Rational and Near-Rational Bubbles Without Drift*

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Abstract

This paper derives a general class of intrinsic rational bubble solutions in a standard Lucas-type asset pricing model. I show that the rational bubble component of the price-dividend ratio can evolve as a geometric random walk without drift, such that the mean of the bubble growth rate is zero. Driftless rational bubbles are part of a continuum of equilibrium solutions that involve an explicit trade-off between the mean and volatility of the bubble growth rate. I also propose a near-rational asset pricing solution in which the representative agent does not construct separate forecasts for the fundamental and bubble components of the asset price. Rather, the agent constructs only a single forecast for the total asset price that is similar in form to the corresponding rational forecast, but involves fewer parameters. The parameters of the agent's forecast rule are chosen to match the moments of observable data. In the near-rational equilibrium, the actual law of motion for the price-dividend ratio is stationary, highly persistent, and nonlinear. The agent's forecast errors exhibit near-zero autocorrelation at all lags, making it difficult for the agent to detect a misspecification of the forecast rule. Unlike a rational bubble, the near-rational solution allows the asset price to occasionally dip below its fundamental value. Under mild risk aversion, the near-rational solution generates pronounced low-frequency swings in the price-dividend ratio, positive skewness, excess kurtosis, and time-varying volatility—all of which are present in long-run U.S. stock market data. An additional contribution of the paper is to demonstrate an approximate analytical solution for the fundamental asset price that employs a nonlinear change of variables.

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Nowhere does history indulge in repetitions so often or so uniformly as in Wall Street. When you read contemporary accounts of booms or panics the one thing that strikes you most forcibly is how little either stock speculation or stock speculators today differ from yesterday. The game does not change and neither does human nature.

From the thinly-disguised biography of legendary speculator Jesse Livermore, by E. Lefevére (1923, p. 180).

1 Introduction

Stories involving speculative bubbles can be found throughout history in various countries and asset markets.[1] The dramatic rise in U.S. stock prices during the late 1990s, followed similarly by U.S. house prices during the mid 2000s, are episodes that have both been described as bubbles. The term "bubble" was coined in England in 1720 following the famous price run-up and crash of shares in the South Sea Company. The run-up led to widespread public enthusiasm for the stock market and a proliferation of highly suspect companies attempting to sell shares to investors. One such venture notoriously advertised itself as "a company for carrying out an undertaking of great advantage, but nobody to know what it is." The epidemic of fraudulent stock-offering schemes led the British government to pass the so-called "Bubble Act" in 1720, which was officially named "An Act to Restrain the Extravagant and Unwarrantable Practice of Raising Money by Voluntary Subscription for Carrying on Projects Dangerous to the Trade and Subjects of the United Kingdom."[2]

Numerous empirical studies starting with Shiller (1981) and LeRoy and Porter (1981) have demonstrated that stock prices appear to exhibit "excess volatility" when compared to the discounted stream of ex post realized dividends.[3] Bubble models offer a potential explanation for excess volatility because they allow stock prices to become detached from fundamentals. So-called "rational bubble" models say that agents are fully cognizant of the fundamental asset price, but nevertheless they may be willing to pay more than this amount. This can occur if expectations of future price appreciation are large enough to satisfy the rational agents's required rate of return. In the typical rational bubble model, the stock price grows faster than dividends (or cash flows) in perpetuity, i.e., the price-dividend ratio exhibits positive drift. This is clearly an unrealistic prediction for long-run stock market behavior. Indeed, Hall (2001, p. 3) dismisses the idea that "intelligent people [would] believe that the value of a stock will become larger and larger in relation to all other quantities in the economy." A more elaborate model assumes that the bubble will periodically crash according to some universally known probability function, but this is an ad hoc feature that is determined completely outside of the model.[4] LeRoy (2004, p.784) maintains that "[rational] bubbles are a viable candidate

[1] See, for example, the collection of papers in Hunter, Kaufman, and Pomerleano (2003). Gürkaynak (2007) reviews the vast literature on econometric tests for the presence of asset price bubbles.

[2] See Gerding (2006).

[3] Shiller (2003) provides a recent update on this literature.

[4] For examples, see Blanchard (1979), Blanchard and Watson (1982), Evans (1991), Fukata (1998), and Van Norden and Schaller (1999), among others.

for an explanation for the volatility of asset prices, even if it is not entirely clear how bubbles should be modeled."

This paper derives a rational bubble solution that is less susceptible to some of the above criticisms. The framework for the analysis is a standard Lucas (1978) type asset pricing model. For any given value of risk aversion, I show that there are two distinct rational bubble solutions for which the bubble component of the price-dividend ratio evolves as a geometric random walk without drift, i.e., the unconditional mean of the bubble growth rate is zero. Under each solution, the volatility of bubble innovations depends exclusively on fundamentals. Starting from an arbitrarily small positive value, a driftless bubble expands and contracts over time in a irregular, wholly endogenous fashion. Although the price-dividend ratio remains non-stationary, the equilibrium trajectory is less explosive than a bubble with positive drift. I show that driftless rational bubbles are part of a continuum of equilibrium solutions that involve an explicit trade-off between the mean and volatility of the bubble drift rate. The positive drift solution derived by Froot and Obstfeld (1991) can be recovered as a special case along this continuum.

Rational bubble models assume that agents always know the size of the bubble—to the point of constructing separate forecasts for the fundamental and bubble components of the asset price. An agent with limited computational resources may be inclined to construct only a single forecast that predicts the movement of the total asset price (fundamental plus bubble). As noted by Nerlove (1983, p. 1255): "Purposeful economic agents have incentives to eliminate errors up to a point justified by the costs of obtaining the information necessary to do so...The most readily available and least costly information about the future value of a variable is its past value." Adopting this type of approach, I solve for a near-rational equilibrium in which the agent's forecast rule is based on a geometric random walk without drift. The innovations to the random walk are linked to observable fundamentals, i.e., consumption/dividend growth. The agent's forecast rule is similar in form to the corresponding rational forecast, but it involves fewer parameters. The parameters of the forecast rule are chosen to match the moments of observable data.

In the near-rational equilibrium, the actual law of motion for the price-dividend ratio is stationary, highly persistent, and nonlinear. The agent's forecast errors exhibit near-zero autocorrelation at all lags, making it difficult for the agent to detect a misspecification of the forecast rule. Unlike a rational bubble, the near-rational solution allows the asset price to occasionally dip below its fundamental value. Under mild risk aversion, the near-rational solution generates pronounced low-frequency swings in the price-dividend ratio, positive skewness, excess kurtosis, and time-varying volatility—all of which are present in long-run U.S. stock market data.

An additional contribution of the paper is to demonstrate an approximate analytical solution for the fundamental asset price. The solution employs a change of variables that captures more of the model's nonlinearity relative to the change of variables employed by Calin, et. al (2005). The behavior of the changed variable is well-captured by a simple exponential function, as opposed to the high-order polynomial function employed by Calin, et. al (2005). I show that the approximate solution yields results that are very close to the exact theoretical solution derived by Burnside (1998) for the case of autocorrelated dividend growth.

The near-rational asset pricing solution developed here builds on earlier research that seeks to explain stock market behavior using some type of distorted belief mechanism or misspecified forecast rule in a representative agent framework. Examples along these lines include Barsky and Delong (1993), Timmerman (1996), Barberis, Schleifer, and Vishney (1998), Cecchetti, Lam, and Mark (2000), Abel (2002), Lansing (2006), and Branch and Evans (2006), among others. Bubble models that involve the interaction of rational and non-rational agents in the same economy include Delong et al. (1990), Brock and Hommes (1998), Abreu and Brunnermeier (2003), and Scheinkman and Xiong (2003).

2 The Model

Equity shares are priced using the frictionless pure exchange model of Lucas (1978). There is a representative agent who can purchase shares to transfer wealth from one period to another. Each share pays an exogenous stream of stochastic dividends in perpetuity.

The agent's problem is to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\alpha} - 1}{1-\alpha} \right], \tag{1}$$

subject to the budget constraint

$$c_t + p_t s_t = (p_t + d_t)\, s_{t-1}, \qquad c_t,\ s_t \geq 0 \tag{2}$$

where c_t is the agent's consumption in period t, β is the subjective time discount factor, and α is the coefficient of relative risk aversion (the inverse of the intertemporal elasticity of substitution). When $\alpha = 1$, the within-period utility function can be written as $\log(c_t)$. The symbol E_t represents the mathematical expectation operator evaluated using the objective distribution of dividend growth. The symbol p_t denotes the ex-dividend price of the equity share, d_t is the dividend, and s_t is the number of shares held in period t.

The growth rate of dividends $x_t \equiv \log(d_t/d_{t-1})$ is governed by the following stochastic process

$$x_t \;=\; \overline{x} + \rho\,(x_{t-1} - \overline{x}) + \varepsilon_t, \qquad \varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right), \tag{3}$$

where $|\rho| < 1$. The mean growth rate is $\overline{x}$ and the variance is given by $\sigma_\varepsilon^2/(1-\rho^2)$.

