A Quantitative Discursive Dilemma*

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Abstract

The typical judgment aggregation problem in economics and other fields is the following: A group of people has to judge (estimate) the value of an uncertain variable y which is a function of k other variables, i.e. $y = D(x_1, ...x_k)$. We analyze when it is possible for the group to arrive at collective judgments on the variables that respect D. We consider aggregators that fulfill Arrow's IIA-condition and neutrality. We show how possibility and impossibility depend on the functional form of D, and generalize Pettit's (2001) binary discursive dilemma.

Keywords: Judgment aggregation, Dependent variables, Impossibility, Possibility

JEL Classification: D71

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1 Introduction

The typical decision problem – or judgment aggregation situation – in economics and other fields is the following: A group of people have to judge or estimate the size of a variable which is a function of some other variables. For example, a monetary policy committee's interest rate decisions depend on judgments about inflationary pressures and financial fragility; a cabinet's judgment of the future budget balance depends on its judgments of future revenues and costs; a corporate board's investment decisions depend on judgments of future cash flows and cost of capital, and so on. In this paper we analyze when a group of people with different judgments on the variables can make a collective judgment on the variables that respects the dependence between the variables.

To illustrate the issue and its importance, consider a corporate board assessing the profitability of an investment project. The profitability is measured as the project's expected net present value (NPV), which per definition is the discounted cash flow (DCF) less the investment cost (IC). Thus, the dependence between the variables is given by $NPV = DCF - IC$. Suppose that the corporate board has three members with estimates as in Table 1. Let the members of the board vote on the size of each variable,

Table 1: Example of aggregate inconsistency for a corporate board assessing the net present value of an investment project.

	Discounted cash flow (DCF)	Investment cost (IC)	Net present value (NPV)
Member A	10	8	2
Member B	10	11	-1
Member C	13	12	1
Board	10	11	1

and assume that the outcome of the vote is the median of the individual estimates. Then a vote on the conclusion-variable gives $NPV = 1$. But this is not consistent with the majority's judgments on the two 'premise-variables', since $10 - 11 = -1$. Thus, the aggregate judgments do not respect the dependence between the variables. As a consequence the board faces a discursive dilemma (Pettit (2001)). A premise-based procedure, where they vote on the two premise variables and let the conclusion follow from the dependence function, gives $NPV = -1$. A conclusion-based procedure, where they vote directly on NPV, gives $NPV = 1$.

In this paper we want to check if the example illustrates a general problem for groups aggregating judgments on dependent variables. We therefore construct a general social choice theoretic model and ask the following question: Under which conditions are there combinations of individual judgments that give aggregate judgments that do not respect the dependence between the variables (impossibility), and under which conditions are there no such combinations (possibility)? By using a general social choice theoretic framework we can treat all aggregation methods fulfilling some general conditions simultaneously. Examples of aggregators fulfilling our conditions are pairwise majority voting over the alternative values for each variable, and an agenda-setting method whereby the aggregate ranking of any two alternatives for one variable is the ranking of the agenda setter (the same each time) unless a supermajority has another judgment.

In our model a group of people have to conclude on the value of a dependent variable x_{k+1} when the value of this variable depends on the value of k independent variables

x_1,..., x_k by some general dependence function D:

$$x_{k+1} = D(x_1, ..., x_k)$$

The dependence function can be a reaction function derived from maximizing an objective function; it can be a rule-of-thumb, a causal relationship between economic variables, a definition, or any mapping from values of the independent variables to the dependent variable. The arguments in the dependence function are the variables and parameters on which the members of the group may have different judgements. Variables and parameters that are relevant for the dependent variable, but which the group always agrees on, may be represented by the functional form of D. Suppose, for instance, that $y = \alpha x$ is a policy rule where y is a policy instrument (e.g. the central bank's key interest rate), x is an economic variable (e.g. the rate of underlying inflation), and α is a parameter that says how much a change in x should affect y. Then the dependence function is the policy rule (with x the argument) if all individuals always agree on the value of α. Otherwise the dependence function has two arguments: x and α.

We assume that each member has one preference relation for each of the $k+1$ variables. A member's estimate of a variable is, if it exsists, the peak of his preference relation for the variable. To reach an aggregate preference relation for a variable the group uses an aggregator that takes profiles of individual preference relations (one relation for each member) as inputs, and fulfill a set of standard general conditions. The conditions are 'pareto', 'independence' and a strong and a weaker form of 'neutrality'. We then derive a characterization result for when there exist non-dictatorial aggregators such that the aggregate estimates respect the dependence function. It is seen that possibility arises only in the special case when $k = 1$ and the dependence function is strictly monotonic.

The paper has four sections. In Section 2 we present the model. In Section 3 we give the main characterization. We conclude with a discussion of our framework and key assumptions in Section 4.

Relation to the literature

Considering the aggregation of different interconnected variables is not new. A variety of aggregation problems has been proposed and solved in production theory, see Blackorby & Schworm (1984) for an overview. In opinion pooling the probability assignments of different individuals are to be merged into collective probability assignments. Genest & Zidek (1986) give an overview of classical results in opinion pooling. See Mongin (1995) and Dietrich & List (2007b) for more recent results. Rubinstein & Fishburn (1986) consider the problem of aggregating the entries in n rows in an $n \times m$ matrix into a summary row, where every entry is an element in an algebraic field. They find that if the entries always form a hyperplane, then every consistent aggregator is an aggregator whereby the aggregate estimate of a variable is the (normalized) linear sum of the individual estimates. If the entries do not form a hyperplane there is no consistent non-dictatorial aggregator.

In an earlier paper on quantitative discursive dilemmas we (Claussen and Røisland 2005) study a situation somewhat similar to the situation studied by Rubinstein & Fishburn (1986), but where we assign one variable the role as a dependent variable and the other variables the role as independent variables. Furthermore we have less strict domain restrictions. In that paper we find that if the group aggregates by taking the mean of the individual estimates, then the boundary between possibility and impossibility lies in whether or not the dependence function is linear. If the group aggregates by taking the median of the individual estimates, the boundary lies in whether or not the dependence function is strictly monotonic. In the current paper we step out of the model of our previous paper and the literature on the aggregation of different interconnected variables by considering the aggregation of $k+1$ preference relations rather than aggregation of $k+1$ estimates. By this move we are able to study the situation when a group aggregates by some voting method, rather than by just combining estimates.

