB episodes in the targeted equilibrium, Dordal-i-Carreras, et al. (2016) develop a model with large, infrequent, and long-lived shocks.[24]

When $\omega = 0.459$, the exponentially-weighted moving average of quarterly inflation $\overline{\pi}_t$ computed from equation (11) provides a very good approximation of the 4-quarter inflation rate. Although not shown in Table 4, the mean, standard deviation, and first-order autocorrelation of $\overline{\pi}_t$ in the switching model are 0.89%, 1.63%, and 0.81, respectively. These values are close to the corresponding statistics for $\pi_{4,t}$ of 0.88%, 1.58%, and 0.91.

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[24]In a New Keynesian model with physical capital, Dennis (2016) shows that the introduction of capital adjustment costs can help to generate infrequent and long-lived ZLB episodes in the targeted equilibrium.

The mean weight on the targeted forecast rules in the switching model is 0.68 with a standard deviation of 0.30. Larger values for the window length T_w that is used to compute the forecast fitness measure from equation (14) serve to reduce the frequency of regime switches and thereby raise the mean 4-quarter inflation rate. For example, when T_w is increased to 16 quarters, the mean value of μ_t is higher at 0.78 and the standard deviation is lower at 0.23. With $T_w = 16$, the ZLB binding frequency in the switching model drops to 10.1% and the average ZLB duration is lower at 8.2 quarters. The mean value of $\pi_{4,t}$ increases to 1.24% from 0.88%.

Figure 9 plots simulations from each of the three model versions: targeted, deflation, and switching. All three versions employ the same sequence of stochastic shocks. When the weight on the targeted forecast rules starts dropping towards zero around period 1725, the switching model generates a negative desired nominal policy rate, a binding ZLB, brief deflation followed by below-target inflation, and a highly negative output gap, reminiscent of the U.S. Great Recession and its aftermath (Figure 4). The severity of the recession in the switching model is due to the larger response coefficient on the state variable $r_t - E_t r_t^*$ in the deflation equilibrium decision rule for y_t. Specifically, the response coefficient in the deflation equilibrium is 2.33 versus 0.82 in the targeted equilibrium (Appendices B and C). The deflation equilibrium response coefficient receives more weight as $\mu_t \to 0$, causing the effects of an adverse real rate shock to be transmitted more forcefully to the output gap.

Evans, Honkapohja, and Mitra (2016) argue that the deflation equilibrium does not provide a convincing explanation of the sluggish output recovery following the Great Recession because the steady state level of real activity in the deflation equilibrium is not much below the steady state level of real activity in the targeted equilibrium. However, their analysis does not take into account that the real rate gap $r_t - E_t r_t^*$ in U.S. data has remained significantly negative since the recession ended, as can be seen in the top left panel of Figure 4. A negative real rate gap puts stronger downward pressure on y_t in the deflation equilibrium, thus helping to explain the sluggish output recovery in U.S. data.

Table 5 summarizes the properties of the agent's forecast errors in each of the three model versions. The forecast error is given by $err_{t+1}^x = x_{t+1} - F_t x_{t+1}$ for $x_{t+1} \in \{y_{t+1}, \ \pi_{t+1}\}$, where $F_t x_{t+1}$ is the value predicted by the local linear forecast rule or, in the case of the switching model, the weighted-average forecast rule, (12) or (13). As noted earlier in Section 2.3, the agent's use of linear forecast rules in a nonlinear environment that is subject to an occasionally binding ZLB would be expected to introduce errors, particularly in the more-volatile deflation

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equilibrium. Nevertheless, Table 5 shows that the agent's forecast errors in all three model versions are close to white noise, giving no clear indication to the agent that the forecast rules are misspecified.

Table 5. Properties of Forecast Errors

Statistic	Model Simulations		
	Targeted	Deflation	Switching
$Corr(err^y_{t+1}, err^y_t)$	0.001	−0.008	0.021
$Corr(err^\pi_{t+1}, err^\pi_t)$	0.003	0.002	0.059
$E\left(err^y_{t+1}\right)$	−0.001%	−0.099%	0.001%
$E\left(err^\pi_{t+1}\right)$	−0.004%	−0.006%	0.004%
$\sqrt{E[(err^y_{t+1})^2]}$	1.211%	2.126%	1.496%
$\sqrt{E[\left(err^\pi_{t+1}\right)^2]}$	1.972%	2.012%	1.995%

Notes: Model results are computed from a 300,000 period simulation.

Recall that the long-run endpoint of π_t in the deflation equilibrium is $-r^*_t$. Allowing more negative values of r^*_t in the simulation will therefore serve to increase the mean inflation rate in both the deflation equilibrium and the switching model. In Figure 6, for example, the real rate gap $r_t - E_t r^*_t$ becomes positive around period 1800 because $E_t r^*_t$ becomes negative. At the same time, Figure 9 shows that the 4-quarter inflation rate in the deflation equilibrium becomes positive. The wide confidence intervals around empirical estimates of the U.S. natural rate would not rule out true values that are more negative. To explore this idea further, the simulations are repeated while imposing the wider bounds $-0.015 \leq r^*_t \leq 0.037$, where the lower bound of -1.5% is the long-run value of the natural rate of interest computed by Eggertsson, Mehrotra, and Robbins (2017) using a life cycle model calibrated to the U.S. economy in 2015.

Table 6 compares the results of the original switching model simulation to the alternative simulation with $-0.015 \leq r^*_t \leq 0.037$. The mean 4-quarter inflation rate in the alternative simulation increases to 1.04% from 0.88% in the original simulation. Moreover, the ZLB binding frequency increases to 21.6%, which is closer to the U.S. data value of 24.6% shown in Table 4.

Table 6. Effect of Natural Rate Range in Switching Model

Statistic	$-0.0042 \leq r_t^* \leq 0.037$	$-0.015 \leq r_t^* \leq 0.037$
Mean y_t	0.48%	0.43%
Std. Dev.	2.20%	2.24%
Corr. Lag 1	0.67	0.67
Mean $\pi_{4,t}$	0.88%	1.04%
Std. Dev.	1.58%	1.52%
Corr. Lag 1	0.91	0.90
Mean i_t^*	2.09%	1.75%
Std. Dev.	2.86%	2.84%
Corr. Lag 1	0.99	0.98
% periods $i_t = 0$	18.4%	21.6%
Mean ZLB duration	11.4 qtrs.	11.2 qtrs.
Max. ZLB duration	139 qtrs.	140 qtrs.

Notes: Model results are computed from a 300,000 period simulation $\pi_{4,t} \equiv [\Pi_{j=0}^3 (1 + \pi_{t-j})]^{0.25} - 1$.

4.3 Replicating U.S. data with the switching model

Given the U.S. data counterparts for the model variables i_t, i_t^*, r_t, $E_t r_t^*$, y_t, π_t, and $\overline{\pi}_t \simeq \pi_{4,t}$ (Figure 4), we can use the calibrated switching model to solve for the implied time series of the two stochastic shocks ν_t and u_t using equations (15) and (16). For this computation, the subjective forecasts $\widehat{E}_t y_{t+1}$ and $\widehat{E}_t \pi_{t+1}$ are constructed according to equations (12) and (13) with U.S. data inserted for the state variables that appear in the two sets of linear forecast rules associated with the two local equilibria. The variable i_t is the nominal federal funds rate, i_t^* is the desired nominal funds rate computed using the policy rule (9), $r_t - E_t r_t^*$ is the difference between the real federal funds rate and the Laubach-Williams estimate of the natural rate of interest, y_t is the CBO output gap, and π_t is quarterly PCE inflation. The value of the forecast weight μ_t is computed each period so as to minimize the $RMSFE$ from equation (14), where $T_w = 8$ quarters. The results of the data replication exercise are plotted in Figure 10 for the period 1988.Q1 to 2017.Q2. This figure can be compared to the model simulation results presented earlier in Figure 6.

The top left panel shows the model-implied time series for the shocks ν_t and u_t. Both shocks become strongly negative at the start of the ZLB episode in 2008.Q4. These adverse shock sequences allow the model to exactly replicate the sharp drops in the CBO output gap and quarterly PCE inflation shown earlier in Figure 4. The standard deviations of ν_t and u_t implied by the data replication exercise are 0.023 and 0.016, respectively.

The top right panel of Figure 10 compares the $RMSFE$ of the deflation forecast rules to the $RMSFE$ of the targeted forecast rules. The performance gap between the two sets

of forecast rules starts to narrow considerably after 2008.Q4. Indeed, the deflation forecast rules start outperforming the targeted forecast rules in 2010.Q3. As a result, the weight on the targeted equilibrium forecast rules starts to decline, eventually reaching a minimum value of 0.16 in 2011.Q3 (bottom right panel). Recall that the average value of μ_t in the switching model simulations was 0.68 with a standard deviation of 0.30. After 2011.Q3, the weight on the targeted equilibrium forecast rules starts to increase as U.S. economy recovers from the Great Recession. But even towards the end of the data sample, the value of μ_t remains well below 1.0, helping the switching model to account for the persistent undershooting of the Fed's 2% inflation target since mid-2012. In contrast, the targeted equilibrium would require a long sequence of negative cost push shocks to account for this feature of the data.

The bottom left panel of Figure 10 compares the 1-year expected inflation rate from U.S. inflation swaps (shown earlier in Figure 5) to the path of $\widehat{E}_t\,\pi_{t+1}$ from the switching model. The correlation coefficient between the two series is 0.6. While the model-implied drop in expected inflation is somewhat more pronounced and more persistent than in the data, both series remain below the Fed's 2% inflation target at the end of the data sample in 2017.Q2.

4.4   Effect of raising the inflation target

In a press conference held on June 14, 2017, Fed Chair Janet Yellen stated that determining the appropriate level of the inflation target "is one of the most important questions facing monetary policy around the world in the future." Numerous authors make the case for a higher inflation target using frameworks that ignore the deflation equilibrium.[25] This methodology may understate the benefits of a higher inflation target because the analysis does not take into account the possibility that a higher target could help prevent switching to the volatile deflation equilibrium where recessions are more severe. Aruoba and Schorfheide (2016) consider the welfare implications of a 4% inflation target in a framework that does consider the possibility of switching to the deflation equilibrium via an exogenous sunspot shock. They conclude (p. 395) that "the case for a higher inflation target is not particularly strong." It's worth noting, however, that the probability of switching to the deflation equilibrium in the Aruoba-Schorfheide model is invariant to changes in the inflation target. In contrast, the framework developed here has the potential to reduce the probability of switching to the deflation equilibrium. This potential benefit should be considered as part of any cost-benefit analysis of a higher inflation target.

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[25]See, for example, Blanchard, Dell'Ariccia, and Mauro (2010), Ball and Mazumder (2011), and Ball (2013).

Table 7 shows the effects of raising the central bank's inflation target. As π^* increases, the ZLB binding frequency declines as does the mean duration of ZLB episodes. Higher values of π^* serve to reduce the volatility of y_t because the volatile deflation equilibrium is now avoided more often. But at the same time, higher values of π^* serve to *increase* the volatility of $\pi_{4,t}$. This is because higher values of π^* widen the spread between the two inflation distributions implied by the the two local equilibria. This causes the mean inflation rate to shift by a larger amount when a regime switch inevitably does occur.

