Rational and Near-Rational Bubbles Without Drift

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Overview

Rational bubbles are actually "near-rational" because transversality is not satisfied.

- Rational bubbles in asset prices.
  - Advantages: potential explanation for "excess volatility" and sustained runups above fundamental values.
  - Drawbacks: price-dividend ratio increases without bound or must assume some exogenous crash probability function that is known to all agents.

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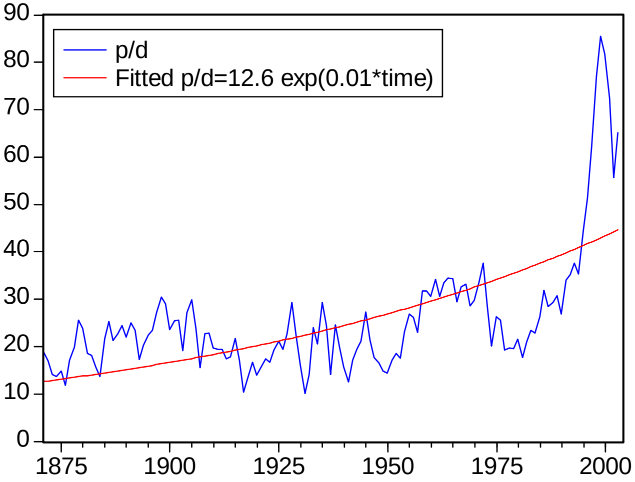
- This paper: bubbles without drift.
  - A driftless rational bubble expands and contracts over time in a irregular, wholly endogenous fashion.
  - Solve for a near-rational solution that is less computationally intensive for the agent.
  - The near-rational solution is stationary, highly persistent, and nonlinear. Does well in matching U.S. stock market data.

U.S. Price Dividend Ratio 1871 to 2003

Empirical estimate of drift rate is small, but statistically significant.



Some Thoughts About Bubbles

"It is difficult to believe that the market is literally stuck for all time on a path along which the price-dividend ratios eventually explode."

Froot and Obstfeld (AER, 1991, p. 1190).

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Hall (AER, 2001, p. 3).

"Bubbles are a viable candidate for an explanation for the volatility of asset prices, even if it is not entirely clear how bubbles should be modeled."

LeRoy (JEL, 2004, p.784)

Empirical Evidence

Bohl and Siklos (2004, QREF) and Coakley and Fuertes (2006, JBF) fit nonlinear time series models to U.S. stock market valuation ratios over the period 1871 to 2001.

$\Rightarrow$ Valuation ratios tend to drift away from fundamentals during bull markets, but these persistent departures are eventually eliminated via downward adjustments during bear markets.

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Empirical tests for nonstationarity of the U.S. price-dividend ratio:

Engsted (2006, FRL) finds support for a rational bubble (i.e., nonstationarity) in U.S. data.

Koustas and Serletis (2005, JBF) reject the rational bubble hypothesis in favor of mean-reverting behavior for the U.S. price-dividend ratio.

Lucas (1978) Asset Pricing Model

Frictionless exchange economy with a representative agent.

FOC:

$$p_t = E_t \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} (p_{t+1} + d_{t+1}),$$

$$\alpha = \text{ CRRA}, \quad \theta \equiv 1 - \alpha \quad y_t \equiv \frac{p_t}{d_t},$$

$$y_t = E_t \left[\beta \exp(\theta x_{t+1}) (y_{t+1} + 1) \right],$$

$$x_{t+1} \equiv \log \frac{d_{t+1}}{d_t}, \quad c_t = d_t$$

$$x_{t+1} = \overline{x} + \rho (x_t - \overline{x}) + \varepsilon_{t+1},$$

$$\varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2), \quad |\rho| < 1.$$

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$|\rho| < 1$.