The setting of this paper is also somewhat parallel to a setting where a group of people aggregate judgments on interconnected propositions. In such situations an aggregation inconsistency similar to the inconsistency in the example of Table 1 may arise. Pettit (2001) coined that inconsistency the 'the discursive dilemma'. Recently, researchers have built general social choice theoretic models to study the aggregation of judgments on propositions. The first example is List & Pettit (2002). They also provided the first impossibility result which was quickly followed by several stronger impossibility and possibility results. Roughly speaking, the impossibility results say that if the propositions under consideration are interlinked, then there is no aggregator that fulfils requirements similar to, but not exactly equal to, the Arrovian requirements that aggregate consistent individual judgments on propositions into consistent collective judgments on these propositions. See Dietrich (2007) for a generalised model of judgment aggregation, and List & Puppe (2008) for an overview of the literature. Compared to the judgment aggregation literature the important novelty our current paper is that variables need not be binary. Thus, we introduce a generalization of Pettit's (2001) discursive dilemma to non-binary and continuous variables.

2 The Model

We consider a group, where $N = \{1, ..., n\}$ denotes the set of members, and $n > 2$.[1] Each member $i \in N$ will be referred to as a 'member' or an 'individual' depending on the context. The group has to evaluate real-valued variables $j = 1, ..., k+1$ where $k \geq 1$ and each variable j takes values in a non-empty set $X_j \subseteq \mathbb{R}$. This set has at least two elements and might be finite or infinite. Examples are $X_j = \mathbb{R}$, $X_j = [0, 1]$, and the binary case where $X_j = \{0, 1\}$ as in standard judgment aggregation. The variables $1, ..., k$ will be denoted 'independent variables', and variable $k+1$ the 'dependent variable'.

Let

$$D : X_1 \times ... \times X_k \rightarrow X_{k+1}$$

be a surjective function, the *dependence function*, representing how the dependent variable $k+1$ depends on the independent variables $1, ..., k$.[2]

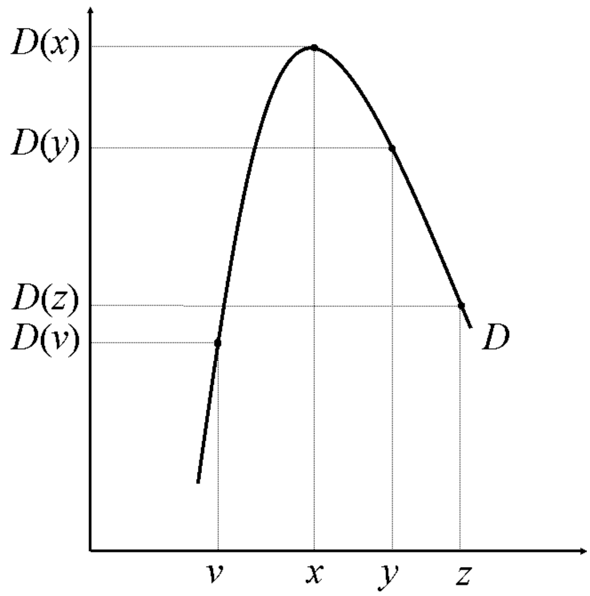
A *ranking* of two alternatives for a variable $x, y \in X_j$, denoted $x \succcurlyeq y$, is a comparison of the two alternatives where $x \succcurlyeq y$ means alternative x is ranked equal to or higher than alternative y. The asymmetric part of $\succcurlyeq$ is denoted by $\succ$, so $x \succ y$ if and only if $x \succcurlyeq y$ holds but $y \succcurlyeq x$ does not.

A set of rankings over the alternatives in X_j is called a *relation* and denoted $\succcurlyeq_{X_j}$. A relation $\succcurlyeq_{X_j}$ is complete if all alternatives in X_j are compared such that for all $x, y \in X_j$, either is $x \succcurlyeq y \in \succcurlyeq_{X_j}$ or $y \succcurlyeq x \in \succcurlyeq_{X_j}$. Let $\mathcal{G}^*_{X_j}$ be the set of all complete relations $\succcurlyeq_{X_j}$. A complete relation $\succcurlyeq_{X_j}$ is a (weak) *order* if it is also transitive. Transitivity means that if an alternative x is considered at least as good as y, and y is at least as good as z, then x

[1] We assume $n > 2$ to make the propositions clearcut (not contingent on n). If $n = 2$, systematicity (see section 3) implies that the dependence must be dictatorial, regardless of the functional form of D.

[2] A function D is said to be *surjective* or *onto*, if its values span its whole codomain; that is, for every $x_{k+1} \in X_{k+1}$, there is at least one vector $x_1, .., x_k \in X_1 \times ... \times X_k$ such that $D(x_1, .., x_k) = x_{k+1}$.

Figure 1: Illustration with $k = 1$ and concave dependence function.



is considered at least as good as z (formally: $\forall\, x, y, z \in X_j$, $(x \succcurlyeq y \wedge y \succcurlyeq z) \rightarrow (x \succcurlyeq z)$). Let $\mathcal{G}_{X_j}$ be the set of all weak orders $\succcurlyeq_{X_j}$. Thus, $\mathcal{G}_{X_j} \subsetneq \mathcal{G}^*_{X_j}$ as the relations in $\mathcal{G}^*_{X_j}$ need not be transitive.

A value $x \in X_j$ is the *unique peak* of variable j under relation $\succcurlyeq_{X_j} \in \mathcal{G}^*_{X_j}$ if $x \succ y$ for all $y \in X_j \backslash x$. Peaks will sometimes, depending on the context, be called estimates. A sequence of relations $(\succcurlyeq_{X_1}, ..., \succcurlyeq_{X_{k+1}}) \in \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ is said to *respect the dependence function* D if $x_{k+1} = D(x_1, ..., x_k)$ whenever $x_1, ..., x_{k+1}$ are unique peaks of $\succcurlyeq_{X_1}, ..., \succcurlyeq_{X_{k+1}}$, respectively.