From Table 7 we see that an increase in π^* can reduce, but not eliminate, the endogenous switches to the deflation equilibrium. Even with an inflation target of 4%, the ZLB binding frequency remains relatively high at 9.9%, the average duration of a ZLB episode is 11.2 quarters, and the maximum duration of a ZLB episode is 132 quarters, or 33 years. Once the possibility of switching to the deflation equilibrium is taken into account, raising the inflation target is a less effective solution for avoiding ZLB episodes.

Table 7. Effect of Raising the Inflation Target in Switching Model

Statistic	$\pi^* = 0.02$	$\pi^* = 0.03$	$\pi^* = 0.035$	$\pi^* = 0.04$	$\pi^* = 0.05$
% periods $i_t = 0$	18.4%	13.9%	11.8%	9.9%	6.6%
Mean ZLB duration	11.4 qtrs.	11.5 qtrs.	11.4 qtrs.	11.2 qtrs.	10.9 qtrs.
Max. ZLB duration	139 qtrs.	139 qtrs.	132 qtrs.	132 qtrs.	134 qtrs.
Std. Dev. y_t	2.20%	2.13%	2.10%	2.06%	1.99%
Std. Dev. $\pi_{4,t}$	1.58%	1.70%	1.75%	1.80%	1.86%
Std. Dev. i_t^*	2.86%	2.98%	3.01%	3.04%	3.02%
Loss value, $\theta = 1$	2.93%	2.78%	2.79%	2.87%	3.21%
Loss value, $\theta = 0.25$	2.23%	2.04%	2.06%	2.16%	2.57%

Note: Model results computed from a 300,000 period simulation. $\pi_{4,t} \equiv [\Pi_{j=0}^{3}(1 + \pi_{t-j})]^{0.25} - 1$.

Coibion, Gorodnichenko, and Wieland (2012) calibrate their model to deliver a ZLB binding frequency equal to that observed in U.S. data going back to the year 1950. From the start of 1950 to 2017.Q2, the ZLB was binding in 29 out of 270 quarters, or 10.7% of the time. The average CPI inflation rate in U.S. data since 1950 is around 4%. Table 7 shows that the switching model with $\pi^* = 0.04$ delivers a ZLB binding frequency of 9.9%—close to the U.S. value of 10.7% going back to 1950. Taking into account the micro-founded welfare costs of positive and variable inflation, Coibion, Gorodnichenko, and Wieland (2012) compute an optimal inflation rate for their model which is relatively low, less than 2% per year. Their analysis is extended by Dordal-i-Carrera et al. (2016), who modify the shock process in the model to match the now-higher ZLB binding frequency implied by additional years of data

for the U.S. and other advanced economies. The optimal inflation rate for the modified model lies in the range of 2.5% to 4%.

Kiley and Roberts (2017) perform stochastic simulations using the DSGE model of Lindé, Smets, and Wouters (2016) which is estimated over the period 1965.Q1 to 2014.Q2. They consider constant values of the natural rate of interest as low as 1% and draws shocks from the estimated distributions of the model. When monetary policy follows a simple Taylor (1999) rule with no inertia, they find that the ZLB binding frequency can be as high as 32.6% with an mean ZLB duration of 12 quarters (p. 22). The very high ZLB binding frequency obtains even though the model solution considers only the targeted equilibrium. In contrast, the simulations here deliver a baseline ZLB binding frequency in the switching model of 18.4%, despite allowing for a natural rate of interest as low as -0.42% and further allowing for the possibility of switches to the deflation equilibrium. The much higher ZLB binding frequency obtained by Kiley and Roberts (2017) appears to be partly due to the shock distributions which are based on the more-volatile U.S. data sample going back to 1965. Here, in contrast, the shock distributions are based the more-recent sample period of consistent monetary policy going back to 1988.

In addition to their use of a more-volatile shock distribution, Kiley and Roberts (2017) employ a Taylor-type rule with no interest rate smoothing. This feature of their model would also appear to contribute to a higher ZLB binding frequency. Table 8 shows the effect of the interest rate smoothing parameter ρ in the switching model. The baseline case with $\rho = 0.8$ is compared to the no smoothing case with $\rho = 0$. For all values of π^*, setting $\rho = 0$ results in a higher ZLB binding frequency, but the episodes exhibit a shorter duration on average. From a ZLB perspective, there appears to be no clear advantage to reducing the degree of interest rate smoothing in the monetary policy rule.

Table 8. Effect of Interest Rate Smoothing in Switching Model

Statistic	$\pi^* = 0.02$	$\pi^* = 0.03$	$\pi^* = 0.04$	$\pi^* = 0.05$
$\rho = 0.8$				
% periods $i_t = 0$	18.4%	13.9%	9.9%	6.6%
Mean ZLB duration	11.4 qtrs.	11.5 qtrs.	11.2 qtrs.	10.9 qtrs.
Max. ZLB duration	139 qtrs.	139 qtrs.	132 qtrs.	134 qtrs.
$\rho = 0$				
% periods $i_t = 0$	25.7%	21.6%	17.6%	13.7%
Mean ZLB duration	3.9 qtrs.	4.0 qtrs.	4.0 qtrs.	4.0 qtrs.
Max. ZLB duration	102 qtrs.	108 qtrs.	113 qtrs.	114 qtrs.

Note: Model results computed from a 300,000 period simulation.

Following Kiley and Roberts (2017), I use a simple loss function approach to quantify the

various trade-offs that are involved in raising the inflation target. The trade-offs here include: (1) reducing the likelihood of endogenous switches to the volatile deflation equilibrium and thereby lowering the ZLB binding frequency, (2) inducing shifts in the volatilities of inflation and the output gap, and (3) introducing economic distortions that come from a higher average inflation. The loss function takes the form

$$Loss \; = \; E\left\{[\pi_{4,t} - 0.02]^2 \; + \; \theta\left[y_t - 0.02\left(1 - \beta\right)/\kappa\right]^2\right\}, \tag{20}$$

where 0.02 and $0.02\left(1 - \beta\right)/\kappa$ are the long-run endpoints in the targeted equilibrium when $\pi^* = 0.02$, as shown in Table 1. The presumption is that the central bank in the baseline calibration with $\pi^* = 0.02$ has chosen to target the "optimal" levels of $\pi_{4,t}$ and y_t. Hence, any shift away from the original target values when adopting $\pi^* > 0.02$ would introduce economic distortions that are taken into account by the loss function. Also following Kiley and Roberts (2017), I consider two values for the weight θ on the second term that captures the loss from output gap deviations. The bottom rows of Table 7 show that the simple loss function approach would favor a modest increase in the central bank's inflation target. Specifically, the loss function is minimized at $\pi^* = 0.03$ when $\theta = 1$ or $\theta = 0.25$.

4.5 Adaptive learning in a simplified model

Up to this point, I have assumed that the representative agent has full knowledge of the forecast rules associated with each of the two local rational expectations equilibria. While the full-knowledge assumption is standard in models with a unique equilibrium, the computational burden on the agent is considerably higher in the present context. To relax the full-knowledge assumption, I introduce an adaptive learning algorithm in a simplified version of the model. Starting from the original model, the simplified model imposes $\rho = 0$, $\omega = 1$, and $\sigma_\eta = 0$. These settings eliminate i^*_{t-1} and $\overline{\pi}_{t-1}$ as state variables and cause the natural rate of interest r^* to be constant. I set $r^* = 0.0184$, corresponding to the average value of the Laubach-Williams one-sided estimate since 1988. Other parameter values are identical to those in the original model. The theoretical decision rules for each of the two local rational expectations equilibria are shown in Appendix D.

The learning agent employs decision rules that are estimated in real-time using model-generated data. The resulting dynamics are self-referential because the agent's current decision rules influence the data generated by the model and thereby influence the estimated decision rules in subsequent periods. An equilibrium is said to be "learnable" if the estimated decision

rules converge to the theoretical decision rules implied by the rational expectations solution.[26]

I consider two versions of the learning algorithm. In the first version, the agent estimates a set of correctly specified decision rules that take the form:

$$y_t = c_{0,t} + c_{1,t}(r_t - r^*) + c_{2,t}\nu_t + c_{3,t}u_t, \tag{21}$$

$$\pi_t = d_{0,t} + d_{1,t}(r_t - r^*) + d_{2,t}\nu_t + d_{3,t}u_t, \tag{22}$$

where the coefficients $c_{i,t}$ and $d_{i,t}$ for $i = 0, 1, 2, 3$ are estimated each period using an ordinary least squares regression applied to a rolling window of the most recent 32 quarters (8 years) of model-generated data. For the first 32 quarters of the simulation, the coefficient values are set halfway between the two values implied by each of the two local rational expectations equilibria. I assume that the agent can observe r_t, ν_t, and u_t and knows the laws of motion for all state variables. In the second version, I assume that the agent estimates a set of misspecified decision rules that omit the terms involving the two white noise shocks ν_t and u_t. In both versions, the agent uses the previous period's estimated decision rules to construct the subjective forecasts $\widehat{E}_t y_{t+1} = c_{0,t-1} + c_{1,t-1}\rho_r(r_t - r^*)$ and $\widehat{E}_t \pi_{t+1} = d_{0,t-1} + d_{1,t-1}\rho_r(r_t - r^*)$, where the values of ρ_r and r^* are assumed known. Use of the previous period's decision rules to construct the subjective forecasts avoids simultaneity in the realized and forecasted values of the output gap and inflation.

The learning agent's subjective forecasts are substituted into the global equilibrium conditions (1) and (2). I allow for an occasionally binding ZLB by making the substitution $i_t = 0.5\,i_t^* + 0.5\sqrt{(i_t^*)^2}$ in the equilibrium condition (1). When combined with a simplified policy rule that imposes $\rho = 0$ and $\omega = 1$, this procedure yields a system of three equations that can be solved each period to obtain the three realizations y_t, π_t, and i_t^*. Given these realizations, the decision rule coefficients are then updated.[27]

Figure 11 plots the estimated coefficients $c_{0,t}$, $c_{1,t}$, $d_{0,t}$, and $d_{1,t}$ from simulations of the learning algorithm. When the agent estimates correctly specified decision rules (black lines), the coefficients quickly converge to the vicinity of the targeted equilibrium values and remain there. This result shows that the targeted equilibrium is learnable. In contrast, when the

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[26] Christiano, Eichenbaum, and Johanssen (2016) consider a more specialized learning experiment in which the ZLB is already binding due to a fundamental shock. Given this condition, they investigate whether various learning algorithms converge to either a high inflation or a low inflation equilibrium.