Define: $$z_t \equiv \beta \exp\left(\theta x_t \right) \left(y_t + 1 \right)$$

FOC: $$z_t = \beta \exp\left(\theta x_t \right) \left[E_t z_{t+1} + 1 \right], \quad \left(\text{note: } y_t = E_t z_{t+1}\right).$$

Exact Fundamental Solution
(Burnside, JEDC, 1998)

$$z_t = \beta \exp(\theta x_t) \left[E_t z_{t+1} + 1 \right], \qquad \text{(FOC)}$$

$$z_t^f = \beta \exp(\theta x_t) \; E_t \left\{ 1 + \beta \exp(\theta x_{t+1}) + \beta^2 \exp(\theta x_{t+1} + \theta x_{t+2}) \ldots \right\},$$

$$= \beta \exp(\theta x_t) \left\{ 1 + \sum_{i=1}^{\infty} \beta^i \exp\left[\kappa_i + \gamma_i \, (x_t - \overline{x})\right] \right\},$$

$$\kappa_i = \theta \overline{x} i + \frac{\theta^2 \sigma_\varepsilon^2}{2(1-\rho^2)} \left[i - \frac{2\rho(1-\rho^i)}{1-\rho} + \frac{\rho^2(1-\rho^{2i})}{1-\rho^2} \right],$$

$$\gamma_i = \frac{\theta\rho(1-\rho^i)}{1-\rho},$$

Approximate Fundamental Solution

Allows analytical calculation of asset pricing moments.

$$z_t^f \simeq \exp\left[a_0 + a_1\left(x_t - \overline{x}\right)\right]$$

where a_1 solves:
$$a_1 = \frac{\theta}{1 - \rho\beta\exp\left[\theta\overline{x} + \frac{1}{2}\left(a_1\right)^2\sigma_\varepsilon^2\right]},$$

and a_0 *is given by*
$$a_0 = \log\left\{\frac{\beta\exp\left(\theta\overline{x}\right)}{1 - \beta\exp\left[\theta\overline{x} + \frac{1}{2}\left(a_1\right)^2\sigma_\varepsilon^2\right]}\right\},$$

$$\begin{aligned} y_t^f &= \frac{p_t^f}{d_t} = E_t z_{t+1}^f, \\ &\simeq \exp\left[a_0 + a_1\rho\left(x_t - \overline{x}\right) + \frac{1}{2}\left(a_1\right)^2\sigma_\varepsilon^2\right] \end{aligned}$$

Comparison with Another Approximate Solution

Calin, Chen, Cosimano, and Himonas, (Econometrica, 2005).

This paper:

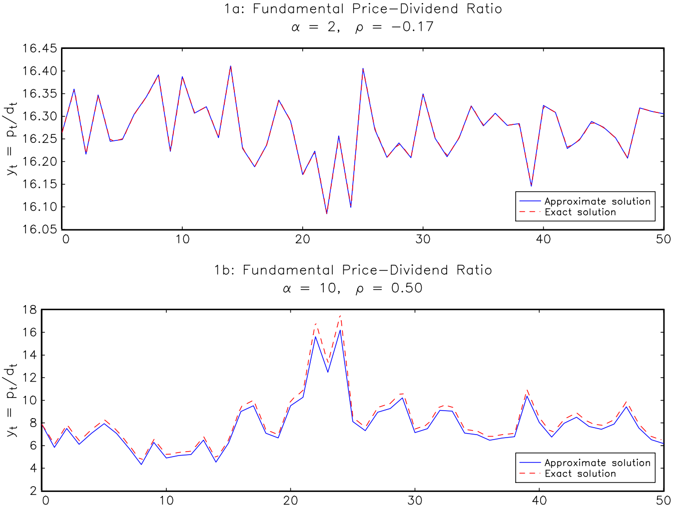
$$\underbrace{\beta \left(\frac{d_t}{d_{t-1}}\right)^{1-\alpha} \left(\frac{p_t^{\mathrm{f}}}{d_t}+1\right)}_{z_t^{\mathrm{f}}} \simeq \exp\left[a_0 + a_1\left(x_t - \overline{x}\right)\right],$$

Calin, et al. (2005):

$$\underbrace{\left(\frac{d_t}{d_{t-1}}\right)^{-\rho(1-\alpha)} \left(\frac{p_t^{\mathrm{f}}}{d_t}\right)}_{q_t^{\mathrm{f}}} \simeq \widehat{a}_0 + \sum_{i=1}^{8} \widehat{a}_i \left(x_t - \overline{x}\right)^i .$$

Approximate versus Exact Fundamental Solution

Highly accurate for realistic calibrations.