The first-order condition that governs the agent's share holdings is given by

$$p_t = E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} (p_{t+1} + d_{t+1}) \right]. \tag{4}$$

Equation (4) can be rearranged to obtain

$$1 = E_t \left[M_{t+1} R_{t+1} \right], \tag{5}$$

where $M_{t+1} \equiv \beta \left(c_{t+1}/c_t\right)^{-\alpha}$ is the stochastic discount factor and $R_{t+1} = \left(p_{t+1} + d_{t+1}\right)/p_t$ is the gross return from holding the equity share from period t to $t+1$. Defining the price-dividend ratio as $y_t \equiv p_t/d_t$, the gross equity return can be written as

$$R_{t+1} = \left(\frac{y_{t+1}+1}{y_t}\right) \exp\left(x_{t+1}\right). \tag{6}$$

Without loss of generality, shares are assumed to exist in unit net supply. Market clearing therefore implies $s_t = 1$ for all t. Substituting this equilibrium condition into the budget constraint (2) yields, $c_t = d_t$ for all t. In equilibrium, equation (4) can now be written as

$$y_t = E_t \left[\beta \exp\left(\theta x_{t+1}\right)\left(y_{t+1}+1\right)\right], \tag{7}$$

where $\theta \equiv 1 - \alpha$. Equation (7) shows that the price-dividend ratio in period t depends on the agent's joint forecast of next period's dividend growth rate x_{t+1} and next period's price-dividend ratio y_{t+1}. It is convenient to transform equation (7) using a nonlinear change of variables to obtain

$$z_t = \beta \exp\left(\theta x_t\right)\left[E_t z_{t+1} + 1\right], \tag{8}$$

where $z_t \equiv \beta \exp\left(\theta x_t\right)\left(y_t + 1\right)$. Under this formulation, z_t represents a composite variable that depends on both the growth rate of dividends and the price-dividend ratio. Equation (8) shows that the value of z_t in period t depends on the agent's conditional forecast of that same variable. By making use of the definition of z_t, equation (7) can be written as $y_t = E_t z_{t+1}$. Hence, the equilibrium price-dividend ratio is the conditional forecast of the composite variable z_{t+1}.[5]

3 Fundamental Solution

The fundamental value of the share price is uniquely pinned down by the agent's rational forecast of the discounted stream of future dividends. Equation (8) can be iterated forward to substitute out z_{t+1+k} for $k = 0, 1, 2, ...$ Applying the law of iterated expectations and imposing a transversality condition yields the following present-value pricing equation

$$\begin{aligned} z_t^{\mathrm{f}} \;&=\; \beta \exp\left(\theta x_t\right) E_t \Big\{ 1 \;+\; \beta \exp\left(\theta x_{t+1}\right) \;+\; \beta^2 \exp\left(\theta x_{t+1} + \theta x_{t+2}\right) \;+ \\ &\quad\; \beta^3 \exp\left(\theta x_{t+1} + \theta x_{t+2} + \theta x_{t+3}\right) ... \Big\}, \end{aligned} \tag{9}$$

where z_t^{f} is the fundamental value of the forecast variable. Following Burnside (1998), the expectation of the infinite sum in (9) can be explicitly evaluated to yield the following exact

[5] The appendix outlines a version of the model that allows $c_t \neq d_t$. By an appropriate change of variables, the forms of equations (3) and (8) are retained. All of the paper's theoretical results go through when expressed in terms of the transformed variables.

analytical solution

$$z_t^{\mathrm{f}} = \beta \exp(\theta x_t) \left\{ 1 + \sum_{i=1}^{\infty} \beta^i \exp\left[\kappa_i + \gamma_i \left(x_t - \overline{x}\right)\right] \right\}, \tag{10}$$

$$\kappa_i = \theta \overline{x} i + \frac{\theta^2 \sigma_\varepsilon^2}{2\left(1-\rho^2\right)} \left[i - \frac{2\rho\left(1-\rho^i\right)}{1-\rho} + \frac{\rho^2\left(1-\rho^{2i}\right)}{1-\rho^2} \right], \tag{11}$$

$$\gamma_i = \frac{\theta\rho\left(1-\rho^i\right)}{1-\rho}. \tag{12}$$

Given z_t^{f}, we can recover the fundamental price-dividend ratio by applying the definitional relationship $y_t^{\mathrm{f}} = \beta^{-1} \exp(-\theta x_t)\, z_t^{\mathrm{f}} - 1$. This procedure yields the result that y_t^{f} is equal to the infinite sum inside the curly brackets in equation (10). In the special case when $\rho = 0$, we have $\gamma_i = 0$ such that y_t^{f} is constant.[6]

In model simulations, computation of the (truncated) infinite sum in equation (10) for each realization of x_t is quite time consuming. Moreover, equation (10) does not lend itself to analytical moment calculations for the asset pricing variables of interest. To avoid these drawbacks, the following proposition presents an approximate analytical solution for z_t^{f}.

Proposition 1. *An approximate analytical solution for the fundamental value of the forecast variable is given by*

$$z_t^{\mathrm{f}} = \exp\left[a_0 + a_1\left(x_t - \overline{x}\right)\right],$$

where a_1 solves the following nonlinear equation

$$a_1 = \frac{\theta}{1 - \rho\beta \exp\left[\theta\overline{x} + \frac{1}{2}\left(a_1\right)^2 \sigma_\varepsilon^2\right]},$$

and a_0 is given by

$$a_0 = \log\left\{ \frac{\beta \exp(\theta \overline{x})}{1 - \beta \exp\left[\theta\overline{x} + \frac{1}{2}\left(a_1\right)^2 \sigma_\varepsilon^2\right]} \right\},$$

provided that $\beta \exp\left[\theta\overline{x} + \frac{1}{2}\left(a_1\right)^2 \sigma_\varepsilon^2\right] < 1$.

Proof: See appendix.

Two values of a_1 satisfy the nonlinear equation. The inequality restriction selects the value of a_1 with lower magnitude to ensure that the non-stochastic steady-state level of z_t^{f} is positive, as given by $\exp(a_0)$. The approximate solution in Proposition 1 is much simpler in

[6] Throughout the paper, I use Greek letters such as κ, γ, and λ, to represent constants implied by exact analytical solutions and English letters such as a, b, k, and m to represent constants implied by approximate analytical solutions.

structure than the one derived by Calin, et. al (2005) for their corresponding model with no habit formation. These authors numerically approximate the law of motion of the changed variable $q_t^{\mathrm{f}} \equiv \exp(-\rho\theta x_t)\, y_t^{\mathrm{f}}$ using a polynomial of the form

$$\underbrace{\left(\frac{d_t}{d_{t-1}}\right)^{-\rho(1-\alpha)}\left(\frac{p_t^{\mathrm{f}}}{d_t}\right)}_{q_t^{\mathrm{f}}} = \widehat{a}_0 + \sum_{i=1}^{8} \widehat{a}_i \left(x_t - \overline{x}\right)^i, \tag{13}$$

which involves a total of nine Taylor-series coefficients.[7] In contrast, Proposition 1, analytically approximates the law of motion of the changed variable $z_t^{\mathrm{f}} \equiv \beta \exp(\theta x_t)\left(y_t^{\mathrm{f}} + 1\right)$ using the exponential form

$$\underbrace{\beta\left(\frac{d_t}{d_{t-1}}\right)^{1-\alpha}\left(\frac{p_t^{\mathrm{f}}}{d_t} + 1\right)}_{z_t^{\mathrm{f}}} = \exp\left[a_0 + a_1\left(x_t - \overline{x}\right)\right], \tag{14}$$

which involves only two Taylor-series coefficients, a_0 and a_1. The change of variables that defines z_t^{f} captures more of the model's nonlinearity relative to that employed by Calin, et. al (2005). Moreover, the approximation in Proposition 1 exploits the curvature of the exponential function rather than relying on a very high-order polynomial in $(x_t - \overline{x})$ to capture curvature.

We can recover an approximate solution for the fundamental price-dividend ratio by applying the equilibrium relationship $y_t^{\mathrm{f}} = E_t z_{t+1}^{\mathrm{f}}$, yielding

$$y_t^{\mathrm{f}} = E_t z_{t+1}^{\mathrm{f}} = \exp\left[a_0 + a_1\rho\left(x_t - \overline{x}\right) + \frac{1}{2}\left(a_1\right)^2 \sigma_\varepsilon^2\right]. \tag{15}$$

Figure 1 compares the approximate and exact analytical solutions for two different calibrations of the model. Throughout the paper, the agent's discount factor is set equal to $\beta = 0.96$, a typical value for annual time periods. In Figure 1a, the risk coefficient is set equal to $\alpha = 2$ and the consumption growth process is calibrated to match the mean, standard deviation, and first-order autocorrelation of U.S. annual data for the growth of real per capita consumption of nondurables and services from 1890 to 2003.[8] This procedure yields $\overline{x} = 0.019$, $\sigma_\varepsilon = 0.030$, and $\rho = -0.166$. In Figure 1b, the risk coefficient is increased to $\alpha = 10$ while the persistence parameter for consumption growth is increased to $\rho = 0.5$, with the value of σ_ε adjusted downward to maintain the same volatility of consumption growth as in Figure 1a.

In Figure 1a, the approximate solution is virtually indistinguishable from the exact fundamental solution. For this calibration, the standard deviation of the fundamental price dividend ratio is tiny—only 0.07 versus a whopping 13.0 in long-run U.S. data. In Figure 1b, where the model calibration is less plausible, the price dividend ratio is more volatile, but still below the U.S. value. In this case, the approximate solution is somewhat less accurate, exhibiting a root mean squared percentage error of 4.7 percent. Collard and Juillard (2001) also find that approximation errors increase with risk aversion and the persistence of the consumption

[7] See Calin, et. al (2005), Table 1, p. 977.

[8] Long-run annual data for U.S. consumption and U.S. stock market variables are from Robert Shiller's website: http://www.econ.yale.edu/~shiller/.