[27] To ensure stability of the learning algorithm, I impose the following projection facility: If the estimated values of $c_{0,t}$ or $d_{0,t}$ fall outside the range of two times the targeted equilibrium value on the upside or two times the deflation equilibrium value on the downside, then the agent does not update the affected decision rule for that period only. Intuitively, the agent is assumed to apply a common sense economic restriction on the magnitude of the intercept coefficients in the estimated decision rules.

agent estimates misspecified decision rules (blue lines), the coefficients exhibit low frequency oscillations that occasionally approach, reach, or go beyond the deflation equilibrium values. The agent's failure to control for the shocks ν_t and u_t in the regressions causes the estimated values of the other coefficients to fluctuate, preventing the algorithm from converging to either of the two equilibria. The low frequency oscillations in the decision rule coefficients shown in Figure 11 induce movements in the macroeconomic variables that are qualitatively similar to those observed in the switching model with full-knowledge. Hence, the basic nature of the switching model dynamics can be preserved in a setting that departs from the full-knowledge assumption.[28]

It does not seem unreasonable to think that real world economic agents might fail to control for some relevant but difficult-to-observe state variables when constructing their conditional forecasts. Along these lines, a recent paper by Busetti, et al. (2017) shows that learning agents' use of simple autogressive forecast rules (that omit some relevant state variables) can lead to persistent undershooting of inflation from the central bank's inflation target. Similarly, Bullard and Cho (2005) show that "certain types of seemingly minor misspecifications along with agent learning might combine to change the global dynamics of the economy in unexpected ways." (p. 1842).

5   Conclusion

Standard New Keynesian models subject to a ZLB exhibit two long-run endpoints (steady states) associated with two local rational expectations equilibria. Most studies employing New Keynesian models focus on the targeted equilibrium and ignore the possibility of switching to the deflation equilibrium. But there appears to be no clear theoretical or empirical reason why the deflation equilibrium should be ruled out. Indeed, Bullard (2010) concludes that by "promising to remain at zero for a long time," central banks may inadvertently coordinate private-sector expectations to select the deflation equilibrium.

I examine a version of the New Keynesian model with a time-varying natural rate of interest and endogenous forecast rule switching based on past performance. The model can produce severe recessions when the real interest rate gap is persistently negative, causing the representative agent to place a significant weight on the forecast rules associated with the deflation equilibrium. Escape from the deflation equilibrium occurs endogenously when the

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[28] Arifovic, Schmitt-Grohé, and Uribe (2017) demonstrate that the deflation equilibrium can be stable under a form of social learning that is governed by a genetic algorithm. A key feature for their result is the initial history of the economy which determines the relative performance among a set of heterogenous forecast rules.

real interest rate gap eventually starts rising. But even in normal times, a non-trivial weight on the deflation forecast rules may cause the central bank to undershoot its inflation target and raise the volatilities of macro variables relative to the targeted equilibrium.

A simple loss function approach favors a modest increase in the central bank's inflation target to around 3%. But even with an inflation target of 4%, the ZLB binding frequency remains relatively high at 9.9% and the average duration of a ZLB episode is 11.2 quarters. These results suggest that concerns about getting stuck in a deflation trap for an extended period are legitimate if one believes that the standard New Keynesian model provides a good description of advanced economies with inflation targeting central banks.

A Appendix: Kalman filter estimate of r-star

Straightforward computations using the laws of motion (3) and (4) yield the following uncon-
ditional moments for the first difference of the exogenous real interest rate:

$$Var\left(\Delta r_t\right) = \sigma_\varepsilon^2 \left[\frac{(1-\rho_r)\,\phi + 2}{1+\rho_r}\right], \tag{A.1}$$

$$Var\left(\Delta^2 r_t\right) = \sigma_\varepsilon^2 \left[\frac{2\,(1-\rho_r)^2\,\phi + 6}{1+\rho_r}\right], \tag{A.2}$$

$$Corr\left(\Delta r_t,\ \Delta r_{t-1-k}\right) = \frac{(\rho_r)^k\,(\rho_r\,\phi - 1)}{\phi + 2/\left(1-\rho_r\right)}, \tag{A.3}$$

where $\Delta r_t \equiv r_t - r_{t-1}$, $\Delta^2 r_t \equiv \Delta r_t - \Delta r_{t-1}$, and $\phi \equiv \sigma_\eta^2/\sigma_\varepsilon^2$. Equation (A.3) shows that
the the values of ρ_r and ϕ can be inferred from the autocorrelation structure of Δr_t, which is
observable. Given ρ_r and ϕ, the value of σ_ε can be inferred from (A.1) using the observable
value of $Var\left(\Delta r_t\right)$. Given ϕ and σ_ε, we have $\sigma_\eta = \sigma_\varepsilon\sqrt{\phi}$.

Solving equation (3) for r_t^* yields:

$$r_t^* = \underbrace{\frac{r_t - \rho_r r_{t-1}}{1 - \rho_r}}_{\text{Signal}} - \underbrace{\varepsilon_t}_{\text{Noise}}, \tag{A.4}$$

where the first term represents the signal and the second term represents the noise. Equation
(4) shows that the Kalman filter estimate of r_t^*, denoted by $E_t r_t^*$, is a weighted average of the
signal and the previous period's estimate $E_{t-1}\, r_{t-1}^*$, where the weight assigned to the signal is
the Kalman gain parameter λ.

The one step ahead forecast error for r_{t+1} is given by

$$
\begin{aligned}
err_{t+1} &= r_{t+1} - E_t r_{t+1}, \\
&= r_{t+1} - \left[\rho_r r_t + (1 - \rho_r)\, E_t r_t^*\right], \\
&= \varepsilon_{t+1} + (1 - \rho_r)\,\eta_{t+1} + (1 - \rho_r)\,(r_t^* - E_t r_t^*), \tag{A.5}
\end{aligned}
$$

where the last term in (A.5) represents the estimation error for r_t^*. The optimal value of λ
minimizes the mean squared forecast error, as given by

$$E\left(err_{t+1}^2\right) = \sigma_\varepsilon^2 \left[1 + (1-\rho_r)^2\,\phi\right] + (1-\rho_r)^2\,Var\left(r_t^* - E_t r_t^*\right). \tag{A.6}$$

where $Var\left(r_t^* - E_t r_t^*\right)$ is the unconditional variance of the estimation error.

The law of motion for the estimation error follows directly from equation (7) and can be
written as

$$r_t^* - E_t r_t^* = \lambda\left[\frac{z_t - \rho_r\,z_{t-1}}{1 - \rho_r}\right] + (1 - \lambda)\,(r_{t-1}^* - E_{t-1}\,r_{t-1}^*) - \left[\frac{1 - \lambda - \rho_r}{1 - \rho_r}\right]\eta_t, \tag{A.7}$$

where $z_t \equiv r_t - r_t^*$ is the actual real rate gap. The law of motion for z_t follows directly from equations (3) and (4) and can be written as

$$z_t = \rho_r \, z_{t-1} - \rho_r \, \eta_t + \varepsilon_t. \tag{A.8}$$

Starting from equations (A.7) and (A.8), we can compute the following expression for the unconditional variance of the estimation error

$$Var\left(r_t^* - E_t r_t^*\right) = \sigma_\varepsilon^2 \left\{ \frac{\lambda \left(\rho_r^2 \phi + 1\right) + (1 - \lambda - \rho_r)\left[(1 - \lambda)(1 - \rho_r)/\lambda + \rho_r\right]\phi}{(2 - \lambda)(1 - \rho_r)^2} \right\}, \tag{A.9}$$

which can be substituted into equation (A.6) to obtain a complicated expression for $E\left(err_{t+1}^2\right)$ in terms of λ. From this expression, we can compute the gradient

$$\frac{\partial E\left(err_{t+1}^2\right)}{\partial \lambda} = \frac{2\left[\lambda^2 - (1 - \lambda)(1 - \rho_r)^2 \phi\right]}{(2 - \lambda)^2 \lambda^2}. \tag{A.10}$$

Setting the gradient equal to zero yields a quadratic equation in λ. The root that minimizes $E\left(err_{t+1}^2\right)$ is given by equation (8).

B   Appendix: Targeted equilibrium

To solve for the local linear forecast rules associated with the targeted equilibrium, I assume that $i_t^* = i_t > 0$ for all t, i.e., the ZLB is never binding. Starting from equation (9) we have:

$$i_t^* = \rho i_{t-1}^* + (1 - \rho)\left[E_t r_t^* + \pi^* + g_\pi \, \omega\left(\pi_t - \pi^*\right) + g_\pi\left(1 - \omega\right)\left(\overline{\pi}_{t-1} - \pi^*\right) + g_y\left(y_t - y^*\right)\right] \tag{B.1}$$

where I have used equation (10) to eliminate $\overline{\pi}_t$.

Equation (B.1) together with the Euler equation (1) and the Phillips curve (2) form a linear system of three equations in the three unknown decision rules for y_t, π_t, and i_t^*. The state variables are: r_t, $E_t r_t^*$, $\overline{\pi}_{t-1}$, i_{t-1}^*, ν_t, and u_t. Standard techniques yield a set of linear decision rules of the form

$$\begin{bmatrix} y_t - \pi^*\left(1 - \beta\right)/\kappa \\ \pi_t - \pi^* \\ i_t^* - \left(E_t r_t^* + \pi^*\right) \end{bmatrix} = \mathbf{A} \begin{bmatrix} r_t - E_t r_t^* \\ \overline{\pi}_{t-1} - \pi^* \\ i_{t-1}^* - E_t r_t^* - \pi^* \\ \nu_t \\ u_t \end{bmatrix}, \tag{B.2}$$

where $\mathbf{A}$ is a 3×5 matrix of decision rule coefficients. For the parameter values shown in Table 2, the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 0.819 & -0.209 & -0.551 & 0.858 & -0.178 \\ 0.084 & -0.021 & -0.042 & 0.014 & 0.982 \\ 0.175 & 0.117 & 0.684 & 0.174 & 0.100 \end{bmatrix}. \tag{B.3}$$

Iterating the linear decision rules in (B.2) ahead one period and then taking the conditional expectation of both sides yields the following set of linear forecast rules associated with the targeted equilibrium:

$$E_t y_{t+1} = y^* + \mathbf{A}_{11}\rho_r\left(r_t - E_t r_t^*\right) + \mathbf{A}_{12}\left(\overline{\pi}_t - \pi^*\right) + \mathbf{A}_{13}\left(i_t^* - E_t r_t^* - \pi^*\right), \quad \text{(B.4)}$$

$$E_t \pi_{t+1} = \pi^* + \mathbf{A}_{21}\rho_r\left(r_t - E_t r_t^*\right) + \mathbf{A}_{22}\left(\overline{\pi}_t - \pi^*\right) + \mathbf{A}_{23}\left(i_t^* - E_t r_t^* - \pi^*\right), \quad \text{(B.5)}$$

where $\mathbf{A}_{ij}$ represents the corresponding element of the matrix $\mathbf{A}$ and I have substituted in $E_t\left(r_{t+1} - E_{t+1}r_{t+1}^*\right) = \rho_r\left(r_t - E_t r_t^*\right)$. Notice that the forecast rules depend on the realization of π_t because $\overline{\pi}_t$ depends on π_t via equation (10). Also, the forecast rules depend on the realization of i_t^* due to the interest rate smoothing term in (B.1). Hence, the model allows for simultaneity between the forecasted and realized values of π_t and i_t^*. Neither the agent or the central bank are required to forecast i_{t+1}^*.

The linear forecast rules (B.4) and (B.5) are derived under the assumption that the ZLB is never binding. However, in the stochastic simulation of the targeted equilibrium, I allow for an occasionally binding ZLB. When simulating the model, I substitute the local linear forecast rules given by (B.4) and (B.5) into the global equilibrium conditions (1) and (2). I allow for an occasionally binding ZLB by making the substitution $i_t = 0.5\,i_t^* + 0.5\,\sqrt{(i_t^*)^2}$ in the equilibrium condition (1). Together with the monetary policy rule (B.1), this procedure yields a system of three equations that are solved each period to obtain the three realizations y_t, π_t, and i_t^*.