We will also use a (rationality) requirement for alternatives ranked lower than the peaks. To get some intuition for this requirement, put $k = 1$, $X_1 = \{v, x, y, z\}$, and suppose D is concave as illustrated in 1. Suppose an individual has the order $v \succcurlyeq x \succcurlyeq y \succcurlyeq z$ on X_1. Our requirement will then allow for individual sequences with rankings over the corresponding alternatives in X_2 like

$$D(v) \succcurlyeq D(x) \succcurlyeq D(y) \succcurlyeq D(z)$$

and

$$D(v) \succcurlyeq D(z) \succcurlyeq D(y) \succcurlyeq D(x).$$

It will rule out arbitrary individual sequences where the order on the alternatives for X_2 for which D is strictly monotonic change its relative order, for instance sequences containing the rankings,

$$D(v) \succcurlyeq D(y) \succcurlyeq D(x) \succcurlyeq D(z).$$

Formally, let $\boldsymbol{x}' = (x'_1, ..., x'_k)$, $\boldsymbol{x}'' = (x''_1, ..., x''_k)$, and $\boldsymbol{x}''' = (x'''_1, ..., x'''_k)$ be three vectors in $X_1 \times ... \times X_k$ such that $x'_m > x''_m > x'''_m$ for $m \in \{1, ..., k\}$ and $x'_j = x''_j = x'''_j$ for all $j \in \{1, ..., k\} \backslash m$. Call D strictly monotonic for $\boldsymbol{x}', \boldsymbol{x}'', \boldsymbol{x}'''$ if $D(\boldsymbol{x}') > D(\boldsymbol{x}'') > D(\boldsymbol{x}''')$ or $D(\boldsymbol{x}') < D(\boldsymbol{x}'') < D(\boldsymbol{x}''')$. We say that a sequence of orders $(\succcurlyeq_{X_1}, ..., \succcurlyeq_{X_{k+1}}) \in \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ is *arbitrary* if (i) D is strictly monotonic for some $\{\boldsymbol{x}', \boldsymbol{x}'', \boldsymbol{x}'''\} \subseteq X_1 \times ... X_{k+1}$, (ii) $x'_m \succcurlyeq x''_m \succcurlyeq x'''_m$ or $x'_m \preccurlyeq x''_m \preccurlyeq x'''_m$, and (iii) $D(\boldsymbol{x}'')$ ranked above or below both $D(\boldsymbol{x}''')$ and $D(\boldsymbol{x}')$. A sequence is *non-arbitrary* if it is not arbitrary.

Let $\mathcal{G}$ be the set of all non-arbitrary sequences $(\succcurlyeq_{X_1}, ..., \succcurlyeq_{X_{k+1}}) \in \mathcal{G}_{X_1} \times ... \times \mathcal{G}_{X_{k+1}}$ that respect D. A *profile (of sequences)*, denoted g, is an $(k+1)n$-tuple in $\mathcal{G}^n$ with one sequence for each member.

An *aggregator* f is a mapping

$$f : \mathcal{G}^n \to \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}.$$

Notice that the aggregator takes (profiles of) *orders* as inputs but produce a *relation*. Thus, we do not require the outcome of the aggregation to be transitive. Furthermore, we do not require the outcome of the aggregation to be non-arbitrary. The *aggregator respects the dependence function* D if, for every profile $g \in \mathcal{G}^n$, $f(g)$ respects D.

Denote individual orders by $\succcurlyeq_{i,X_j}$ and aggregate relations by $\succcurlyeq_{N,X_j}$. We say that the aggregator is *non-dictatorial* if there is no $i \in N$ such that for all profiles $g \in \mathcal{G}^n$, $f(g) = (\succcurlyeq_{i,X_j})_{i,j=1,...,k+1}$.

3 The Characterization

We will now see when there is a $g \in \mathcal{G}^n$ such that $f(g)$ does not respect the dependence function (impossibility), and when $f(g)$ respects the dependence function for all $g \in \mathcal{G}^n$ (possibility).

We will consider aggregators that fulfill a set of standard conditions. The first condition is the unanimity principle which says that if every member of the group find that $x \in X_j$ is at least as good as $y \in X_j$ then the collective view should also be that x is at least as good as y. The second condition is Arrow's independence condition (IIA). This condition says that the aggregator obtains aggregate relations by comparing two alternatives at a time taken in isolation from the other alternatives. Thus, the aggregate ranking of any pair of alternatives for a variable will depend exclusively on the individual rankings over that pair. Consequently, the aggregation is independent between variables and independent for each variable seen in isolation. Formally, the conditions are as follows.

Unanimity principle/Pareto: For any profile $g \in \mathcal{G}^n$, and any ranking $x \succcurlyeq y$, $x, y \in X_j$, $j \in \{1, ..., k+1\}$, if $(x \succcurlyeq y) \in \succcurlyeq_{i,X_j}$ for all $i \in N$, then $x \succcurlyeq y \in f(g)$.

Independence (of Irrelevant Alternatives): For any two profiles in $\mathcal{G}^n$, $g^a = (\succcurlyeq^a_{i,X_j})_{i \in N, j=1,...,k+1}$, $g^b = (\succcurlyeq^b_{i,X_j})_{i \in N, j=1,...,k+1}$, any variable j, and any alternatives $x, y \in X_j$, if for all individuals i the restriction of $\succcurlyeq^a_{i,X_j}$ to the pair $\{x, y\}$ equals the restriction of $\succcurlyeq^b_{i,X_j}$ to the pair $\{x, y\}$ (that is, $x \succcurlyeq^a_{i,X_j} y$ if and only if $x \succcurlyeq^b_{i,X_j} y$ and $y \succcurlyeq^a_{i,X_j} x$ if and only if $y \succcurlyeq^b_{i,X_j} x$), then $x \succcurlyeq^a_{N,X_j} y$ if and only if $x \succcurlyeq^b_{N,X_j} y$.

In addition to fulfilling independence, we require the aggregator to be neutral in two respects. First, if the aggregate ranking on two alternatives for one variable is determined by some method, for example a pair-wise majority vote, then the aggregate order on any other two alternatives for the same variable shall be determined by the same method. Second, if the aggregate order on one variable is determined by some method, then the aggregate order on any other variable shall be determined by the same method. Neutrality and independence give the following requirement:[3]

[3]The systematicity-condition is inspired by a similar concept from the literature on the aggregation of judgments on propositions where it was first introduced by List & Pettit (2002). It will be relaxed somewhat in Section 4.