Rational Bubble Solutions

A potential explanation for "excess volatility."

$$z_t = \beta \exp(\theta x_t) \left[E_t z_{t+1} + 1 \right], \qquad \text{(FOC)}$$

$$z_t = z_t^{\text{f}} + z_t^{\text{b}} \quad \Rightarrow \quad \begin{aligned} z_t^{\text{f}} &= \beta \exp(\theta x_t) \left[E_t z_{t+1}^{\text{f}} + 1 \right] \\ z_t^{\text{b}} &= \beta \exp(\theta x_t) \; E_t z_{t+1}^{\text{b}} \end{aligned}$$

$$\underbrace{E_t z_{t+1}}_{y_t} = \underbrace{E_t z_{t+1}^{\text{f}}}_{y_t^{\text{f}}} + \underbrace{E_t z_{t+1}^{\text{b}}}_{y_t^{\text{b}}}$$

On the Transversality Condition
(and other technical arguments against bubbles).

"It is a testament to economists' capacity for abstraction that they have accepted without question that an intricate theoretical argument against bubbles has somehow migrated from the pages of *Econometrica* to the floor of the New York Stock Exchange."

LeRoy (JEL, 2004, p. 801).

Intrinsic Rational Bubbles

Evolution of the bubble depends exclusively on fundamentals.

Proposition. *There exists a continuum of intrinsic rational bubbles of the form*

$$z_t^{\text{b}} = z_{t-1}^{\text{b}} \exp\left[\lambda_0 + \lambda_1 \left(x_t - \overline{x}\right) + \lambda_2 \left(x_{t-1} - \overline{x}\right)\right], \quad z_0^{\text{b}} > 0,$$

$$y_t^{\text{b}} = y_{t-1}^{\text{b}} \exp\left[\lambda_0 + \left(\lambda_1 - \theta\right)\left(x_t - \overline{x}\right) + \left(\lambda_2 + \theta\right)\left(x_{t-1} - \overline{x}\right)\right],$$

where λ_0, λ_1, and λ_2 are any three constants that satisfy the following two equilibrium conditions

$$\tfrac{1}{2}\left(\lambda_1\right)^2 \sigma_\varepsilon^2 + \theta\overline{x} + \log\left(\beta\right) + \lambda_0 = 0,$$ drift-volatility trade-off

$$\lambda_2 = -\left(\rho\lambda_1 + \theta\right), \quad \text{recall: } \theta = 1 - \overbrace{\alpha}^{\text{CRRA}}.$$

Intrinsic Rational Bubbles Without Drift

Bubble component of the price-dividend ratio is a geometric random walk without drift.

Imposing the zero-drift restriction $\lambda_0 = 0$ yields:

$$z_t^{\mathrm{b}} = z_{t-1}^{\mathrm{b}} \exp\left[\lambda_1 \left(x_t - \overline{x}\right) + \lambda_2 \left(x_{t-1} - \overline{x}\right)\right], \quad z_0^{\mathrm{b}} > 0,$$