C Appendix: Deflation equilibrium

To solve for the local linear forecast rules associated with the deflation equilibrium, I assume $i_t^* \leq 0$ such that $i_t = 0$ for all t, i.e., the ZLB is always binding. Equation (B.1) applies unchanged to the deflation equilibrium, as does the Phillips curve (2). However, due to the binding ZLB, the Euler equation (1) now becomes

$$y_t = E_t\,y_{t+1} + \alpha[E_t\,\pi_{t+1} + r_t] + \nu_t. \quad \text{(C.1)}$$

Equation (C.1) together with equations (B.1) and (2) form a linear system of three equations in the three unknown decision rules for y_t, π_t, and i_t^*. The state variables are: r_t, $E_t r_t^*$, $\overline{\pi}_{t-1}$, i_{t-1}^*, ν_t, and u_t. The minimum state variable (MSV) solution yields a set of linear deci-

sion rules of the form

$$
\begin{bmatrix}
y_t - \left(-E_t r_t^*\right)\left(1 - \beta\right)/\kappa \\
\pi_t - \left(-E_t r_t^*\right) \\
i_t^* - \left(E_t r_t^* + \pi^*\right)\left[1 - g_\pi - g_y\left(1 - \beta\right)/\kappa\right]
\end{bmatrix}
= \mathbf{B}
\begin{bmatrix}
r_t - E_t r_t^* \\
\overline{\pi}_{t-1} - \left(-E_t r_t^*\right) \\
i_{t-1}^* - \left(E_t r_t^* + \pi^*\right)\left[1 - g_\pi - \frac{g_y\left(1 - \beta\right)}{\kappa}\right] \\
\nu_t \\
u_t
\end{bmatrix},
$$

(C.2)

where $\mathbf{B}$ is a 3×5 matrix of constant coefficients. The MSV solution implies $\mathbf{B}_{12} = \mathbf{B}_{22} = 0$ and $\mathbf{B}_{13} = \mathbf{B}_{23} = 0$. For the parameter values shown in Table 2, the matrix $\mathbf{B}$ is

$$
\mathbf{B} =
\begin{bmatrix}
2.330 & 0 & 0 & 1 & 0 \\
0.394 & 0 & 0 & 0.025 & 1 \\
0.520 & 0.162 & 0.8 & 0.203 & 0.138
\end{bmatrix}.
$$

(C.3)

Comparing the first column of matrix $\mathbf{B}$ in (C.3) to the first column of matrix $\mathbf{A}$ in (B.3) shows that a shock to $r_t - E_t r_t^*$ will be transmitted more forcefully to macro variables in the deflation equilibrium than in targeted equilibrium. Specifically, we have

$$
\frac{\mathbf{B}_{11}}{\mathbf{A}_{11}} = 2.8, \qquad \frac{\mathbf{B}_{21}}{\mathbf{A}_{21}} = 4.7, \qquad \frac{\mathbf{B}_{31}}{\mathbf{A}_{31}} = 3.0.
$$

(C.4)

For the special case when $\rho = 0$ and $\omega = 1$, it is straightforward to derive the following analytical relationship between the decision rule coefficients for the two local equilibria:

$$
\frac{\mathbf{B}_{11}}{\mathbf{A}_{11}} = \frac{\mathbf{B}_{21}}{\mathbf{A}_{21}} = \frac{\mathbf{B}_{31}}{\mathbf{A}_{31}} = 1 + \frac{\alpha\left[\kappa g_\pi + \left(1 - \beta\rho_r\right)g_y\right]}{\left(1 - \beta\rho_r\right)\left(1 - \rho_r\right) - \alpha\kappa\rho_r} > 1.
$$

(C.5)

Iterating the linear decision rules in (C.2) ahead one period and then taking the conditional expectation of both sides yields the following set of local linear forecast rules for the deflation equilibrium:

$$
E_t\, y_{t+1} = -E_t r_t^*\left(1 - \beta\right)/\kappa + \mathbf{B}_{11}\rho_r\left(r_t - E_t r_t^*\right),
$$

(C.6)

$$
E_t \pi_{t+1} = -E_t r_t^* + \mathbf{B}_{21}\rho_r\left(r_t - E_t r_t^*\right),
$$

(C.7)

where the MSV solution implies $\mathbf{B}_{12} = \mathbf{B}_{22} = 0$ and $\mathbf{B}_{13} = \mathbf{B}_{23} = 0$ and I have substituted in $E_t\left(r_{t+1} - E_{t+1}r_{t+1}^*\right) = \rho_r\left(r_t - E_t r_t^*\right)$. Neither the agent or the central bank are required to forecast i_{t+1}^*.

The linear forecast rules (C.6) and (C.7) are derived under the assumption that the ZLB is always binding. However, in the stochastic simulation of the deflation equilibrium, I allow for an occasionally binding ZLB. When simulating the model, I substitute the local linear

forecast rules given by (C.6) and (C.7) into the global equilibrium conditions (1) and (2). I allow for an occasionally binding ZLB by making the substitution $i_t = 0.5\,i_t^* + 0.5\sqrt{(i_t^*)^2}$ in the equilibrium condition (1). Together with the monetary policy rule (B.1), this procedure yields a system of three equations that are solved each period to obtain the three realizations y_t, π_t, and i_t^*.

D   Appendix: Simplified model

This appendix provides the theoretical equilibrium decision rules for the simplified model that is used in the adaptive learning algorithm described in Section 4.5. Starting from the original model, the simplified model imposes $\rho = 0$, $\omega = 1$, and $\sigma_\eta = 0$. These settings eliminate i_{t-1}^* and $\overline{\pi}_{t-1}$ as state variables and cause the natural rate of interest r^* to be constant.

The targeted equilibrium decision rules are

$$
\begin{bmatrix}
y_t - \pi^*\,(1-\beta)\,/\kappa \\
\pi_t - \pi^* \\
i_t^* - (r^* + \pi^*)
\end{bmatrix}
= \mathbf{A}
\begin{bmatrix}
r_t - r^* \\
\nu_t \\
u_t
\end{bmatrix},
\tag{D.1}
$$

where the elements of the matrix $\mathbf{A}$ are given by

$$
\mathbf{A} =
\begin{bmatrix}
\dfrac{\alpha\,(1-\beta\rho_r)}{(1-\rho_r+\alpha g_y)(1-\beta\rho_r)+\alpha\kappa\,(g_\pi-\rho_r)} & \dfrac{1}{1+\alpha\kappa g_\pi+\alpha g_y} & \dfrac{-\alpha g_\pi}{1+\alpha\kappa g_\pi+\alpha g_y} \\[4mm]
\dfrac{\alpha\kappa}{(1-\rho_r+\alpha g_y)(1-\beta\rho_r)+\alpha\kappa\,(g_\pi-\rho_r)} & \dfrac{\kappa}{1+\alpha\kappa g_\pi+\alpha g_y} & \dfrac{1+\alpha g_y}{1+\alpha\kappa g_\pi+\alpha g_y} \\[4mm]
\dfrac{\alpha\kappa g_\pi+\alpha g_y\,(1-\beta\rho_r)}{(1-\rho_r+\alpha g_y)(1-\beta\rho_r)+\alpha\kappa\,(g_\pi-\rho_r)} & \dfrac{\alpha\kappa g_\pi+g_y}{1+\alpha\kappa g_\pi+\alpha g_y} & \dfrac{g_\pi}{1+\alpha\kappa g_\pi+\alpha g_y}
\end{bmatrix}.
\tag{D.2}
$$

The deflation equilibrium decision rules are

$$
\begin{bmatrix}
y_t - (-r^*)\,(1-\beta)\,/\kappa \\
\pi_t - (-r^*) \\
i_t^* - (r^* + \pi^*)\,[1 - g_\pi - g_y\,(1-\beta)\,/\kappa]
\end{bmatrix}
= \mathbf{B}
\begin{bmatrix}
r_t - r^* \\
\nu_t \\
u_t
\end{bmatrix},
\tag{D.3}
$$

where the elements of the matrix $\mathbf{B}$ for the MSV solution are given by

$$
\mathbf{B} =
\begin{bmatrix}
\dfrac{\alpha\,(1-\beta\rho_r)}{(1-\rho_r)(1-\beta\rho_r)-\alpha\kappa\rho_r} & 1 & 0 \\[4mm]
\dfrac{\alpha\kappa}{(1-\rho_r)(1-\beta\rho_r)-\alpha\kappa\rho_r} & \kappa & 1 \\[4mm]
\dfrac{\alpha\,(1-\beta\rho_r)}{(1-\rho_r)(1-\beta\rho_r)-\alpha\kappa\rho_r} & \kappa g_\pi + g_y & g_\pi
\end{bmatrix}.
\tag{D.4}
$$

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## Figure 1: Nominal Interest Rates and Inflation in Four Countries
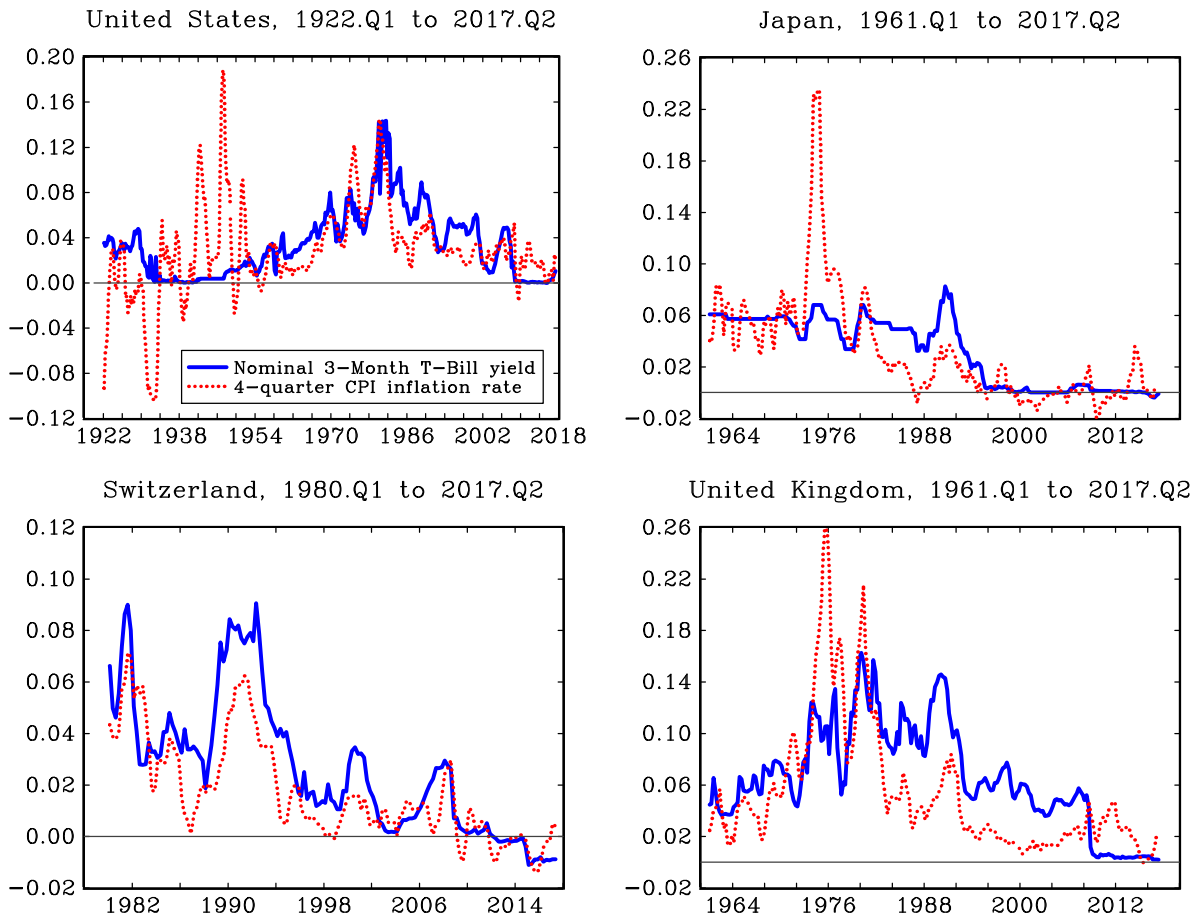
Nominal interest rates in the United States encountered the zero lower bound during the 1930s and from 2008.Q4 though 2015.Q4. Nominal interest rates in Japan have remained near zero since 1998.Q3, except for the relatively brief period from 2006.Q4 to 2008.Q3. Nominal interest rates in Switzerland have been zero or slightly negative since 2008.Q4. Nominal interest rates in the United Kingdom have been approximately zero since 2009.Q1. Outside of these episodes, all four countries exhibit a strong positive correlation between nominal interest rates and inflation, consistent with the Fisher relationship.