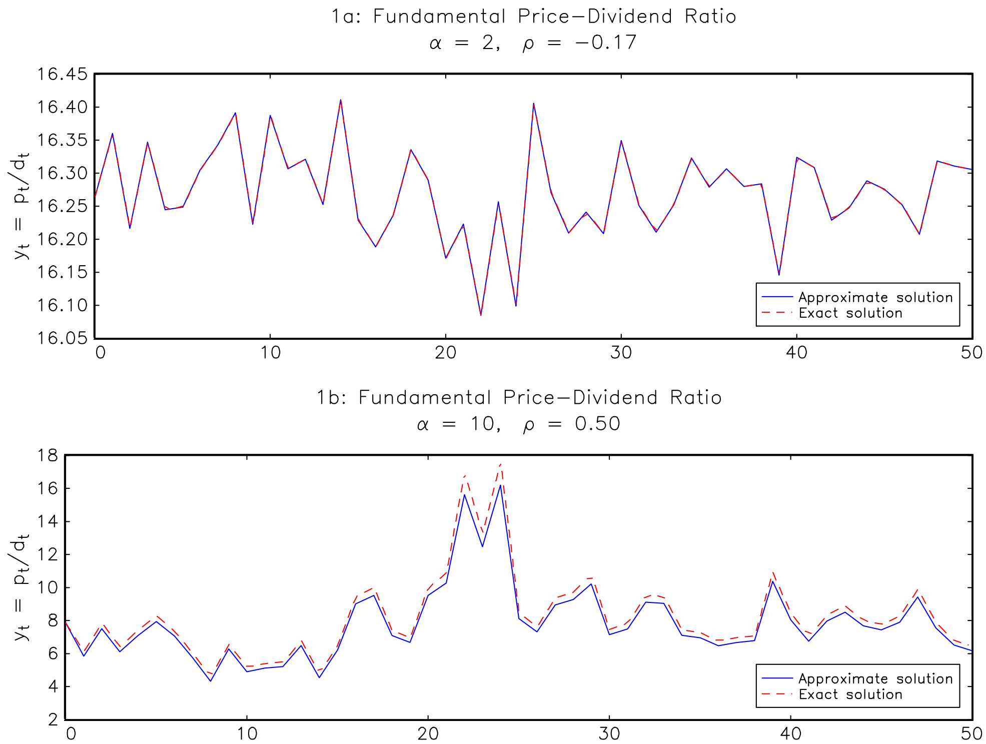

Figure 1: Approximate and exact solutions for fundamental price dividend ratio.

growth process. A more accurate approximation could be obtained by increasing the order of the polynomial that appears inside the exponential function on the right-side of equation (14). Experiments with the model show that a quadratic polynomial inside the exponential is successful in reducing the approximation error to nearly zero for the calibration of Figure 1b.

As shown in the appendix, the approximate fundamental solution can be used to derive the following expressions for the unconditional moments of the asset pricing variables

$$E\left[\log\left(y_t^{\mathrm{f}}\right)\right] = a_0 + \tfrac{1}{2}\left(a_1\right)^2\sigma_\varepsilon^2, \tag{16}$$

$$Var\left[\log\left(y_t^{\mathrm{f}}\right)\right] = \frac{\left(a_1\rho\right)^2\sigma_\varepsilon^2}{1-\rho^2}, \tag{17}$$

$$E\left[\log\left(R_{t+1}^{\mathrm{f}}\right)\right] = -\log\left(\beta\right) \; + \; \alpha\overline{x} \; - \; \tfrac{1}{2}\left(a_1\right)^2\sigma_\varepsilon^2, \tag{18}$$

$$Var\left[\log\left(R_{t+1}^{\mathrm{f}}\right)\right] = \left[\frac{\alpha^2}{1-\rho^2} + \left(a_1\right)^2 + 2\alpha a_1\right]\sigma_\varepsilon^2. \tag{19}$$

Given equations (16) through (19), the unconditional moments of y_t^{f} and R_{t+1}^{f} can be computed by making use of the properties of the log-normal distribution.[9]

4 Rational Bubble Solutions

The present-value pricing equation (9) imposes a no-arbitrage condition across all future time periods whereas equation (8) imposes a no-arbitrage condition only from period t to $t+1$. Since equation (8) does not enforce a transversality condition, it admits solutions where z_t can deviate from the fundamentals-based value. These so-called "rational bubble" solutions have been proposed as a way to account for the empirical observation that stock prices appear to be excessively volatile relative to a discounted stream of dividends or cash flows. The underlying assumption is that agents are forward-looking, but not to the extreme degree implied by the transversality condition.

Santos and Woodford (1997), Kamigashi (1998), and Montruccio and Pivileggi (2001) all discuss the many theoretical caveats that govern the existence of rational bubbles in an intertemporal competitive equilibrium. A basic intuition is that rational bubbles can usually be ruled out for the simple reason that, if a bubble existed, then an infinitely-lived agent could achieve a gain by permanently selling shares at the bubble price and then foregoing dividends on those shares. Since the rational bubble solution assumes $s_t = 1$ for all t, the solution fails to maximize infinite-horizon utility as required by the equilibrium concept. In light of such arguments, the term "rational bubble" should perhaps be considered a misnomer. Nevertheless, rational bubbles can still be viewed as a possible descriptive model of asset pricing, even if these solutions do not maximize infinite-horizon utility. Along these lines, LeRoy (2004, p. 801), remarks "It is a testament to economists' capacity for abstraction that they have accepted without question that an intricate theoretical argument against bubbles has somehow migrated from the pages of *Econometrica* to the floor of the New York Stock Exchange."

The forecast variable z_t that appears in equation (8) can be disaggregated as follows

$$z_t \;=\; z_t^{\mathrm{f}} \;+\; z_t^{\mathrm{b}}, \tag{20}$$

where z_t^{f} satisfies the present-value pricing equation (9) and hence (8). The bubble component of the forecast variable is defined as $z_t^{\mathrm{b}} \equiv \beta \exp\left(\theta x_t\right) y_t^{\mathrm{b}}$, where y_t^{b} is the bubble component of the price-dividend ratio. Substituting equation (20) into (8) yields the following expectational difference equation that governs the evolution of the bubble component

$$z_t^{\mathrm{b}} = \beta \exp\left(\theta x_t\right) E_t z_{t+1}^{\mathrm{b}}. \tag{21}$$

Together, equations (8) and (21) imply

$$\underbrace{E_t z_{t+1}}_{y_t} \;=\; \underbrace{E_t z_{t+1}^{\mathrm{f}}}_{y_t^{\mathrm{f}}} \;+\; \underbrace{E_t z_{t+1}^{\mathrm{b}}}_{y_t^{\mathrm{b}}}, \tag{22}$$

[9] If a random variable w_t is log-normally distributed, then $E\left(w_t\right) = \exp\left\{E\left[\log\left(w_t\right)\right] + \frac{1}{2} Var\left[\log\left(w_t\right)\right]\right\}$ and $Var\left(w_t\right) = E\left(w_t\right)^2 \left\{\exp\left(Var\left[\log\left(w_t\right)\right]\right) - 1\right\}$.

which shows that $E_t z_{t+1}$ is the sum of two separate forecasts that pertain to the fundamental and bubble components, respectively.

4.1 Intrinsic Rational Bubbles With Drift

The typical rational bubble solution requires the equity price to grow faster than dividends in perpetuity, i.e., the bubble component of price-dividend ratio exhibits positive drift. To illustrate the idea, Proposition 2 generalizes the intrinsic rational bubble solution of Froot and Obstfeld (1991) to allow for risk aversion and autocorrelated dividend growth. Froot and Obstfeld (1991) consider the special case of $\rho = 0$ and $\alpha = 0$ (such that $\theta = 1$).[10] The bubble is "intrinsic" because the stochastic drift rate depends exclusively on fundamentals; there is no exogenous crash probability appended to the model.

Proposition 2. *The Froot-Obstfeld intrinsic rational bubble takes the form*

$$\begin{aligned} z_t^{\mathrm{b}} &= \eta \left[\frac{d_t^{\lambda_1}}{d_{t-1}^{\rho\lambda_1+\theta}} \right], \qquad \eta > 0, \\ &= z_{t-1}^{\mathrm{b}} \exp\left[\lambda_1 x_t - (\rho\lambda_1 + \theta)\, x_{t-1}\right], \qquad z_0^{\mathrm{b}} > 0, \end{aligned}$$

where d_t is the level of dividends, η is an arbitrary positive constant that determines z_0^{b}, and λ_1 is a root of the quadratic equation

$$\tfrac{1}{2} (\lambda_1)^2 \sigma_\varepsilon^2 + \lambda_1 \overline{x} (1-\rho) + \log(\beta) = 0.$$

Proof: See appendix.

Given z_t^{b}, we can recover the bubble component of the price-dividend ratio by applying the definitional relationship $y_t^{\mathrm{b}} = \beta^{-1} \exp(-\theta x_t)\, z_t^{\mathrm{b}}$, yielding

$$\begin{aligned} y_t^{\mathrm{b}} &= \frac{\eta}{\beta} \left[\frac{d_t^{\lambda_1-\theta}}{d_{t-1}^{\rho\lambda_1}} \right], \\ &= y_{t-1}^{\mathrm{b}} \exp\left[(\lambda_1 - \theta)\, x_t - \rho\lambda_1 x_{t-1}\right], \qquad y_0^{\mathrm{b}} > 0. \end{aligned} \tag{23}$$

From equation (23), we see that the rational bubble must be positive and must exist from the first day of trading onwards, as demonstrated originally by Diba and Grossman (1988). Defining the stochastic bubble drift rate as $\mu_t^{\mathrm{b}} \equiv \log\left(y_t^{\mathrm{b}}/y_{t-1}^{\mathrm{b}}\right)$, we have

$$E\left(\mu_t^{\mathrm{b}}\right) = \left[\lambda_1 (1-\rho) - \theta\right] \overline{x}, \tag{24}$$

$$Var\left(\mu_t^{\mathrm{b}}\right) = \left[(\lambda_1 - \theta)^2 + (\rho\lambda_1)^2 - 2\rho^2 \lambda_1 (\lambda_1 - \theta)\right] \frac{\sigma_\varepsilon^2}{1-\rho^2}. \tag{25}$$

The quadratic equation that determines the value of λ_1 has two roots—one positive and one negative. The positive root is associated with an expanding bubble $E\left(\mu_t^{\mathrm{b}}\right) > 0$ while

[10] Bidarkota and Dupoyet (2007) generalize the Froot-Obstfeld solution to allow for non-Gaussian shocks.

the negative root is associated with a collapsing bubble $E\left(\mu_t^{\mathrm{b}}\right) < 0$. A collapsing bubble will become vanishingly small as $t \to \infty$, so attention is typically restricted to the positive root.[11] Starting from an arbitrarily small positive value $y_0^{\mathrm{b}} > 0$, the positive root solution predicts that price-dividend ratio $y_t = y_t^{\mathrm{f}} + y_t^{\mathrm{b}}$ will increase without bound, never returning to the vicinity of the fundamental value y_t^{f}.

Proposition 3 presents an alternate rational bubble solution that exists only when $\rho \neq 0$.