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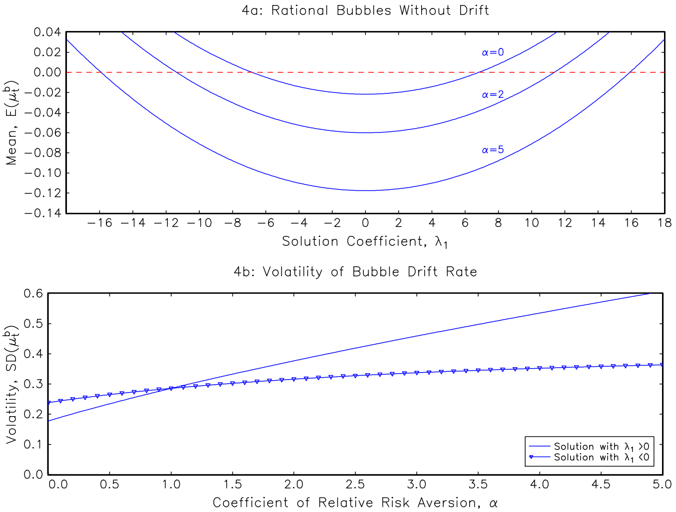
Define bubble drift rate $\mu_t^b \equiv \log\left(y_t^b / y_{t-1}^b\right), \quad y_0^b > 0,$

$$E\left(\mu_t^b\right) = 0, \qquad Var\left(\mu_t^b\right) > 0.$$

Properties of Two Rational Bubbles Without Drift

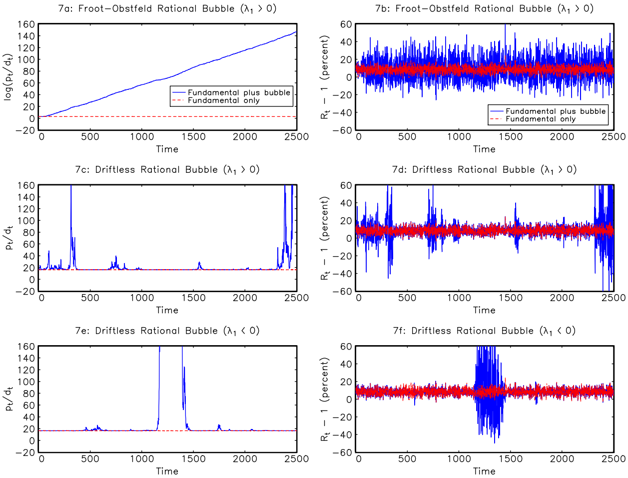
Volatility increases with the degree of risk aversion.



Rational Bubble Simulations

Left panels: Price-dividend ratio. Right panels: Equity return.

$$\overline{x} = 0.019,\ \sigma_{\varepsilon} = 0.030,\ \rho = -0.166,\ \alpha = 2,\ \text{and}\ \beta = 0.96.$$



Drawbacks of Rational Bubble Solutions

- Non-stationary behavior of price-dividend ratio (even when mean drift rate is zero).
- How select among a continuum of rational bubble equilibria?
- Computationally intensive: two separate forecasts with numerous parameters.

$$
\begin{aligned}
E_t z^{\mathrm{f}}_{t+1} &= \exp\left[a_0 + a_1\rho\left(x_t - \overline{x}\right) + \tfrac{1}{2}\left(a_1\right)^2\sigma_\varepsilon^2\right], \\
E_t z^{\mathrm{b}}_{t+1} &= z^{\mathrm{b}}_{t-1}\exp\left\{\left[\lambda_1\left(1+\rho\right) + \lambda_2\right]\left(x_t - \overline{x}\right) + \lambda_2\left(x_{t-1} - \overline{x}\right)\right. \\
&\qquad \left. + \tfrac{1}{2}\left(\lambda_1\right)^2\sigma_\varepsilon^2\right\}.
\end{aligned}
$$

Near-Rational Asset Pricing Solution

In the real world, computational resources are limited.