Figure 2: U.S. Nominal Interest Rates and Inflation
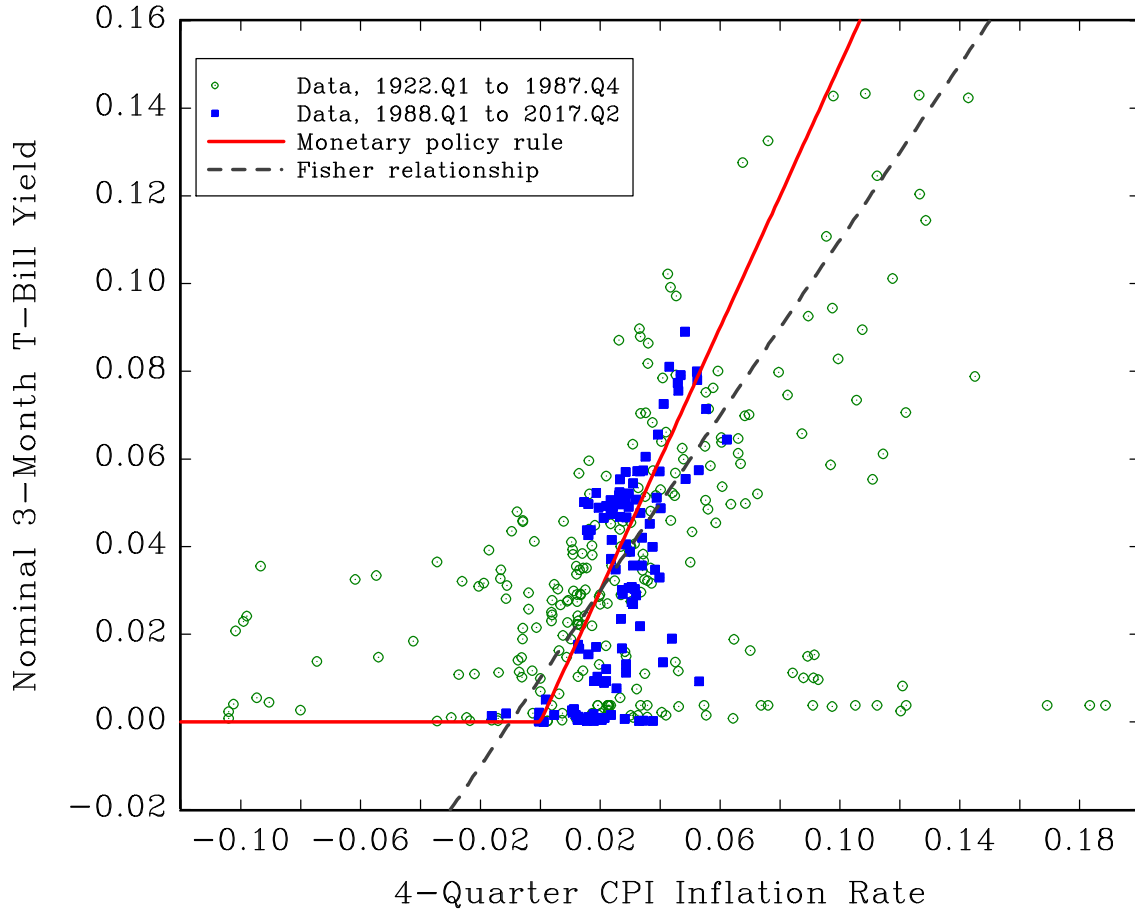
The two intersections of the ZLB-augmented monetary policy rule (solid red line) with the Fisher relationship (dashed black line) define two long-run endpoints, labeled the "targeted equilibrium" and "deflation equilibrium," respectively. The monetary policy rule is $i_t = r^* + \pi^* + g_\pi (\pi_t - \pi^*)$ with $r^* = 0.01$, $\pi^* = 0.02$ and $g_\pi = 1.5$. The Fisher relationship is $i_t = r^* + \pi_t$. Data since 2008.Q4 lie closer to the deflation equilibrium than the targeted equilibrium.

Figure 3: U.S. Real Interest Rates
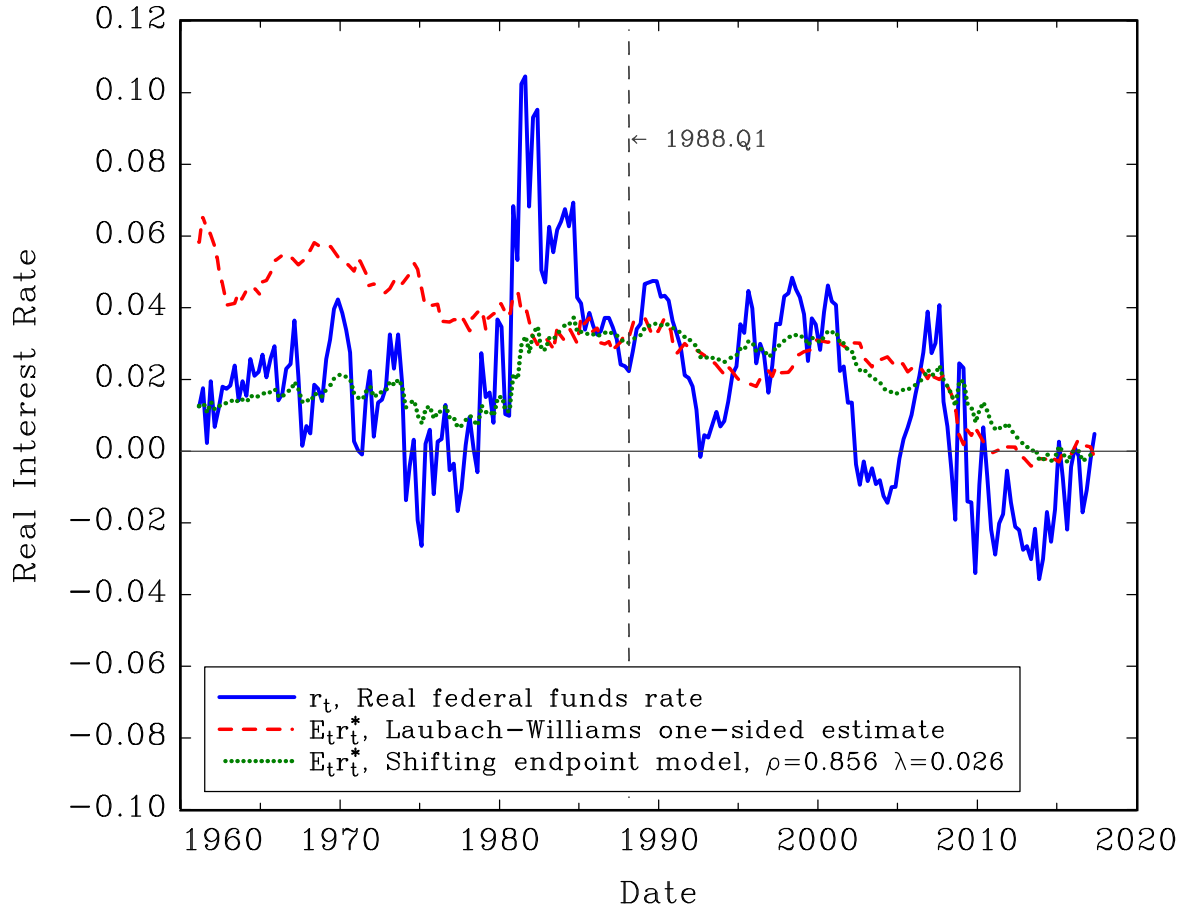
The real federal funds rate (blue line) is defined as the nominal federal funds rate minus expected quarterly inflation computed from a rolling 40-quarter, 4-lag vector autoregression that includes the nominal funds rate, quarterly PCE inflation (annualized), and the CBO output gap. The time series process for the Kalman filter estimate of the natural rate of interest in the model (dotted green line) is calibrated to approximate the one-sided estimate of the U.S. natural rate series (dashed red line) from Laubach and Williams (2016, updated) for the period of consistent monetary policy from 1988.Q1 to 2017.Q2.