Systematicity: For any two profiles in $\mathcal{G}^n$, $g^a = (\succsim^a_{i,X_j})_{i\in N, j=1,...,k+1}$, $g^b = (\succsim^b_{i,X_j})_{i\in N, j=1,...,k+1}$, any variable j and m , and any alternatives $x'_j, x''_j \in X_j$ and $x'_m, x''_m \in X_m$, if for all individuals i the restriction of $\succsim^a_{i,X_j}$ to the pair $\{x'_j, x''_j\}$ equals the restriction of $\succsim^b_{i,X_m}$ to the pair $\{x'_m, x''_m\}$ (that is, $x'_j \succsim^a_{i,X_j} x''_j$ if and only if $x'_m \succsim^b_{i,X_m} x''_m$ and $x''_j \succsim^a_{i,X_j} x'_j$ if and only if $x''_m \succsim^b_{i,X_m} x'_m$), then $x'_j \succsim^a_{N,X_j} x''_j$ if and only if $x'_m \succsim^b_{N,X_m} x''_m$.

The following proposition holds.

Proposition 1 *A non-dictatorial aggregator* $f : \mathcal{G}^n \to \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ *that satisfies the Unanimity principle and Systematicity respects the dependence function* D *if and only if* $k = 1$ *and* D *is strictly monotonic.*

Proof. The proof is in the appendix. ■

It might come as a surprise that the boundary between possibility and impossibility is somewhat simpler in our framework where the group aggregates orders than in a model of aggregating just estimates, c.f. Rubinstein & Fishburn (1986). In the latter case, a crucial question is whether the dependence function is linear. The reason why we get the simpler boundary is that in our case only the relative ranking of the estimates for a variable matter in the aggregation. In the literature on the aggregation of estimates the relative size of the individuals' estimates matters for the aggregate estimate. This is because the authors assume that the aggregate estimate is some linear or non-linear combination of the individual estimates. An exception is Claussen and Røisland (2005) where we assume that the aggregate estimate is the median of the individual estimates. With this aggregator, the crucial question is whether the dependence function is strictly monotonic or not, as it is in the case with aggregating orders.

4 Discussion

We will conclude by a brief discussion of our framework and some key assumptions.

Aggregation of preference relations

Our framework is based on the assumption that each member of the group has an order (an ordinal ranking) over the alternatives for each variable. We think of the individual orders the following way. The cost of making judgement errors will typically be increasing in the size of the error. In monetary policy, for instance, the cost of setting the key interest rate to high or to low is typically larger the larger is the deviation of the actual and the unknown optimal key rate. In climate policies, the cost of wrongly estimating the effect of climate change on the sea level is likely to be larger the larger is the error. Similarily, outsiders' judgments of the competence of a committee member may be decreasing in the committee's judgment errors. Thus, committee members will aim at judgments that minimize the expected judgment errors. With such an aim, an order over the alternative values for a variable will follow as soon as a member has an estimate which he believes minimize the expected estimation error. To see this, suppose the committee members seek to minimize $E(x - x^*)^2$ where x^* is the true but unknown alternative in X_j. Then an order $\succsim_{i,X_j}$ will follow from $E(x_j - x^*)^2$ as soon as member i has an $x \in X_j$ which he finds minimize $E(x - x^*)^2$.

The aggregator used in the model takes the individual orders as inputs, and produces an aggregate relation for each variable. We think it is relevant to study the properties of this aggregator for at least three reasons: First, if the members of the group can not agree, but have to reach a decision, they have to use some aggregation method. Many groups resort to majority voting or some other method that implicitly take ordinal rankings as the input and output of the aggregation. They do this even though the primary interest is the highest ranked estimate. The method is often 'implicit' as the group does not pursue an explicit aggregation and spell out the aggregate ranking over pairs where the aggregate ranking is obvious. Furthermore, the aggregate ranking of some pairs may not be of interest and are therefore not explicitly spelled out. Monetary policy committees, for instance, will never make an explicit aggregation over all possible values for the key interest rate, but only pursue an explicit vote over the rankings where there is disagreement.[4] Second, in theoretical models of economics and political economy, methods where the alternatives are cast against each other in a pairwise vote is typically assumed to be the aggregation method (see e.g. Persson & Tabellini (2000)). It is therefore useful to have a characterization for such aggregation methods. Third, we want to relate to the existing literature on binary judgement aggregation and introduce a generalization of Pettit's (2001) binary discursive dilemma to non-binary quantitative judgements.

Systematicity

Our Systematicity requirement implies that our characterization only regards situations where the aggregation on each variable is independent of the aggregation on the other variables. However, our result is relevant also if this condition is violated, especially from a normative perspective. Suppose, for instance, that $k = 1$, D is concave as in Figure 1, and that the group aggregate by pairwise majority voting. Then a premise-based procedure —a procedure where the aggregate estimate for the conclusion variable follows from the aggregate estimates of the independent variable and the dependence function— will tend to give a higher estimate of the dependent variable than a conclusion based procedure where the group aggregate directly over the judgements for the dependent variable. Thus, even though the group may in fact use a premise-based procedure (which violates variable wise independence), there exist an alternative procedure, the conclusion-based procedure, which will tend to give a higher estimate than a premise based procedure. Similar effects arise when $k > 1$. Thus, our results highlight when groups face a choice between different procedures that may give different expected outcomes.

The Systematicity requirement also implies that our characterization only regards situations where the group uses the same aggregation method on each variable. We suspect that relaxing this condition will give a general impossibility. The case when the group might use different aggregation methods for different variables is instructive. Consider the following weakening of Systematicity allowing for different aggregation methods on different variables.

Weak systematicity: For any two profiles in $\mathcal{G}^n$, $g^a = (\succcurlyeq^a_{i,X_j})_{i\in N, j=1,\dots,k+1}$, $g^b = (\succcurlyeq^b_{i,X_j})_{i\in N, j=1,\dots,k+1}$, any variable j, and any alternatives $x, y \in X_j$ and $x', y' \in X_j$, if for all individuals i the restriction of $\succcurlyeq^a_{i,X_j}$ to the pair $\{x, y\}$ equals the restriction of

[4] Notice also that our framework does not require the members to have judgements on irrelevant values of a variable, and it does not require the group to aggregate judgments over irrelevant alternatives. To see this, consider again the example of Table 1. For our model to apply, it is sufficient that each member have an order over the estimates in the table, and that these are the alternatives considered in the aggregation. Thus, the set X_j may be the set of alternatives for the variable that has been put on the table, or the set of all values that the variable may take in any hypothetical world. Our results apply in both cases.