- Perceived Law of Motion (PLM):

$$z_t = z_{t-1} \exp\left[b\left(x_t - \overline{x}\right)\right], \qquad b \text{ is the only parameter.}$$

- Subjective forecast (using lagged information):

$$\widehat{E}_t z_{t+1} = z_{t-1} \exp\left[b\left(1+\rho\right)\left(x_t - \overline{x}\right) + \tfrac{1}{2} b^2 \sigma_\varepsilon^2\right],$$

- Actual Law of Motion (ALM):

$$z_t = \beta \exp\left(\theta x_t\right)\left[\overbrace{\widehat{E}_t z_{t+1}}^{\downarrow} + 1\right] \quad \text{(FOC)}$$

$$\simeq z_{t-1}^{k} \overline{z}^{1-k} \exp\left[m\left(x_t - \overline{x}\right)\right]$$

$$\text{where } k = k\left(b\right) \text{ and } m = m\left(b\right).$$

Restricted Perceptions Equilibrium (RPE)

Agent's forecast rule is parameterized using observable data.

- The PLM implies that b can be estimated by

$$b = \frac{Cov\left[\Delta \log\left(z_t\right),\, x_t\right]}{Var\left(x_t\right)},$$

- The ALM implies that the covariance is given by

$$\frac{Cov\left[\Delta \log\left(z_t\right),\, x_t\right]}{Var\left(x_t\right)} = \frac{\left(1-\rho\right) m\left(b\right)}{1-\rho k\left(b\right)},$$

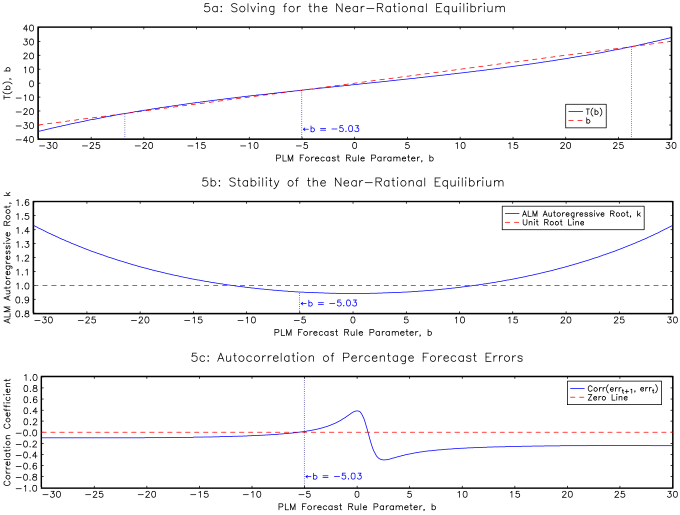
- An RPE is defined as the fixed point of the nonlinear map

$$b \;=\; T\left(b\right) \;\equiv\; \frac{\left(1-\rho\right)\; m\left(b\right)}{1-\rho\, k\left(b\right)}$$

provided $k\left(b\right) \leq 1$ so that $\Delta \log\left(z_t\right)$ remains stationary.

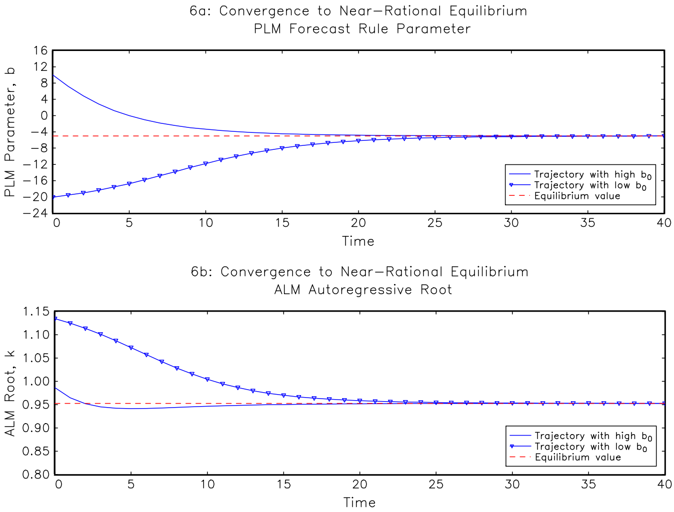
Solving for the Near-Rational Equilibrium

Forecast errors are close to white noise.