Figure 4: U.S. Data
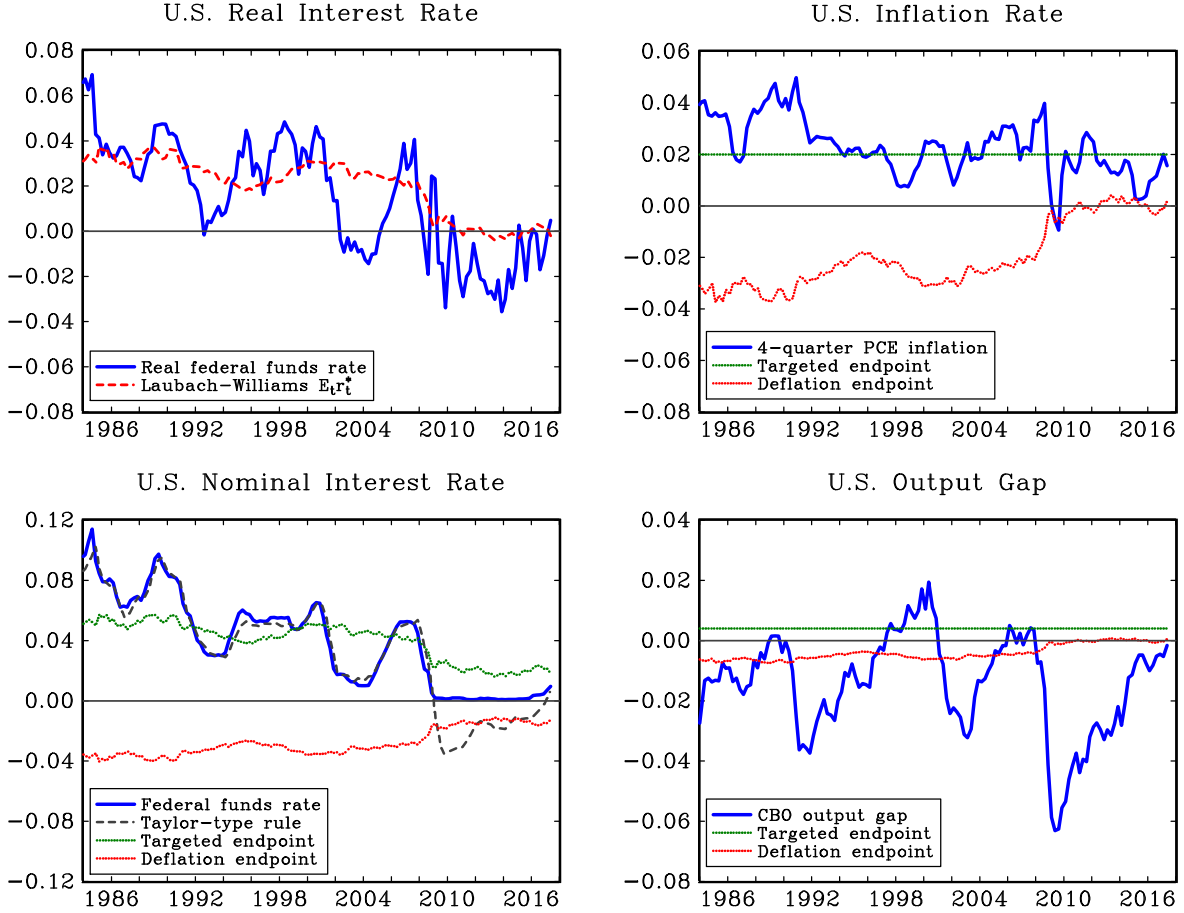
The U.S. real federal funds rate has remained mostly below the Laubach-Williams estimate of r_t^* since early 2009. The nominal federal funds rate was approximately zero from 2008.Q4 through 2015.Q4. A Taylor-type rule of the form (9) using the parameter values in Table 2, with $E_t r_t^*$ given by Laubach-Williams estimate, $\overline{\pi}_t$ given by the 4-quarter PCE inflation rate, and y_t given by the CBO output gap predicts that the desired nominal funds rate was negative starting in 2009.Q1 and remains negative through 2016.Q4. The 4-quarter PCE inflation rate was briefly negative in 2009 and has remained below the Fed's 2% inflation target since 2012.Q2. The Great Recession was very severe, pushing the CBO output gap down to −6.3% at the business cycle trough in 2009.Q2. The output gap remains negative at −0.2% in 2017.Q2, eight years after the Great Recession ended. The various endpoints plotted in the figure are computed using the expressions in Table 1, with r_t^* given by the Laubach-Williams estimate. As r_t^* approaches zero or becomes negative, the "deflation" equilibrium is characterized by zero or low inflation, allowing this equilibrium to provide a better fit of recent U.S. inflation data.

## Figure 5: Measures of Expected U.S. Inflation
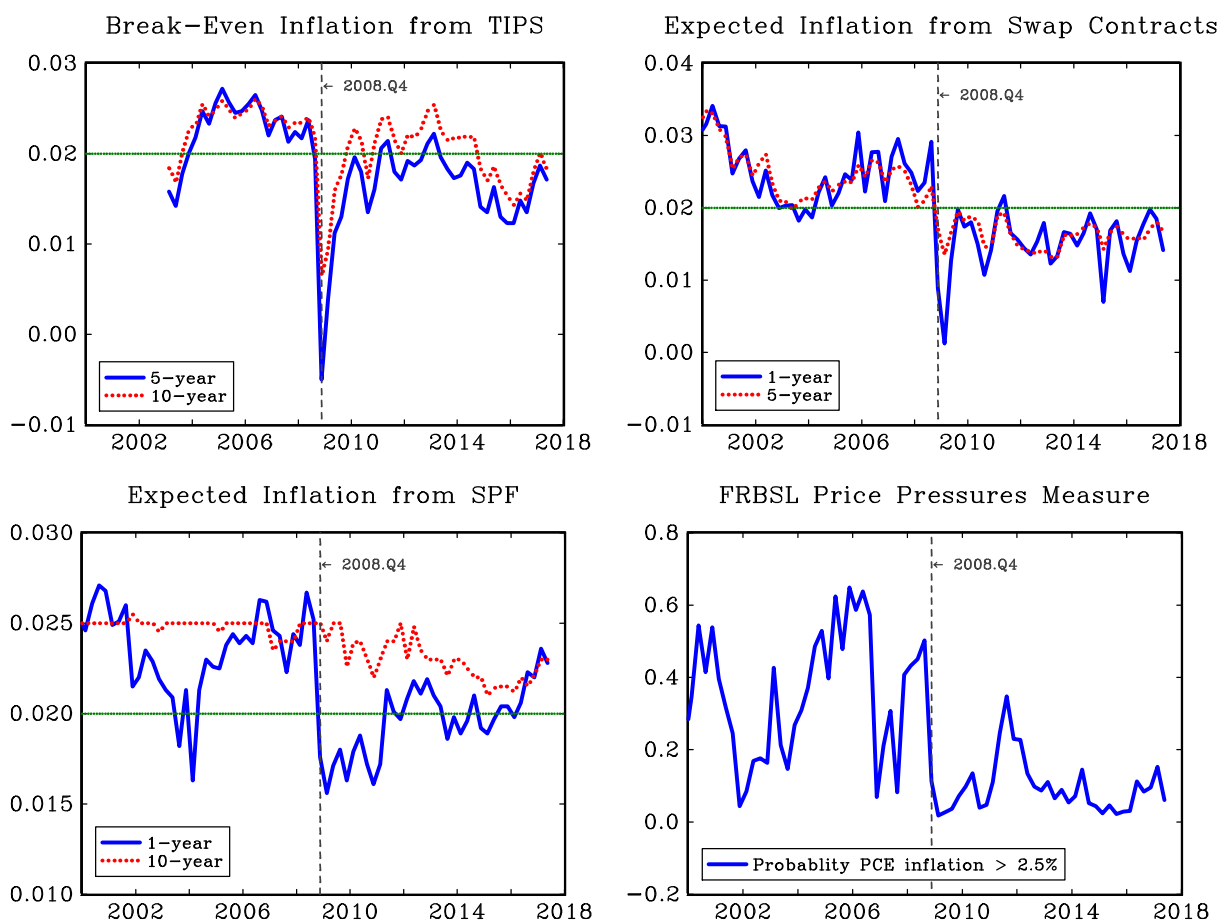
Market-based measures of expected inflation from Treasury Inflation Protected Securities (TIPS) and zero coupon inflation swap contracts dropped sharply in 2008.Q4, coinciding with the start of the ZLB episode. The market-based measures remain below the Fed's 2% inflation target at the end of the data sample. The median 1-year expected inflation rate from the Survey of Professional Forecasters (SPF) dropped sharply in 2008.Q4 and has recovered slowly to a level that is below its pre-recession range. The 10-year survey measure does not exhibit a sharp drop in 2008.Q4, but has since trended downward to a level that is below its pre-recession range. The Federal Reserve Bank of St. Louis' Price Pressures Measure (PPM) represents the probability that the 4-quarter PCE inflation rate over the next year will exceed 2.5%. The PPM dropped sharply in 2008.Q4 and is currently hovering around a probability of 10%.

Figure 6: Model Simulation: Endogenous Regime Switching
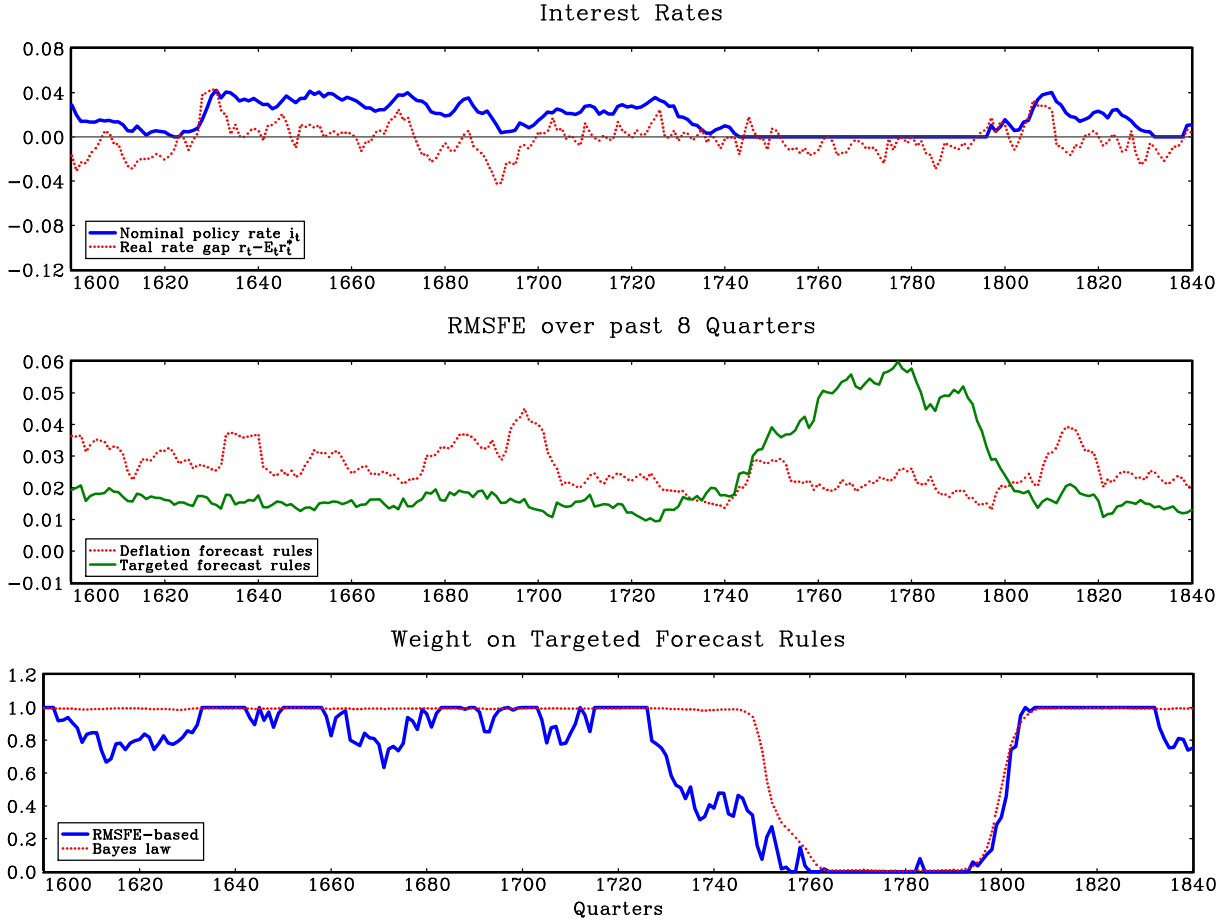
When the exogenous real interest gap $r_t - E_t r_t^*$ is negative for a sustained period (top panel), the resulting downward pressure on y_t and π_t serves to reduce the recent $RMSFE$ of the deflation forecast rules and increase the recent $RMSFE$ of the targeted forecast rules (middle panel). Around period 1725, the shift in relative forecast performance induces the agent to place a substantially lower weight on the targeted equilibrium forecast rules, causing the deflation equilibrium to become temporarily self-fulfilling (bottom panel). Around period 1800, the real rate gap once again becomes positive, causing the $RMSFE$ of the deflation forecast rules to exceed the $RMSFE$ of the targeted forecast rules. The agent now increases the weight on the targeted forecast rules, causing the targeted equilibrium to be restored. Qualitatively similar results are obtained if the agent employs Bayes law to compute the likelihood that a string of recent quarterly inflation observations are drawn from one equilibrium or the other.