Proposition 3. *Provided that $\rho \neq 0$, there exists an intrinsic rational bubble of the form*

$$z_t^{\mathrm{b}} \;=\; z_{t-1}^{\mathrm{b}} \exp\left[\lambda_0 + \lambda_1 \left(x_t - \overline{x}\right)\right], \qquad z_0^{\mathrm{b}} \;>\; 0,$$

where λ_0 and λ_1 are given by

$$\lambda_0 \;=\; -\log\left(\beta\right) \;-\; \theta\overline{x} \;-\; \frac{\theta^2\sigma_\varepsilon^2}{2\rho^2},$$

$$\lambda_1 \;=\; -\frac{\theta}{\rho}.$$

Proof: See appendix.

Again solving for $y_t^{\mathrm{b}} = \beta^{-1}\exp\left(-\theta x_t\right) z_t^{\mathrm{b}}$ and defining $\mu_t^{\mathrm{b}} \equiv \log\left(y_t^{\mathrm{b}}/y_{t-1}^{\mathrm{b}}\right)$, Proposition 3 implies

$$E\left(\mu_t^{\mathrm{b}}\right) \;=\; -\log\left(\beta\right) \;-\; \theta\overline{x} \;-\; \frac{\theta^2\sigma_\varepsilon^2}{2\rho^2}, \tag{26}$$

$$Var\left(\mu_t^{\mathrm{b}}\right) \;=\; \left[\frac{1}{1-\rho^2} + 2\rho\right]\frac{\theta^2\sigma_\varepsilon^2}{\rho^2}. \tag{27}$$

Figure 2 plots the mean and volatility of the bubble drift rate for the foregoing bubble solutions as the risk coefficient α is varied. As before, $\beta = 0.96$ and the consumption growth process is calibrated to match U.S. data from 1890 to 2003. As risk aversion increases, the Froot-Obstfeld expanding bubble grows faster and is more volatile. The Froot-Obstfeld collapsing bubble exhibits very high volatility for any value of α. The high volatility (which is known to the agent) raises the value of $E_t z_{t+1}^{\mathrm{b}}$ via Jensen's inequality, thereby allowing equation (21) to be satisfied with a negative mean drift rate. The alternate bubble solution in Proposition 3 yields $E\left(\mu_t^{\mathrm{b}}\right) > 0$ for low levels of risk aversion, but $E\left(\mu_t^{\mathrm{b}}\right) < 0$ for high levels of risk aversion. Volatility is zero when $\alpha = 1$ (the case of logarithmic utility) but increases with risk aversion thereafter. These results illustrate the existence of a trade-off between the mean and the volatility of the bubble drift rate in order to satisfy the intertemporal no-arbitrage condition (21).

Figure 3 plots the U.S. price-dividend ratio from 1871 to 2003 together with an estimated exponential time trend. The estimated annual drift rate is 0.0096 (s.e. = 0.001). If this trend

[11] The sum and product of expanding and collapsing bubble components can also be valid solutions to equation (21). Ikeda and Shibata (1992) examine bubble solutions of this type.

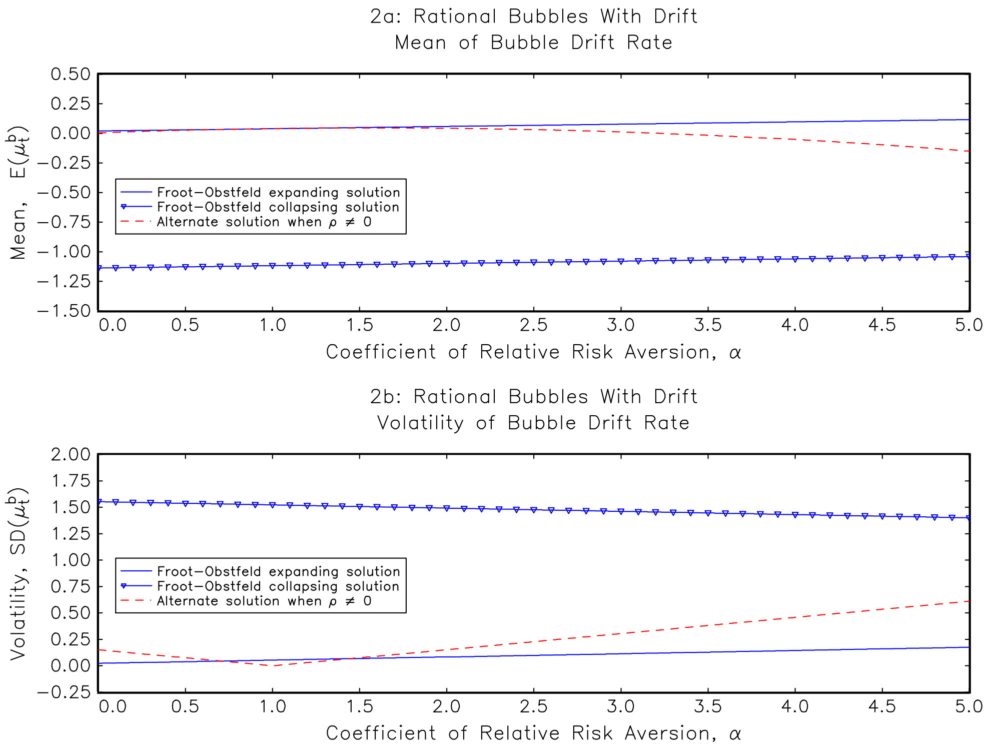

Figure 2: Mean and volatility of bubble drift rate for three different rational bubble solutions.

were to continue indefinitely, as implied by a rational bubble with drift, then the U.S. ratio would double every 72 years. When $\alpha = 2$, equation (24) predicts a positive drift rate of 0.059, while equation (26) predicts a positive drift rate of 0.043. These drift rates imply doubling times of only 12 to 16 years. Smaller predicted drift rates and longer predicted doubling times could be obtained by increasing the calibrated value of β or, in the case of equation (26), increasing the degree of risk aversion α. Froot and Obstfeld (1991, p. 1190) acknowledge that "It is difficult to believe that the market is literally stuck for all time on a path along which price-dividend ratios eventually explode." They argue, however, that explosive price-dividend ratios would not necessarily be observed over a finite sample period. Driffill and Sola (1998) augment the Froot-Obstfeld model to allow for regime-switching dividends. They argue that the incremental explanatory contribution of the expanding bubble component is low, relative to the regime-switching fundamentals. Their data set only extends through 1988, however, and thus does not include the dramatic, bubble-like rise in the U.S. price-dividend ratio that appears at the end of the sample in Figure 3.

4.2 Intrinsic Rational Bubbles Without Drift

Proposition 4 presents a solution to equation (21) where the mean drift rate is zero by construction.

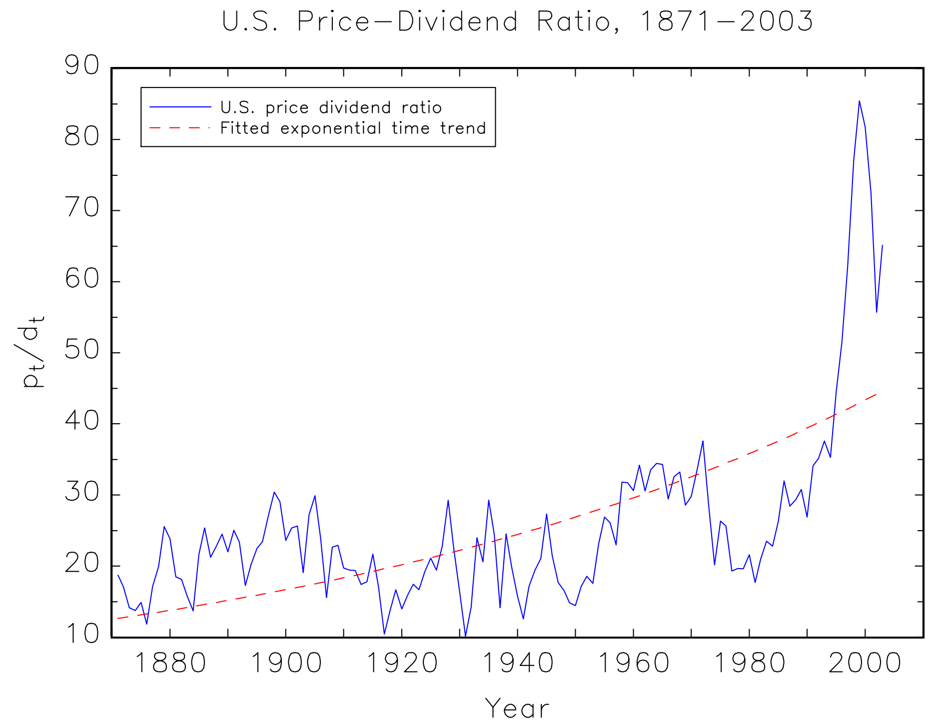

Figure 3: The annual drift rate in the U.S. price-dividend ratio is about 0.01.

Proposition 4. *An intrinsic rational bubble without drift takes the form*

$$z_t^{\mathrm{b}} \;=\; z_{t-1}^{\mathrm{b}} \exp\left[\lambda_1 \left(x_t - \overline{x}\right) + \lambda_2 \left(x_{t-1} - \overline{x}\right)\right], \qquad z_0^{\mathrm{b}} \;>\; 0,$$

where λ_1 *and* λ_2 *are given by*

$$\lambda_1 \;=\; \pm\sqrt{\frac{-2\log\left(\beta\right) - 2\theta\overline{x}}{\sigma_\varepsilon^2}},$$

$$\lambda_2 \;=\; -\;\left(\rho\lambda_1 \;+\; \theta\right).$$

Proof: See appendix.