$\succcurlyeq^b_{i,X_j}$ to the pair $\{x', y'\}$ (that is, $x \succcurlyeq^a_{i,X_j} y$ if and only if $x' \succcurlyeq^b_{i,X_j} y'$ and $x \succcurlyeq^a_{i,X_j} y$ if and only if $x' \succcurlyeq^b_{i,X_j} y'$), then $x \succcurlyeq^a_{N,X_j} y$ if and only if $x' \succcurlyeq^b_{N,X_j} y'$.

Notice that the set of aggregators fulfilling Systematicity is entailed in the set of aggregators fulfilling Weak systematicity. A straight forward corollary from our proposition is therefore that there is impossibility for $k > 1$ and for non-monotonic dependence functions also under Weak systematicity. Furthermore, a corollary (less straight forward) is that there is impossibility also for cases when there is possibility under Systematicity.

Corollary 1 *If* $|X_j| > 2$ *for some* $j \in \{1, ..., k+1\}$*, no non-dictatorial aggregator* $f : \mathcal{G}^n \to \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ *that satisfies the Unanimity principle and Weak systematicity respects the dependence function.*

Proof. The proof is in the appendix. ■

It is easy to come up with examples where aggregate peaks do not respect the dependence function if $|X_1| = |X_2| = 2$. However, there is no general corollary for this case.

Notice also that there are sometimes normative or epistemic reasons for using the same aggregation method for all variables. If all members are more or less equally skilled in judging all variables then the aggregator that is optimal for one variable is presumably also optimal for the other variables.

Non-arbitrariness

The need for non-arbitrariness as a minimal rationality requirement is clear. We will now argue that a stronger requirement whereby the orders on the independent variables 'pin down' the order on X_{k+1} is too strong.[5]

Consider again the illustration in Figure 1 where $k = 1$, $X_1 = \{v, x, y, z\}$ and $X_2 = \{D(v), D(x), D(y), D(z)\}$. Say that the order on X_1 pins down the order on X_2 if $x' \succcurlyeq x'' \Leftrightarrow D(x') \succcurlyeq D(x'')$ where $x', x'' \in X_1$. Suppose someone has the order $v \succcurlyeq x \succcurlyeq y \succcurlyeq z$ over X_1. If this order pins down the order on X_2, we have that $D(v) \succcurlyeq D(x) \succcurlyeq D(y) \succcurlyeq D(z)$. These two orders forms a sequence that often will not make sense. In particular, it will not make sense if the dependent variable is the value of a policy instrument (a key interest rate, a tax rate, etc.). To see this, let φ be a target variable (inflation, pollution, etc.). Let the relationship between the policy instrument p and the target variable φ be given by $\varphi = \alpha p + \varepsilon$, where ε is a factor that affects φ which is exogenous to the policy and α is the effect of policy. Let the committee's aim be to minimize a standard objective function $W = [(\varphi - \varphi^*)^2 + \lambda p^2]$, where φ^* is the desired (target) level of variable φ, and λ is the cost changing the policy instrument. Then optimal policy is given by (first order condition)[6]

$$p = \frac{\alpha}{\alpha^2 + \lambda} (\varphi^* - \varepsilon) . \tag{1}$$

[5] A formal definition of such a condition, which entails the definition of respecting D in section 2 and non-arbitraryness, is the following: Call $(x_1, ..., x_k) \in X_1 \times ... \times X_k$ an *explanation* for $x \in X_{j+1}$ if $x = D(x_1, ..., x_k)$. Given the orders $\succcurlyeq_{X_1} \in \mathcal{G}_{X_1}, ..., \succcurlyeq_{X_{k+1}} \in \mathcal{G}_{X_{k+1}}$, say that $(x_1, ..., x_k) \in X_1 \times ... \times X_k$ dominates $(x'_1, ..., x'_k) \in X_1 \times ... \times X_k$ if $x_1 \succcurlyeq_{X_1} x'_1 \& ... \& x_k \succcurlyeq_{X_k} x'_k$. Say that a sequence $(\succcurlyeq_{X_1}, ..., \succcurlyeq_{X_{k+1}})$ *respects* D if, for all $x, x' \in X_{k+1}$ we have that $x \succcurlyeq_{X_{k+1}} x'$ whenever x has an explanation that dominates every explanation of x'.

[6] The same type of dependence function appears if only the first argument $(\varphi - \varphi^*)^2$ enters the objective function, but where the members take uncertainty into account, see Brainard (1967).

If the members agree on ε and λ, but disagree on α, we have that $p = D(\alpha)$, where $D(\alpha)$ is given by (1). The function $D(\alpha)$ is non-monotonic and concave. Turning back to Figure 1, let D illustrate $D(\alpha)$. Suppose a member has alternative v as the peak of his order over the alternatives $\{v, x, y, z\}$. For this person we have that $W = [(vp + \varepsilon - \varphi^*)^2 + \lambda p^2]$ which has its minimum at $p = D(v)$. Furthermore,

$$\frac{dW}{dp} = 2[(v - \lambda)\,p + v\,(\varepsilon - \varphi^*)] > 0.$$

Then $D(v) < D(x) < D(y) < D(z)$,and the order on X_1 must be $D(v) \succcurlyeq D(z) \succcurlyeq D(y) \succcurlyeq D(x)$, not the order that is 'pinned down'.

Similarly, requiring the orders on the independent variables pin down the order on X_{k+1} will tend to be too strong if $k > 2$. Suppose, for instance, that an advisory committee of three persons is assessing the profitability of an investment project. The members are asked to summarize their assessment in one estimate of the net present value of the project. Each member knows that ex-post, his competence will be measured by some measure that is decreasing in the absolute distance between the committee's estimate and the actual profitability of the project. Thus, each member has a single peaked order over the alternative estimates for the net present value with his estimate being the peak. Let the committee members have estimates similar to the estimates in Table 1, and let member A has the following order over the estimates for the independent variables: DCF: $10 \succcurlyeq 13$; IC: $8 \succcurlyeq 11 \succcurlyeq 12$. If the order on the independent variables should dictate the order on the dependent variable it then follows that the order on the alternatives for the NPV is $2 \succcurlyeq -1 \succcurlyeq 1$. This order is clearly at odds with his single peaked orders.