Figure 7: Model Simulations: Distributions of Endogenous Variables
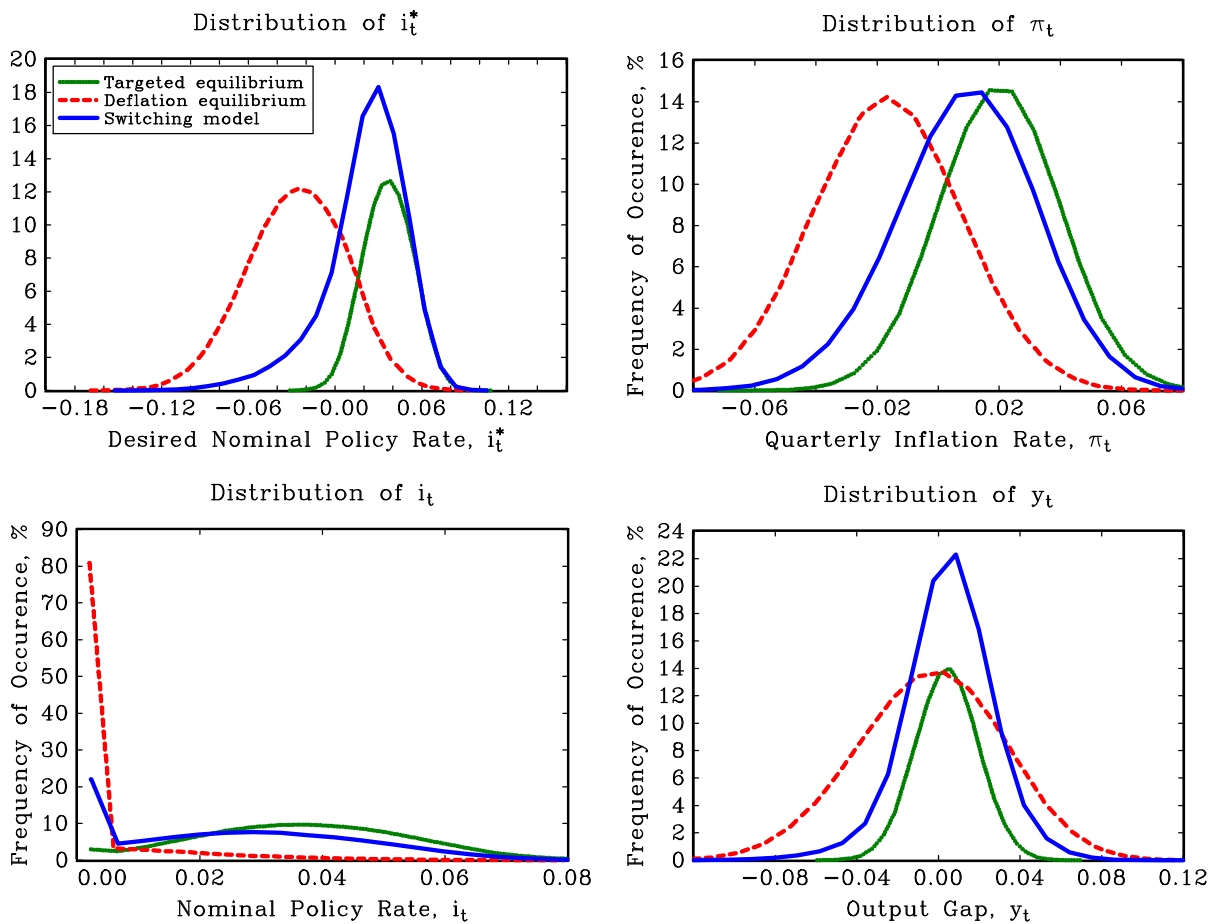
Model variables in the deflation equilibrium have distributions with lower means but higher variances than those in the targeted equilibrium. But the significant overlap in the various distributions creates a dilemma for an agent who seeks to determine the likelihood that a string of recent data observations are drawn from one equilibrium or the other. Variables in the switching model have means that are somewhat lower and variances that are somewhat higher than those in the targeted equilibrium. Consequently, the central bank in the switching model undershoots its inflation target and the volatilities of the output gap and inflation are both higher relative to the targeted equilibrium.

Figure 8: Model Simulations: Distribution of ZLB Durations
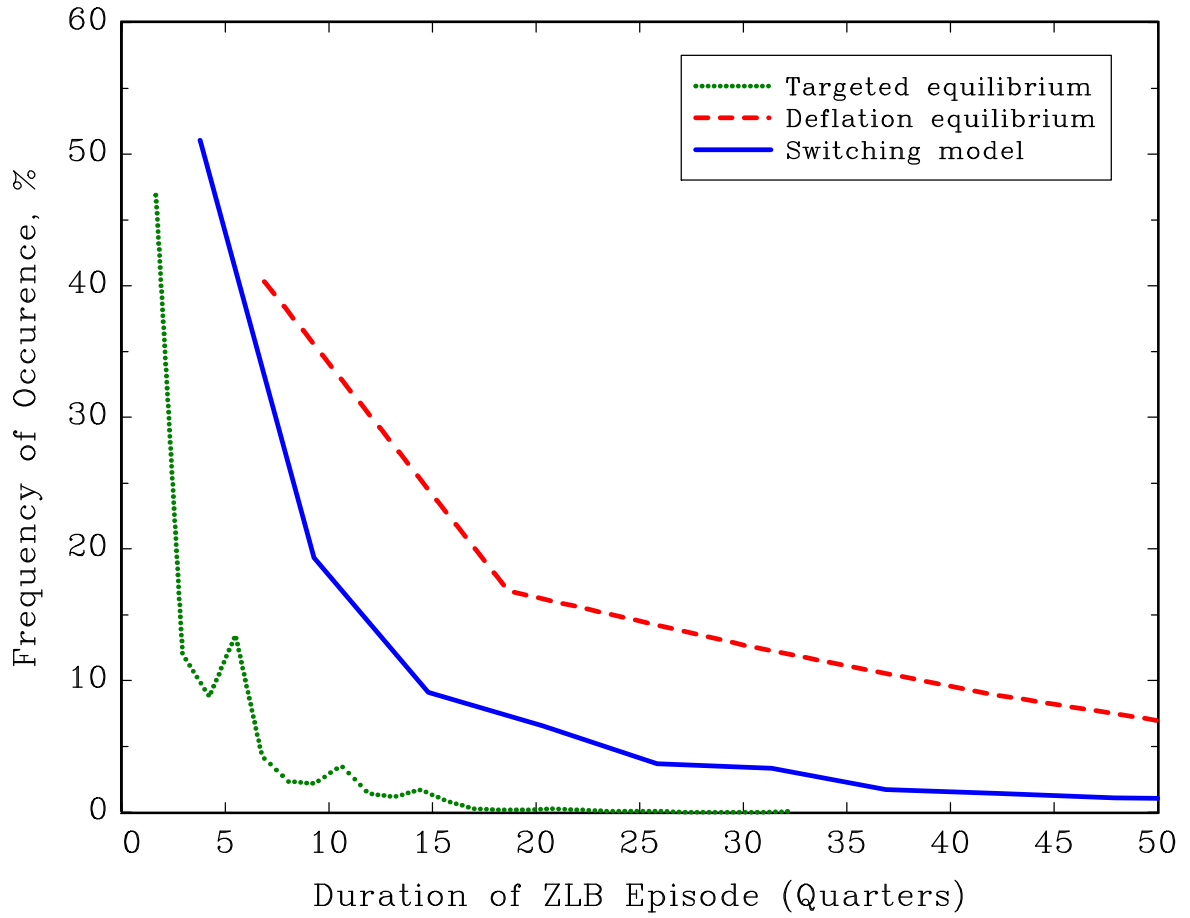
Unlike the targeted equilibrium, the switching model can produce infrequent, but long-lived ZLB episodes in response to small, normally distributed shocks. To account for infrequent but long-lived ZLB episodes, the targeted equilibrium would require large shocks that are themselves infrequent and long-lived, as in Dordal-i-Carreras, et al. (2016).