The results of Proposition 4 are the same, regardless of whether the agent is assumed to make use of the contemporaneous or lagged realization of z_t^{b} when forming $E_t z_{t+1}^{\mathrm{b}}$. Solving for $y_t^{\mathrm{b}} = \beta^{-1}\exp\left(-\theta x_t\right) z_t^{\mathrm{b}}$ yields the following law of motion for the bubble component of the price-dividend ratio

$$y_t^{\mathrm{b}} \;=\; y_{t-1}^{\mathrm{b}} \exp\left[\left(\lambda_1 - \theta\right)\left(x_t - \overline{x}\right) \;-\; \rho\lambda_1\left(x_{t-1} - \overline{x}\right)\right], \qquad y_0^{\mathrm{b}} \;>\; 0, \tag{28}$$

where I have made the substitution $\lambda_2 = -\left(\rho\lambda_1 + \theta\right)$. The above equation shows that y_t^{b}

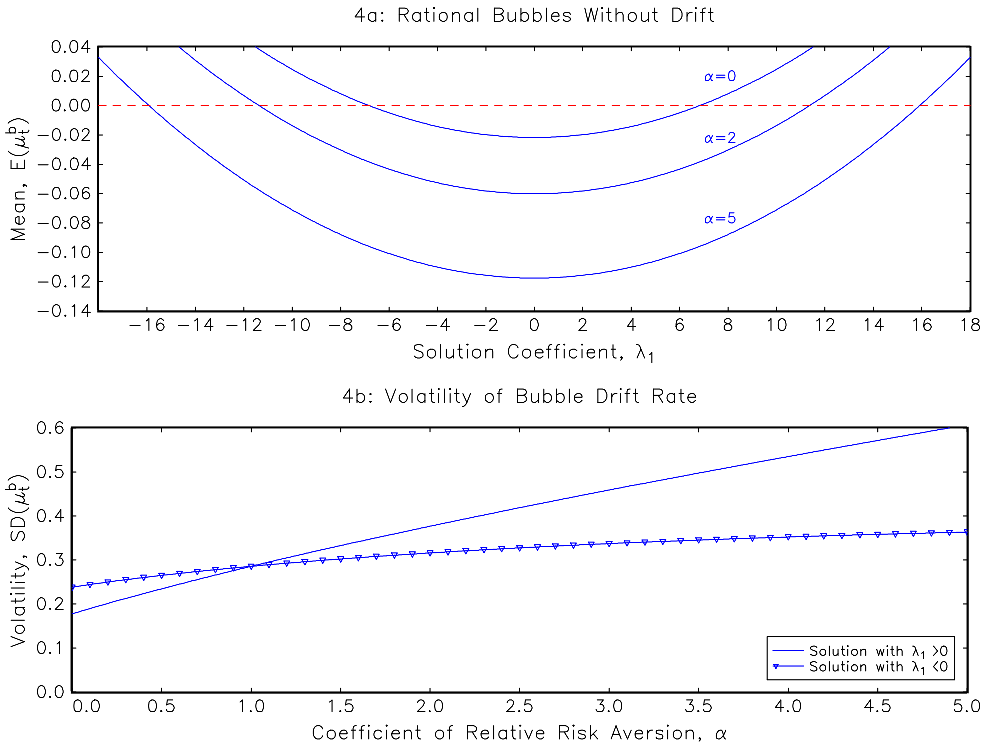

Figure 4: Properties of two rational bubble solutions without drift.

follows a geometric random walk without drift, where bubble innnovations are linked to consumption/dividend growth. Equation (28) implies

$$E\left(\mu_t^{\mathrm{b}}\right) = 0, \tag{29}$$

$$Var\left(\mu_t^{\mathrm{b}}\right) = \left[(\lambda_1-\theta)^2+(\rho\lambda_1)^2-2\rho^2\lambda_1\left(\lambda_1-\theta\right)\right]\frac{\sigma_\varepsilon^2}{1-\rho^2}. \tag{30}$$

Figure 4a plots the values of λ_1 that satisfy the no-drift equilibrium condition $(\lambda_1)^2\,\sigma_\varepsilon^2/2+\theta\overline{x}+\log(\beta)=0$. The equilibrium values are given by the intersections with the horizontal zero line. For each value of the risk coefficient α, there are two values of λ_1 that yield a driftless bubble such that $E\left(\mu_t^{\mathrm{b}}\right)=0$. Figure 4b plots the volatility of the bubble drift rate as a function of the risk coefficient. The bubble solution with $\lambda_1>0$ has lower volatility for $\alpha<1$, but higher volatility for $\alpha>1$. Both solutions exhibit more volatility as risk aversion increases.

Proposition 5 shows that the bubble solutions presented in Propositions 2, 3, and 4 are special cases along a continuum of rational bubble equilibria.

Proposition 5. *There exists a continuum of intrinsic rational bubbles of the form*

$$z_t^{\mathrm{b}} \;=\; z_{t-1}^{\mathrm{b}} \exp\left[\lambda_0 + \lambda_1\,\left(x_t - \overline{x}\right) + \lambda_2\,\left(x_{t-1} - \overline{x}\right)\right], \qquad z_0^{\mathrm{b}} \;>\; 0,$$

where λ_0, λ_1, *and* λ_2 *are any three constants that satisfy the following two equilibrium conditions*

$$\tfrac{1}{2}\left(\lambda_1\right)^2 \sigma_\varepsilon^2 \;+\; \theta\overline{x} \;+\; \log\left(\beta\right) \;+\; \lambda_0 \;=\; 0,$$

$$\lambda_2 \;=\; -\;\left(\rho\lambda_1 \,+\, \theta\right).$$

Proof: See appendix.

The proof of Proposition 5 shows that the results are the same, regardless of whether the agent is assumed to make use of the contemporaneous or lagged realization of z_t^{b} when forming the rational expectation $E_t z_{t+1}^{\mathrm{b}}$. A continuum of solutions exists because the agent's forecast rule for $E_t z_{t+1}^{\mathrm{b}}$ is overparameterized, relative to what is needed to satisfy the intertemporal no-arbitrage condition (21). The equilibrium condition that relates the drift parameter λ_0 to the volatility parameter λ_1 formalizes the trade-off between the mean and the volatility of the bubble drift rate. To recover the Froot-Obstfeld bubble in Proposition 2, we impose $\lambda_0 = \left(\lambda_1 + \lambda_2\right)\overline{x}$. To recover the bubble with drift in Proposition 3, we impose $\lambda_2 = 0$. To recover the driftless bubble in Proposition 4, we impose $\lambda_0 = 0$. Other valid solutions can be obtained by imposing $\lambda_1 = 0$ or say, by imposing the arbitrary restriction $\lambda_1 = \lambda_2$. The volatility of the bubble drift rate $Var\left(\mu_t^{\mathrm{b}}\right)$ can be minimized by imposing $\lambda_1 = \theta$. A so-called "time bubble" occurs when $Var\left(\mu_t^{\mathrm{b}}\right) = 0$. If $\rho = 0$, a time bubble can be obtained by setting $\lambda_1 = \theta$. If $\rho \neq 0$ and $\alpha = 1$ (such that $\theta = 0$), a time bubble can be obtained by setting $\lambda_2 = 0$.

5 A Near-Rational Asset Pricing Solution

All of the rational bubble solutions derived in the previous section imply non-stationary behavior of the price-dividend ratio. The solutions require the representative agent to construct both a fundamental forecast $E_t z_{t+1}^{\mathrm{f}}$ and bubble forecast $E_t z_{t+1}^{\mathrm{b}}$ each period. Furthermore, the model is silent on how the agent would choose among a continuum of rational bubble forecasts.

As an alternative to a rational bubble, this section presents a near-rational asset pricing solution that: (1) requires the agent to construct only a single forecast each period, (2) involves a parsimonious forecast rule that is parameterized by matching the moments of the observable data, and (3) yields a stationary, but highly persistent nonlinear law of motion for the price-dividend ratio.

I assume that the agent's perceived law of motion (PLM) for the total forecast variable $z_t = z_t^{\mathrm{f}} + z_t^{\mathrm{b}}$ is given by

$$z_t \;=\; z_{t-1} \exp\left[b\left(x_t - \overline{x}\right)\right], \qquad z_0 \;>\; 0, \tag{31}$$

which is a geometric random walk without drift. The functional form of the PLM bears similarity to both the approximate fundamental solution in Proposition 1 and the driftless rational bubble solution in Proposition 4. For an agent with limited computational resources, equation (31) is an attractive candidate PLM because it allows for nonstationary bubble behavior and involves only a single parameter b that can be readily estimated from observable data. The estimated version of the PLM can be used to construct a single forecast that predicts the movement of the total asset price (fundamental plus bubble).

In constructing the subjective forecast $\widehat{E}_t z_{t+1}$, I assume that the agent cannot make use of the contemporaneous realization z_t, but rather uses the lagged realization z_{t-1}. Use of the lagged realization ensures that the forecast is "operational." Since equation (8) implies that z_t depends on the agent's own forecast, it is not clear how the agent could make use of z_t when constructing the forecast in real-time. A lagged information assumption is commonly used in adaptive learning models because it avoids simultaneity in the determination of the actual and expected values of the forecast variable.

As in a rational solution, I assume the representative agent is endowed with the knowledge of the stochastic process for dividends. The underlying assumption is that enough time has elapsed for the agent to correctly identify the stochastic process from observable data. With the above assumptions, the PLM can be iterated ahead two periods to compute the following subjective forecast:

$$\widehat{E}_t z_{t+1} \;=\; z_{t-1} \exp\left[b\,(1+\rho)\,(x_t - \overline{x}) + \tfrac{1}{2} b^2 \sigma_\varepsilon^2\right], \tag{32}$$

where the exponential term can be interpreted as a time-varying extrapolation factor applied to the most recent observation.[12] For comparison, the rational forecast implied by Propositions 1 and 4 is given by

$$\begin{aligned} E_t z_{t+1} \;=\; & \underbrace{\exp\left[a_0 + a_1 \rho\,(x_t - \overline{x}) + \tfrac{1}{2}(a_1)^2 \sigma_\varepsilon^2\right]}_{E_t z_{t+1}^{\mathrm{f}}} \\ & + \underbrace{z_{t-1}^{\mathrm{b}} \exp\left\{\left[\lambda_1 (1+\rho) + \lambda_2\right](x_t - \overline{x}) + \lambda_2 (x_{t-1} - \overline{x}) + \tfrac{1}{2}(\lambda_1)^2 \sigma_\varepsilon^2\right\}}_{E_t z_{t+1}^{\mathrm{b}}}. \end{aligned} \tag{33}$$

where, for symmetry, I have assumed that the rational bubble forecast $E_t z_{t+1}^{\mathrm{b}}$ makes use of the lagged realization z_{t-1}^{b}. Notice that the rational bubble forecast also involves the application of a time-varying extrapolation factor to the recent observation z_{t-1}^{b}. Not counting $\overline{x}$, ρ, and σ_ε which are presumed known, the rational forecast rule (33) involves four separate parameters $(a_0,\ a_1,\ \lambda_1,\ \text{and}\ \lambda_2)$, as opposed to the subjective forecast rule (32) which involves only a single parameter b. An agent with limited computational resources might be inclined to adopt the more parsimonious forecast rule (32).