References

Blackorby, C. & Schworm, W. (1984). The structure of economies with aggregate measures of capital: A complete characterization. *The Review of Economic Studies*, 51(4), 633–650.

Brainard, W. C. (1967). Uncertainty and the effectiveness of policy. *The American Economic Review*, 57(2), 411–425.

Claussen, C. A. & Røisland, i. (2005). *Collective economic decisions and the discursive dilemma*. Working Paper 2005/3, Norges Bank.

Dietrich, F. (2007). A generalised model of judgment aggregation. *Social Choice and Welfare*, 28(4), p529 – 565.

Dietrich, F. & List, C. (2007a). Arrow's theorem in judgment aggregation. *Social Choice & Welfare*, 29(1), p19 – 33.

Dietrich, F. & List, C. (2007b). *Opinion pooling on general agendas*. Research Memoranda 038, Maastricht : METEOR, Maastricht Research School of Economics of Technology and Organization.

Genest, C. & Zidek, J. V. (1986). Combining probability distributions: A critique and an annotated bibliography. *Statistical Science*, 1(1), 114–135.

List, C. & Pettit, P. (2002). Aggregating sets of judgments: An impossibility result. *Economics and Philosophy*, 18, 89–110.

List, C. & Puppe, C. (2008). Judgment aggregation: a survey. In P. Anand, C. Puppe, & P. Pattaniak (Eds.), *Oxford Handbook of Rational and Social Choice*. Oxford University Press. Forthcoming.

Mongin, P. (1995). Consistent bayesian aggregation. *Journal of Economic Theory*, 66(2), 313–351.

Persson, T. & Tabellini, G. (2000). *Political Economics - Explaning Economic Policy*. MIT Press: Cambridge.

Pettit, P. (2001). Deliberative democracy and the discursive dilemma. *Philosophical Issues (supplement to Nous)*, 11, 268–99.

Rubinstein, A. & Fishburn, P. C. (1986). Algebraic aggregation theory. *Journal of Economic Theory*, 38(1), 63–77.

Appendix.

Proof of Proposition

For any profile $g \in \mathcal{G}^n$ and any ranking $x'_j \succcurlyeq x''_j \in f(g)$, call the set $\{i : x'_j \succcurlyeq x''_j \in \succcurlyeq_{i,X_j}\}$ a *winning coalition*. Assume that f satisfies the Unaninity principle and Systematicity. Then there is a set $\mathcal{C}$ of winning coalitions such that for all profiles $g = (\succcurlyeq_{i,X_j})_{i\in N, j=1,\dots,k+1} \in \mathcal{G}^n$, $f(g) = \{x'_j \succcurlyeq x''_j, j \in 1,..,k+1 : \{i : x'_j \succcurlyeq x''_j \in \succcurlyeq_{i,X_j}\} \in \mathcal{C}\}$.

Claim 1: $N \in \mathcal{C}$, and for every coalition $C \in N$, $C \in \mathcal{C}$ if and only if $N\backslash C \notin \mathcal{C}$.
Proof: The first part follows from the unanimity principle. The second part follows from complete aggregate relations and the universal domain (complete individual relations).

Possibility: $f : \mathcal{G}^n \rightarrow \mathcal{G}^*_{X_1} \times \dots \times \mathcal{G}^*_{X_{k+1}}$ respects the dependence function if $k = 1$ and D is strictly monotonic.
Proof: Let $k = 1$ and D be monotonic. Suppose x is the unique peak of $\succcurlyeq_{N,1} \in \mathcal{G}^*_{X_1}$. Then there is, for each $y \in X_1\backslash x$, a winning coalition $C^w_{x\succ y}$of members that strictly prefer x to y, i.e. for all $y \in X_1\backslash x$, $C^w_{x\succ y} = \{i : x \succ y \in \succcurlyeq_{i,1}\}$. Since $k = 1$, D is strictly monotonic, and each individual's orders are non-arbitrary, we have that for all members of N, $D(x) \succ D(y)$ if and only if $x \succ y$. Thus, for all $i \in C^w_{x\succ y}$, and all $D(x) \in X_2\backslash D(x)$, $D(x) \succ D(y) \in \succcurlyeq_{i,2}$. Then $f(g)$ must respect the dependence function by the second part of claim 1.

Impossibility: $f : \mathcal{G}^n \rightarrow \mathcal{G}^*_{X_1} \times \dots \times \mathcal{G}^*_{X_{k+1}}$ does not respect the dependence function if $k = 1$ and D is non-monotonic, or if $k > 1$.

Proof: Notice first that if $k > 1$ and $|X_1| = \dots = |X_{k+1}| = 2$ we have a standard binary judgment aggregation problem, and it follows from Proposition 1 in Dietrich & List (2007a) that f does not respect the dependence function. (If $k = 1$ and $|X_1| = |X_2| = 2$, f is monotonic.) We have left to prove impossibility in the cases when $|X_j| > 2$ for at least one $j \in \{1, ..., k+1\}$. The proof will proceed through claim 2-5 below.

Claim 2: For any coalitions $C, C^* \subseteq N$, if $C \in \mathcal{C}$ and $C \subseteq C^*$ then $C^* \in \mathcal{C}$.
Proof: Suppose X_j has three pairwise distinct $x, y, z \in X_j$. Suppose $C \in \mathcal{C}$ and $C \subseteq C' \subseteq N$. We have to show that $C' \in \mathcal{C}$. A partition of N is $\{C, C'\backslash C, N\backslash C'\}$. Consider a profile in $\mathcal{G}^n$ where members have the following order on X_j

$$\begin{array}{ll} z \succcurlyeq x \succcurlyeq y \succcurlyeq \dots, & \text{if } i \in C \\ z \succcurlyeq y \succcurlyeq x \succcurlyeq \dots, & \text{if } i \in C'\backslash C \\ y \succcurlyeq z \succcurlyeq x \succcurlyeq \dots, & \text{if } i \in N\backslash C' \end{array}$$

such that for each member, x, y, z are ranked above all other alternatives in X_j as indicated by "$\succcurlyeq$ ...". Then all rank z over x, hence $z \succcurlyeq_{NX_j} x$ by $N \in \mathcal{C}$. Further, exactly those in C rank x over y, so $x \succcurlyeq_{NX_j} y$ by $C \in \mathcal{C}$. Thus, if $\succcurlyeq_{NX_j}$ is to have a unique peak, $z \succcurlyeq_{NXj} y$. Hence $C' \in \mathcal{C}$, since exactly the members of C' rank z over y.