Convergence to the Near-Rational Equilibrium

The equilibrium is “Iteratively E-stable” (Evans & Honkapohja, 2001).



Model Simulations

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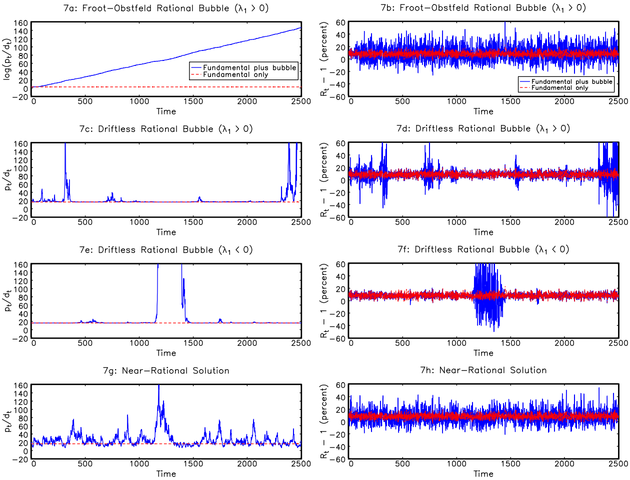


Unconditional Moments

U.S. data 1871-2003 versus model simulations.

	Statistic	U.S. Data	Model Simulations: Fundamental	Model Simulations: Near-Rational
p_t / d_t	Mean	25.7	16.3	26.0
	Std. Dev.	13.0	0.07	15.0
	Skew.	2.55	-0.01	2.52
	Kurt.	10.6	3.00	12.8
	Corr. Lag 1	0.93	-0.17	0.97
$R_t - 1$	Mean	8.30%	8.25%	7.35%
	Std. Dev	17.8%	3.95%	10.8%
	Skew.	-0.02	0.09	0.33
	Kurt.	2.78	3.02	3.19
	Corr. Lag 1	0.03	-0.29	0.15

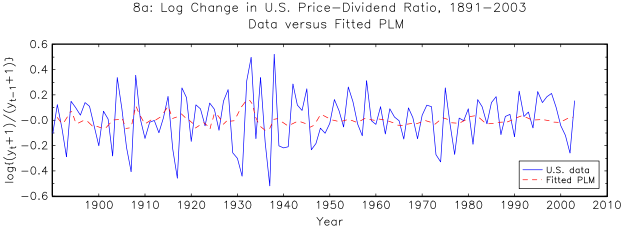
Parameters: $\overline{x} = 0.019$, $\sigma_\varepsilon = 0.030$, $\rho = -0.166$, $\alpha = 2$, and $\beta = 0.96$.

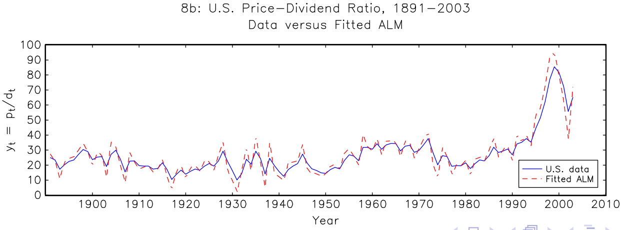
Fitting the PLM and ALM to Long-Run U.S. Data

Estimate the subjective forecast parameter b that appears in both laws of motion.

PLM: $\widehat{b} = -2.11$ (s.e. $= 0.58$) Model value: $b = -5.03$.

ALM: $\widehat{b} = -0.75$ (s.e. $= 0.89$)





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- Rational bubbles are actually "near-rational" because transversality is not satisfied.
- In reality, computational resources are limited, so further small departures from full-rationality seem plausible.
- A near-rational asset pricing solution can match the moments of U.S. data and allow prices to occasionally dip below the fundamental price.