Substituting the subjective forecast rule (32) into equation (8) in place of a rational forecast yields the following actual law of motion (ALM):

$$z_t \;=\; \beta \exp(\theta x_t)\left\{z_{t-1} \exp\left[b\,(1+\rho)\,(x_t - \overline{x}) + \tfrac{1}{2} b^2 \sigma_\varepsilon^2\right] + 1\right\}, \tag{34}$$

[12] Lansing (2006) considers a model in which the agent's PLM is given by $z_t = z_{t-1} \exp(v_t)$, where $v_t \sim N\left(0, \sigma_v^2\right)$ is a perceived exogenous shock that is unrelated to consumption/dividend growth. In this case, the extrapolation factor is constant rather than time-varying.

which is nonlinear and autoregressive. The ALM for the price-dividend ratio can be recovered from the above expression by making use of the near-rational equilibrium relationship $y_t = \widehat{E}_t z_{t+1}$, where $\widehat{E}_t z_{t+1}$ is given by equation (32) with $z_{t-1} = \beta \exp(\theta x_{t-1})(y_{t-1}+1)$. This procedure yields

$$y_t \;=\; (y_{t-1}+1)\,\beta \exp\left[b\,(1+\rho)\,(x_t-\overline{x})+\theta x_{t-1}+\tfrac{1}{2}b^2\sigma_\varepsilon^2\right], \tag{35}$$

which is also nonlinear and autoregressive. The stationarity properties of the price-dividend ratio depend on the value of the forecast parameter b. For comparison, equation (31) can be converted into the following PLM for the price-dividend ratio

$$y_t \;=\; (y_{t-1}+1)\exp\left[(b-\theta)\,(x_t-\overline{x})+\theta\,(x_{t-1}-\overline{x})\right]\,-\,1, \tag{36}$$

which is similar, but not identical, to the form of the ALM (35).

5.1 Near-Rational Equilibrium

This section derives a near-rational, "restricted perceptions equilibrium" in which the forecast parameter b is pinned down using the moments of observable data.[13] Since the agent's PLM (31) implies that z_t is nonstationary, it is natural to assume that the agent's forecast rule is parameterized to match the covariance of $\Delta \log(z_t)$ and x_t.

The PLM implies that b is given by

$$b \;=\; \frac{Cov\left[\Delta \log(z_t)\,,\, x_t\right]}{Var\,(x_t)}, \tag{37}$$

where both $Cov\left[\Delta \log(z_t)\,,\, x_t\right]$ and $Var\,(x_t)$ can be computed from observable data. An analytical expression for the observable covariance can be derived using the following log-linear approximation to the nonlinear ALM (34):

$$z_t \;\simeq\; z_{t-1}^{k}\,\overline{z}^{\,1-k}\,\exp\left[m\,\left(x_t-\overline{x}\right)\right], \tag{38}$$

where k, m, and $\overline{z} \equiv \exp\left(E\left[\log(z_t)\right]\right)$ are Taylor-series coefficients. If $k=1$ and $m=b$, then the approximate ALM (38) will coincide exactly with the PLM (31). Straightforward computations yield the following expressions for the Taylor-series coefficients

$$k \;=\; \beta \exp\left[\theta \overline{x}+\tfrac{1}{2}b^2\sigma_\varepsilon^2\right], \tag{39}$$

$$m \;=\; \theta + b\,(1+\rho)\,\beta \exp\left[\theta \overline{x}+\tfrac{1}{2}b^2\sigma_\varepsilon^2\right], \tag{40}$$

$$\overline{z} \;=\; \frac{\beta \exp\left(\theta \overline{x}\right)}{1-\beta \exp\left[\theta \overline{x}+\tfrac{1}{2}b^2\sigma_\varepsilon^2\right]}, \tag{41}$$

[13] The restricted perceptions equilibrium concept is described in Evans and Honkapohja (2001, Chapter 13).

which all depend in a nonlinear way on the subjective forecast parameter b. The approximate law of motion of $\Delta \log(z_t)$ can be computed directly from equation (38), which in turn yields the following expression for the relevant covariance

$$Cov\left[\Delta \log(z_t),\, x_t\right] \;=\; \left[\frac{(1-\rho)\, m}{1-\rho k}\right] Var\left(x_t\right), \tag{42}$$

which is nonlinear in b via the expressions for k and m. Details are contained in the appendix. Equations (37) and (42) can be combined to form the following definition of equilibrium.

Definition 1. *A near-rational "restricted perceptions equilibrium" is defined as a perceived law of motion (31), an approximate actual law of motion (38), and a subjective forecast rule parameter b, such that the equilibrium value b is given by the fixed point of the nonlinear map*

$$b \;=\; T(b) \;\equiv\; \frac{(1-\rho)\; m(b)}{1-\rho\, k(b)},$$

where $k(b)$ and $m(b)$ are parameters of the approximate actual law of motion that depend on b, as given by equations (39) and (40), provided that $k(b) \leq 1$.

In equilibrium, we require $k(b) \leq 1$ so that $\Delta \log(z_t)$ remains stationary, thereby allowing $Cov\left[\Delta \log(z_t),\, x_t\right]$ to be computed from observable data. If $k(b) < 1$, then $\log(z_t)$ is stationary.

The approximate ALM (38) can be used to derive the following analytical expressions for the unconditional moments of the asset pricing variables

$$E\left[\log(y_t)\right] \;=\; \log\left[\frac{k(b)}{1-k(b)}\right], \qquad k(b) < 1, \tag{43}$$

$$E\left[\log(R_{t+1})\right] \;=\; -\log(\beta) \;+\; \alpha \overline{x} \;-\; \tfrac{1}{2} b^2 \sigma_{\varepsilon}^2, \tag{44}$$

where $k(b)$ is given by equation (39). Notice that the expression for $E\left[\log(R_{t+1})\right]$ has the same form as the fundamental mean return $E\left[\log\left(R^{\mathrm{f}}_{t+1}\right)\right]$ given by equation (18), except that $(a_1)^2$ is replaced here by b^2. At the baseline calibration, we have $a_1 = -0.86$. As shown in the next section, the near-rational equilibrium yields $b = -5.03$, so the near-rational mean return is below that of the fundamental mean return. This result can be traced to a small amount of optimism in the near-rational forecast rule (also shown in the next section). Optimism has an effect on the mean return that is similar to increasing patience about future payoffs via a higher value for the discount factor β. The appendix outlines the derivation of analytical expressions for the unconditional variances $Var\left[\log(y_t)\right]$ and $Var\left[\log(R_{t+1})\right]$.

5.2 Numerical Solution for the Equilibrium

The complexity of the nonlinear map $b = T(b)$ necessitates a numerical solution for the equilibrium. Parameters are set to the same values used in Figure 1a.

Figure 5a plots $T(b)$ over the range $-30 \leq b \leq 30$. There are three fixed points. Only the middle fixed point yields a stationary equilibrium such that $k < 1$, as shown in Figure 5b. At

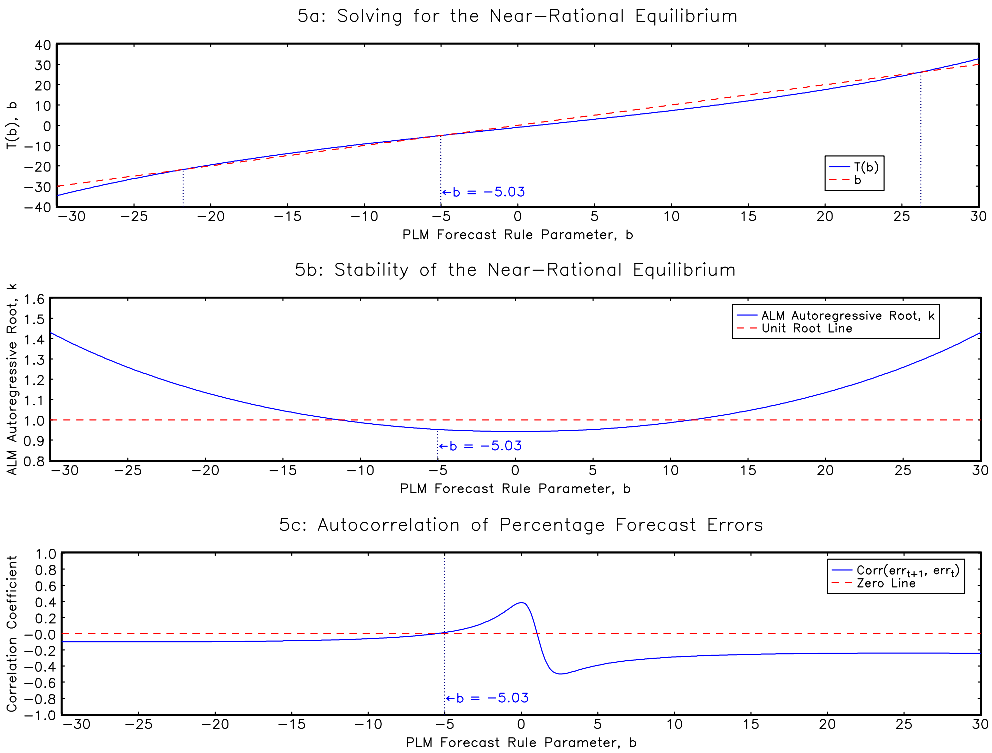

Figure 5: Numerical solution and properties of the near-rational equilibrium.

the middle fixed point, we have $b = -5.03$, and $k = 0.95$. The response coefficient on $(x_t - \overline{x})$ in the approximate ALM is $m = -4.99$. Recall that when $k = 1$ and $m = b$, the approximate ALM (38) coincides exactly with the PLM (31). At the middle fixed point, we have $k \simeq 1$ and $m \simeq b$, such that the equilibrium can be described as "near-rational."