Claim 3: $\mathcal{C}$ contains a ($\subseteq$-)minimal element C^* that has at least 2 members.
Proof: If $\mathcal{C}$ is non-empty, it has an element, hence has also a minimal element C^*. By claim 1, $C^* \neq \emptyset$. By claim 2 C^*can not be singleton as then we obtain dictatorship.

By claim 3, C^* can be partioned into two non-empty disjoint sets C_1, C_2. Define $C_3 := N\backslash C^*$.

Claim 4: The coalitions C_1, C_2, C_3 are individually non-winning, but pairwise unions of them are winning.
Proof: As C^* is a ($\subseteq$-)minimal element, no partition of C^* can be winning, and hence C_1 and C_2 are not winning. As $C_3 = N \backslash C^*$, C_3 can not be winning by claim 1. As $C_1 \cup C_2 = C^*$, $C_2 \cup C_1$ is winning. Furthermore, we have that since C_1 is not winning, $C_2 \cup C_3$ must be winning by claim 1. The union $C_1 \cup C_3$ must be winning for the same reasons.

Claim 5: There is a $g \in \mathcal{G}^n$ such that f does not respect D when $|X_j| > 2$ for at least one $j \in \{1, ..., k+1\}$.
Proof: For the proof we have to go through 5 cases. When going through the cases we consider profiles where the members rank all alternatives not explicitly mentioned below the alternatives explicitly mentioned. For any relation $x \in X_j$, "$x \succcurlyeq ...$" will mean that all $y \in X_j \backslash x$ are ranked lower than x.

Let $\{C_1, C_2, C_3\}$ be a partition of N such that the coalitions C_1, C_2, C_3 are individually non-winning but pairwise unions of them are winning.

Case 1. $k = 1$ and D is non-monotonic and non-injective.

Let x, y, z be three pairwise distinct alternatives in X_1 where $D(x) = D(z)$ and $D(y)$ are the two corresponding distinct alternatives in X_2. Then there are profiles in $\mathcal{G}^n$ such that

$$g = \begin{cases} x \succ y \succ z \succcurlyeq ... & \text{and} \quad D(x) = D(z) \succ D(y) \succcurlyeq ... \quad \text{if } i \in C_1 \\ y \succ x \succcurlyeq z \succcurlyeq ... & \text{and} \quad D(y) \succ D(x) = D(z) \succcurlyeq ... \quad \text{if } i \in C_2 \\ z \succ y \succ x \succcurlyeq ... & \text{and} \quad D(x) = D(z) \succ D(y) \succcurlyeq ... \quad \text{if } i \in C_3 \end{cases}$$

For these profiles, use claim 1 for the neglected parts, we then have $f(g) = (y \succ x \succcurlyeq z \succcurlyeq ..., D(x) = D(z) \succ D(y) \succcurlyeq ...)$. This sequence does not respect the dependence function.

Case 2. $k = 1$ and D is non-monotonic and is injective.

Let x, y, z be three pairwise distinct alternatives in X_1 where $D(x), D(y)$ and $D(z)$ are the three corresponding pairwise distinct alternatives in X_2. Then there are profiles in $\mathcal{G}^n$ such that

$$g = \begin{cases} x \succ y \succ z \succcurlyeq .. & \text{and} \quad D(x) \succ D(z) \succ D(y) \succcurlyeq ... \quad \text{if } i \in C_1 \\ y \succ x \succcurlyeq z \succcurlyeq ... & \text{and} \quad D(y) \succ D(z) \succ D(x) \succcurlyeq ... \quad \text{if } i \in C_2 \\ z \succ y \succ x \succcurlyeq ... & \text{and} \quad D(z) \succ D(x) \succ D(y) \succcurlyeq ... \quad \text{if } i \in C_3 \end{cases}$$

For these profiles, use claim 1 for the neglected parts, we then have $f(g) = (y \succ x \succ z \succcurlyeq ..., D(z) \succ D(x) \succ D(y) \succcurlyeq ...)$. This sequence does not respect the dependence function.

Case 3. $k > 1$, and D is non-monotonic in one or more $j \in \{1, ..., k\}$.

Let D be non-monotonic in some variable $m \in \{1, ..., k\}$. Let $\succcurlyeq'_j$ be an order on variable j that has a unique peak. Let $Y \subseteq \mathcal{G}^n$ be the set of profiles where each member has the (same) order $\succcurlyeq'_j$ for each of the variables in $\{1, ..., k\} \backslash m$, i.e. $Y = \{g \in \mathcal{G}^n : \succcurlyeq_{i,j} = \succcurlyeq'_j \ \forall i \in N \text{ and } \forall j \in \{1, ..., k\} \backslash m\}$. By Claim 1 we then have that for all profiles $g \in Y$, $f(g)$ gives an aggregate order $\succcurlyeq_{N,j} = \succcurlyeq'_j$ for $\forall j \in \{1, ..., k\} \backslash m$. We may then plug the peaks (estimates) of $\succcurlyeq_{N,j}$ $j = \{1, ..., k\} \backslash m$ into D and consider D a function of x_m only. It then follows from case 1 that f does not respect the dependence function.

Case 4. $k > 1$, and D is strictly monotonic in all $j \in \{1, ..., k\}$ and non-injective.