# Figure 9: Model Simulations: Comparing Three Model Versions
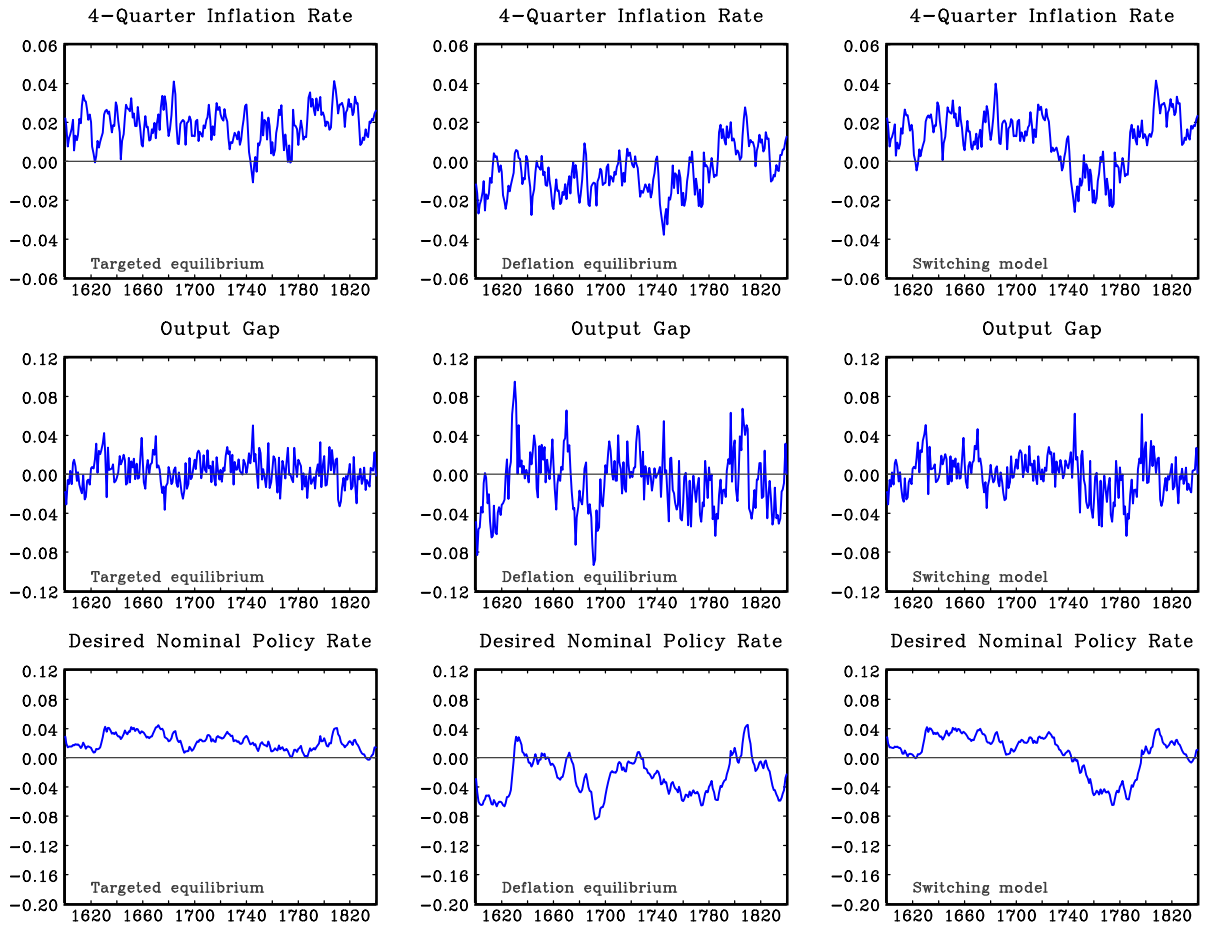
All three model versions employ the same sequence of stochastic shocks. Around period 1725, the weight on the targeted forecast rules in the switching model starts dropping toward zero, causing the deflation equilibrium to become temporarily self-fulfilling. The episode results in a negative desired nominal policy rate, a binding ZLB, a highly negative output gap, and brief deflation followed by below-target inflation, reminiscent of the U.S. Great Recession and its aftermath.

Figure 10: Replicating U.S. Data with the Switching Model

Model Implied Shocks

RMSFE over past 8 Quarters

Expected Inflation

Weight on Targeted Forecast Rules
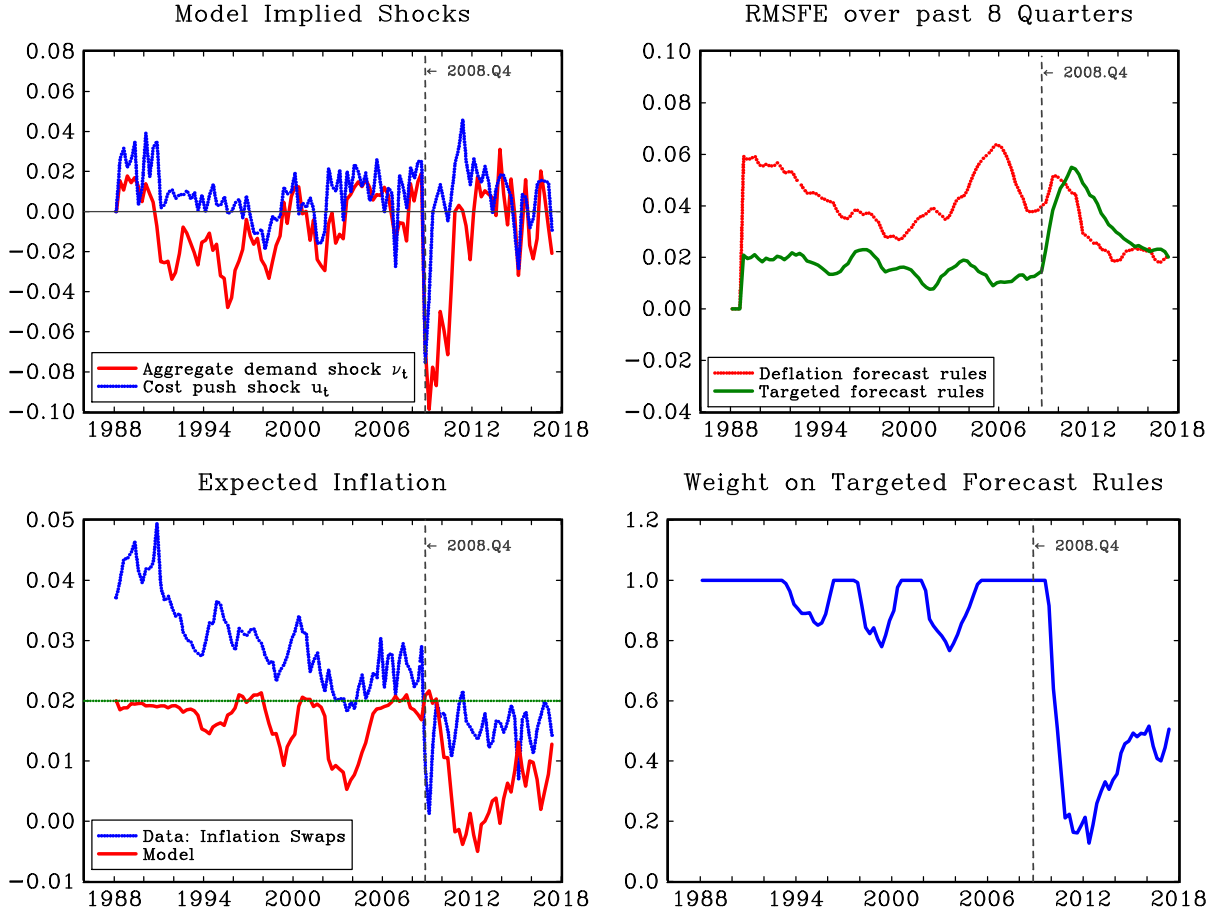
Given the U.S. data counterparts for the model variables, we can solve for the implied time series of stochastic shocks and forecast rule weights that allow the switching model to exactly replicate the observed U.S. time paths of the CBO output gap and PCE inflation since 1988. The model-implied shocks ν_t and u_t become strongly negative at the start of the ZLB episode in 2008.Q4. The top right panel shows that the performance gap between the two sets of forecast rules starts to narrow considerably after 2008.Q4. The deflation forecast rules start outperforming the targeted forecast rules in 2010.Q3. As a result, the weight on the targeted equilibrium forecast rules starts to decline, eventually reaching a minimum value of 0.16 in 2011.Q3. The weight subsequently increases as U.S. economy recovers from the Great Recession. But even towards the end of the data sample, the weight remains well below 1.0, helping the switching model to account for the persistent undershooting of the Fed's 2% inflation target since mid-2012. The bottom left panel compares the 1-year expected inflation rate from U.S. inflation swaps to the path of $\widehat{E}_t\,\pi_{t+1}$ from the switching model. The correlation coefficient between the two series is 0.6. Both series remain below 2% at the end of the data sample in 2017.Q2.

Figure 11: Adaptive Learning in a Simplified Model
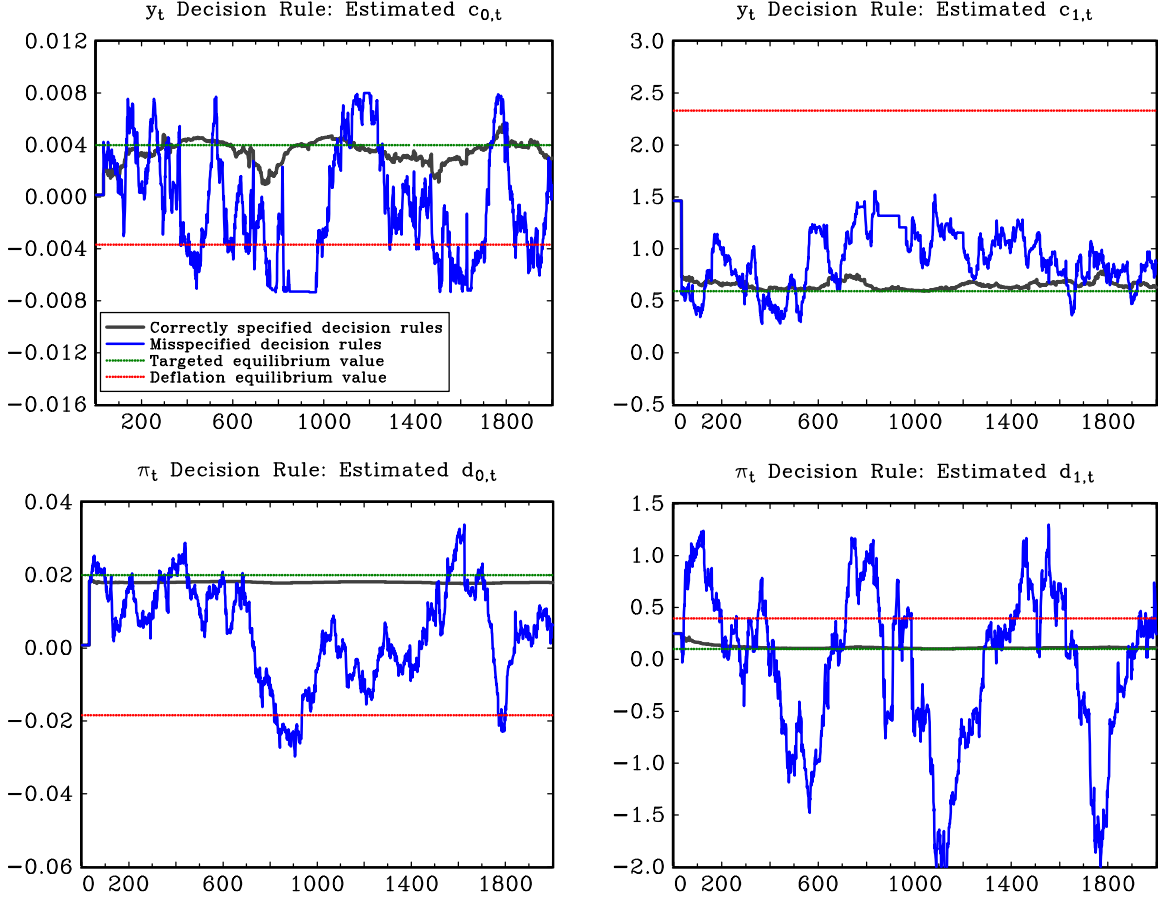
Starting from the original model, the simplified model imposes $\rho = 0$, $\omega = 1$, and $\sigma_\eta = 0$. These settings eliminate i^*_{t-1} and $\overline{\pi}_{t-1}$ as state variables and cause the natural rate of interest r^* to be constant. When the agent estimates correctly specified decision rules (black lines), the coefficients quickly converge to the vicinity of the targeted equilibrium values and remain there. In contrast, when the agent estimates misspecified decision rules that fail to control for the white noise shocks ν_t and u_t (blue lines), the coefficients exhibit low frequency oscillations that can occasionally approach, reach, or go beyond the deflation equilibrium values. The low frequency oscillations in the decision rule coefficients induce movements in the macroeconomic variables that are qualitatively similar to those observed in Figure 9 for the switching model with full-knowledge.