Making use of the approximate ALM (38) and the subjective forecast rule (32), the percentage forecast error observed by the agent is given by

$$
\begin{aligned}
err_{t+1} &= \log\left(\frac{z_{t+1}}{\widehat{E}_t z_{t+1}}\right), \\
&= \log\left\{\frac{z_t^k\,\overline{z}^{\,1-k}\exp\left[m\,(x_{t+1}-\overline{x})\right]}{z_{t-1}\exp\left[b\,(1+\rho)\,(x_t-\overline{x})+\frac{1}{2}b^2\sigma_\varepsilon^2\right]}\right\}, \qquad (45)
\end{aligned}
$$

where k and m are given by equations (39) and (40). Recalling that $\overline{z} = \exp\left(E\left[\log\left(z_t\right)\right]\right)$, the above equation implies

$$
E\left(err_{t+1}\right) = -\tfrac{1}{2}b^2\sigma_\varepsilon^2. \qquad (46)
$$

At the equilibrium value $b = -5.03$, with $\sigma_\varepsilon = 0.030$, we have $E\left(err_{t+1}\right) = -0.012$, indicating a small amount of optimism in the agent's subjective forecast, i.e., the forecast exceeds the realization on average. Optimism about future payoffs makes the agent more willing to defer

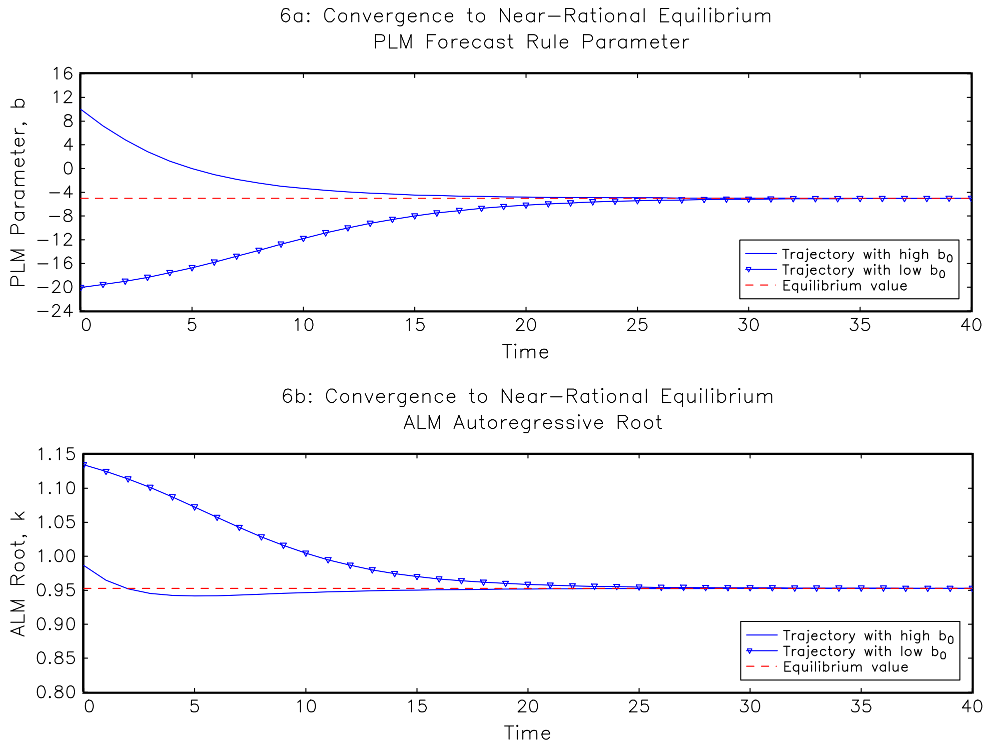

Figure 6: Iterating the nonlinear map $b_{t+1} = T\,(b_t)$ for different starting values b_0.

current consumption and increase saving, thereby driving up the share price and reducing the mean return. Conversely, pessimism about future payoffs serves to increase the mean return, as demonstrated by Abel (2002).

Equation (45) can be used to derive an analytical expression for the autocorrelation of percentage forecast errors $Corr\,(err_{t+1}, err_t)$, as outlined in the appendix. Figure 5c plots $Corr\,(err_{t+1}, err_t)$. At the equilibrium value $b = -5.03$, the autocorrelation is 0.014. The near-zero autocorrelation of the forecast errors makes it difficult for the agent to detect a misspecification of the subjective forecast rule (32).

Figure 6a plots the convergence to the near-rational equilibrium by iterating the nonlinear map as follows: $b_{t+1} = T\,(b_t)$. The map converges for starting values b_0 which are either above or below the equilibrium value $b = -5.03$. Convergence requires only 20 to 25 iterations. Figure 6b plots the corresponding trajectories for the ALM autoregressive root k, as computed from equation (39). The exercise demonstrates that the near-rational equilibrium is stable under a form of real-time learning. The equilibrium can be described as “iteratively E-stable” in the terminology of Evans and Honkapohja (2001, p. 373).

6 Model Simulations

Table 1 presents unconditional moments of asset pricing variables computed from a long-run simulation of the model. The table also reports the corresponding statistics from U.S. data over the period 1871 to 2003.[14] The fundamental solution is simulated using the expressions in Proposition 1. The expressions in Propositions 2 and 4 are used to simulate the rational bubble solutions, which are superimposed on top of the fundamental solution.[15] For the rational bubble solutions, the initial level of the bubble component y_0^{b} is set equal to 1 percent of the steady-state fundamental price-dividend ratio. For the fundamental and near-rational solutions, the initial condition is the corresponding steady-state price-dividend ratio.

The top section of Table 1 shows that the near-rational solution does an excellent job of matching the unconditional moments of the U.S. price-dividend ratio. The U.S. ratio exhibits high volatility, positive skewness, excess kurtosis, and strong positive serial correlation. In contrast, the fundamental solution delivers low volatility, near-zero skewness, no excess kurtosis, and weak negative serial correlation which is inherited directly from the consumption growth process with $\rho = -0.166$. The rational bubble solutions imply that the price-dividend ratio is non-stationary, so the corresponding moments do not exist.

The middle section of Table 1 compares unconditional moments for the drift rate of the price-dividend ratio—a stationary variable for all model solutions. As noted earlier in the discussion of Figure 3, the mean drift rate in U.S. data is 0.01 versus a drift rate of 0.06 for the Froot-Obstfeld solution. The near-rational solution provides a good match with the higher moments of the U.S. drift rate.

The last section of Table 1 compares unconditional moments for the equity return. The mean return for the near-rational solution is about 1 percentage point below that of the fundamental solution (7.35 percent versus 8.30 percent). As noted earlier, this result can be traced to the small amount of optimism in the near-rational forecast. The returns generated by the near-rational solution exhibit only a small amount of positive serial correlation, albeit slightly stronger than in U.S. data.

Figures 7a through 7h plot simulated data for the different solutions of the model. The left-side panels show the price dividend ratio $y_t = y_t^{\mathrm{f}} + y_t^{\mathrm{b}}$, while the right-side panels show the net equity return $R_t - 1$. The explosive price-dividend ratio in the Froot-Obstfeld solution with $\lambda_1 > 0$ can be seen in Figure 7a, which employs a logarithmic scale. The corresponding equity return remains stationary and exhibits time-varying volatility (Figure 7b). The two driftless rational bubble solutions exhibit irregularly-spaced episodes of expanding and contracting bubbles (Figures 7c and 7e). Return volatility increases dramatically during these episodes (Figures 7d and 7f).

[14] The price-dividend ratio in year t is defined as the value of the S&P 500 stock index at the beginning of year $t + 1$, divided by the accumulated dividend over year t.

[15] "Driftless Bubble 1" refers to the solution in Proposition 4 with $\lambda_1 > 0$, while "Driftless Bubble 2" refers to the solution with $\lambda_1 < 0$.

Table 1. Unconditional Moments

Statistic	U.S. Data 1871 – 2003	Model Simulations: Funda-mental	Froot-Obstfeld	Driftless Bubble 1	Driftless Bubble 2	Near-Rational
$y_t = p_t/d_t$			—	—	—	
Mean	25.7	16.3	—	—	—	26.0
Std. Dev.	13.0	0.07	—	—	—	15.0
Skew.	2.55	−0.01	—	—	—	2.52
Kurt.	10.6	3.00	—	—	—	12.8
Corr. Lag 1	0.93	−0.17	—	—	—	0.97
$\log(y_t/y_{t-1})$						
Mean	0.01	0.00	0.06	0.00	0.00	0.00
Std. Dev.	0.20	0.01	0.08	0.06	0.27	0.13
Skew.	−0.06	0.02	−0.02	0.26	0.04	0.03
Kurt.	3.02	3.00	3.00	86.3	4.11	3.00
Corr. Lag 1	−0.09	−0.58	−0.07	−0.11	0.02	0.04
$R_t - 1$						
Mean	8.30%	8.25%	8.70%	8.23%	7.01%	7.35%
Std. Dev	17.8%	3.95%	12.6%	8.20%	26.6%	10.8%
Skew.	−0.02	0.09	0.33	4.40	1.01	0.33
Kurt.	2.78	3.02	3.21	100.1	5.88	3.19
Corr. Lag 1	0.03	−0.29	−0.10	−0.15	0.02	0.15

Note: Model statistics are based on a 10,000 period simulation after dropping 500 periods.
Parameter values: $\overline{x} = 0.019$, $\sigma_\varepsilon = 0.030$, $\rho = -0.166$, $\alpha = 2$, and $\beta = 0.96$.