Let x_1', x_1'' be two pairwise distinct alternatives in X_1, x_2', x_2'' be two pairwise distinct alternatives in X_1, and $x_3' = D(x_1', x_2')$, $x_3'' = D(x_1', x_2'') = D(x_1'', x_2')$, $x_3''' = D(x_1'', x_2'')$ be the three corresponding pairwise distinct alternatives in X_3. Then there are profiles in $\mathcal{G}^n$ such that

$$g = \begin{cases} x_1' \succ x_1'' \succcurlyeq ..., x_2' \succ x_2'' \succcurlyeq ... & \text{and} \quad x_3' \succ x_3'' \succ x_3''' \succcurlyeq ... & \text{if } i \in C_1 \\ x_1' \succ x_1'' \succcurlyeq ..., x_2'' \succ x_2' \succcurlyeq ... & \text{and} \quad x_3'' \succ x_3' \succ x_3''' \succcurlyeq ... & \text{if } i \in C_2 \\ x_1'' \succ x_1' \succcurlyeq ..., x_2' \succ x_2'' \succcurlyeq ... & \text{and} \quad x_3'' \succ x_3''' \succ x_3' \succcurlyeq ... & \text{if } i \in C_3 \end{cases}$$

For these profiles, use claim 1 for the neglected parts, we then have $f(g) = (x_1' \succ x_1'' \succcurlyeq ..., x_2' \succ x_2'' \succcurlyeq ..., x_3'' \succ x_3' \succ x_3''' \succcurlyeq ...)$.This sequence does not respect the dependence function. The extension to cases where $k > 2$ follows from letting each member agree on everything except the rankings on two independent variables and the dependent variable, and claim 1.

Case 5. $k > 1$, and D is strictly monotonic in all $j \in \{1, ..., k\}$ and injective.

Let x_1', x_1'' be two pairwise distinct alternatives in X_1, x_2', x_2'' be two pairwise distinct alternatives in X_1, and $x_3' = D(x_1', x_2')$, $x_3'' = D(x_1', x_2''), x_3''' = D(x_1'', x_2')$ and $x_3'''' = D(x_1'', x_2'')$ be the four corresponding pairwise distinct alternatives in X_3. Put $x_3' < x_3'' < x_3''' < x_3''''$ (The analysis of the other cases is similar when D is monotonic.). Then there are profiles in $\mathcal{G}^n$ such that

$$g = \begin{cases} x_1' \succ x_1'' \succcurlyeq ..., x_2' \succ x_2'' \succcurlyeq ... & \text{and} \quad x_3' \succ x_3'' \succ x_3''' \succ x_3'''' \succcurlyeq ... & \text{if } i \in C_1 \\ x_1' \succ x_1'' \succcurlyeq ..., x_2'' \succ x_2' \succcurlyeq ... & \text{and} \quad x_3'' \succ x_3''' \succ x_3' \succ x_3'''' \succcurlyeq ... & \text{if } i \in C_2 \\ x_1'' \succ x_1' \succcurlyeq ..., x_2' \succ x_2'' \succcurlyeq ... & \text{and} \quad x_3''' \succ x_3'''' \succ x_3'' \succ x_3' \succcurlyeq ... & \text{if } i \in C_3 \end{cases}$$

For this profile, use claim 1 for the neglected parts, we then have $f(g) = (x_1' \succ x_1'' \succcurlyeq ..., x_2' \succ x_2'' \succcurlyeq ..., x_3'' \succ x_3''' \succ x_3' \succ x_3'''' \succcurlyeq ...)$. This sequence does not respect the dependence function. The extension to cases where $k > 2$ follows from letting each member agree on everything except the rankings on two independent variables and the dependent variable, and claim 1.

Proof of Corollary

As the set of aggregators satisfying Systematicity is a subset of the set of aggregators satisfying Weak systematicity it follows from our propossition that no non-dictatorial aggregator $f : \mathcal{G}^n \to \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ that satisfies the Unanimity principle and Weak systematicity respects the dependence function when $k > 1$. Thus, it suffices for the proof to consider the case when $k = 1$ and D is strictly monotonic.

Put $k = 1$ and let D be strictly monotonic. Then there are two distinct elements $x, y \in X_1$ such $D(x) \neq D(y)$.

Put $|X_1| > 2$.

Let f_A and f_B be two aggregators satisfying the Unaminity principle and Systematicity. Say that the two aggregators are *different* if there is a $g \in \mathcal{G}^n$ such that $f_A(g) \neq f_B(g)$. Let 'winning coalition' be defined as in the proof of our proposition above. Denote the set of winning coalitions under f_A by $\mathcal{C}_{f_A}$, and the set of winning coalitions under f_B by $\mathcal{C}_{f_B}$. As $f_A(g) \neq f_B(g)$ we have that there is a coalition C_A such that that $C_A \in \mathcal{C}_{f_A}$ and $C_A \notin \mathcal{C}_{f_B}$. As f_A and f_B satisfy the Unaminity principle and Systematicity claim 1 and 2 in the proof of our proposition also apply to f_A and f_B. By claim 1 $C_A \in \mathcal{C}_{f_A} \Rightarrow N \backslash C_A \notin \mathcal{C}_{f_A}$. By claim 1 and claim 2 ($C_A \in \mathcal{C}_{f_A}$ and $C_A \notin \mathcal{C}_{f_B}$)$\Rightarrow$ $N \backslash C_A \in \mathcal{C}_{f_B}$. Thus, we have that there is a partition of N such that $N = \{C_A, N \backslash C_A\}$ and $C_A \in \mathcal{C}_{f_A}$ and $N \backslash C_A \in \mathcal{C}_{f_B}$.

Let f_C be an aggregator where the aggregate relation on variable 1 is determined by the same aggregation method that is used in f_A and the aggregate relation on variable 2 is determined by the same aggregation method that is used in f_B.

There is a profile $g^* \in \mathcal{G}^n$ where all members of C_A rank x strictly above all other alternatives in X_1 and $D(x)$ strictly above all other alternatives in X_2, and the members of $N \backslash C_A$ rank y strictly above all other alternatives for X_1 and $D(y)$ strictly above all other alternatives in X_2. Then $f_C(g^*)$ have peaks $x, D(y)$. As $D(x) \neq D(y)$ it follows no non-dictatorial aggregator $f : \mathcal{G}^n \rightarrow \mathcal{G}^*_{X_1} \times ... \times \mathcal{G}^*_{X_{k+1}}$ that satisfies the Unanimity principle and Weak systematicity respects the dependence function if $k = 1$, $|X_1| > 2$ and D is monotonic.