Table 2. 20-Year Rolling Volatility of Returns

Std. Dev.	U.S. Data 1871 – 2003	Model Simulations: Funda-mental	Froot-Obstfeld	Driftless Bubble 1	Driftless Bubble 2	Near-Rational
Min 20-Yr	12.5%	1.66%	3.10%	1.66%	1.16%	5.22%
Max 20-Yr	27.9%	7.21%	23.5%	75.6%	52.2%	16.5%
Full Sample.	17.8%	3.95%	12.6%	8.20%	26.6%	10.8%

Notes: Model statistics are based on a 10,000 period simulation after dropping 500 periods.
Parameter values: $\overline{x} = 0.019$, $\sigma_\varepsilon = 0.030$, $\rho = -0.166$, $\alpha = 2$, and $\beta = 0.96$.

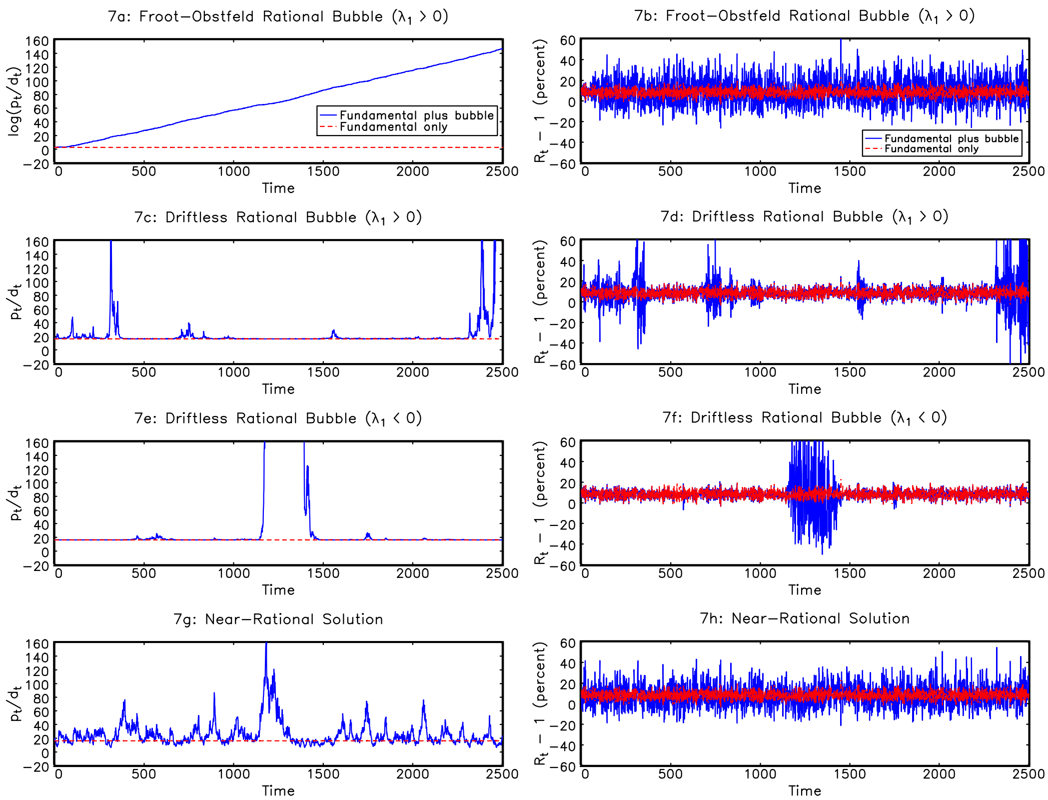

Figure 7: Left panels: price-dividend ratio. Right panels: net equity return.

Bohl and Siklos (2004) and Coakley and Fuertes (2006) fit nonlinear time series models to U.S. stock market valuation ratios over the period 1871 to 2001. Both studies find evidence that valuation ratios drift upwards into bubble territory during bull markets, but these persistent departures from fundamentals are eventually eliminated via downward adjustments during bear markets. Recent empirical tests for nonstationarity of the U.S. price-dividend ratio are inconclusive. Engsted (2006) finds support for a rational bubble in U.S. data. In contrast, a study by Koustas and Serletis (2005) rejects the rational bubble hypothesis in favor of mean-reverting behavior for the U.S. price-dividend ratio.

The near-rational solution generates pronounced low-frequency swings in the price-dividend ratio that occasionally dip below the fundamental value (Figure 7g). In contrast, rational bubble solutions require the price-dividend ratio to always remain above the fundamental.[16] The timing of expanding and contracting bubble episodes in Figure 7g is somewhat similar to that generated by the driftless rational bubble solution with $\lambda_1 < 0$ plotted in Figure 7e. Both solutions exhibit a negative response coefficient on the fundamental term $(x_t - \overline{x})$ in the corresponding law of motion.

[16] Weil (1990) notes that a positive rational bubble can cause the equilibrium asset price to dip below the ex ante fundamental value if there is sufficient feedback from the bubble to either dividends or the discount rate.

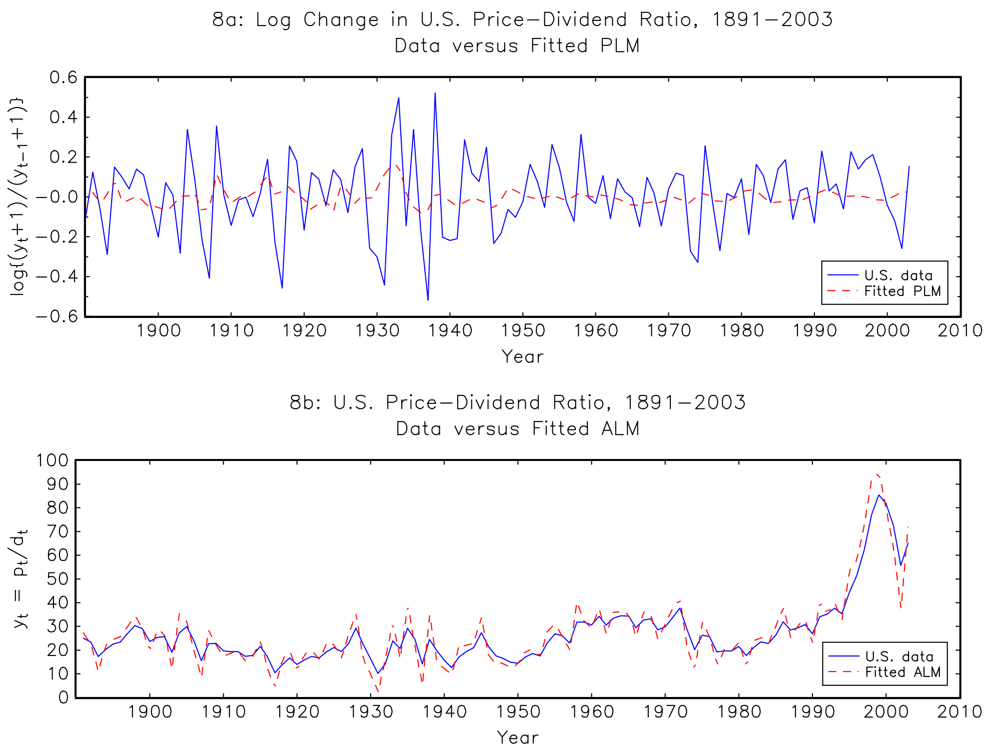

Figure 8: Fitting the perceived and actual laws of motion to long-run U.S. data.

The nonlinear nature of the exact ALM (35) gives rise to time-varying return volatility (Figure 7h). Table 2 provides a quantitative comparison of the return volatilities in U.S. data and the various model solutions. From 1871 to 2003, the 20–year rolling standard deviation of U.S. returns varies from a minimum of 12.5% to a maximum of 27.9%. The Froot-Obstfeld solution provides the best match with the data, followed by the near-rational solution.

Figure 8a shows the results of regressing the PLM (36) in first-difference form on U.S. data for the price-dividend ratio and per capita consumption growth. The regression attempts to replicate the manner in which the representative agent would estimate the parameter b using the covariance (37). For the regression, I impose $\alpha = 2$ and $\overline{x} = 0.019$.[17] The regression yields $b = -2.11$ (s.e. $= 0.58$), which is reasonably close to the theoretical value of $b = -5.03$ implied by the near-rational equilibrium. Figure 8b shows the results of regressing the ALM (35) in level form on U.S. data. I impose $\beta = 0.96$, $\alpha = 2$, $\overline{x} = 0.019$, $\sigma_\varepsilon = 0.030$, and $\rho = -0.166$. This regression yields $b = -0.753$ (s.e. $= 0.89$), which has the same sign as the theoretical value, but is not statistically different from zero. The excellent fit shown in Figure 8b is a consequence of the highly persistent nature of the ALM (35) for any value of b. From the model's perspective, the substantial run-up in U.S. price dividend ratio at the end of the

[17] With these parameter restrictions, the regression equation becomes $\Delta \log(y_t + 1) = (b+1)(x_t - 0.019) - (x_{t-1} - 0.019)$, where y_t is the U.S. price-dividend ratio and x_t is U.S. per capita consumption growth.

sample can be interpreted as a long swing that results when random shocks impinge upon a highly-persistent law of motion.[18]

Table 3 provides a quantitative comparison of forecast errors between the fundamental and near-rational solutions. As expected, the fundamental solution delivers a lower root mean squared percentage error. However, the near-rational forecast errors are close to white noise at all lags—giving no discernible indication to the agent that his subjective forecast rule (32) is misspecified.

Table 3: Comparison of Percentage Forecast Errors

	Model Simulations	
Statistic	Fundamental	Near-Rational
$E\left(err_{t+1}\right)$	0.00	-0.01
$\sqrt{E\left(err_{t+1}^2\right)}$	0.03	0.16
$Corr\left(err_{t+1},\ err_t\right)$	-0.01	0.01
$Corr\left(err_{t+1},\ err_{t-1}\right)$	-0.01	-0.03
$Corr\left(err_{t+1},\ err_{t-2}\right)$	-0.01	-0.02

Notes: Model statistics are based on a 10,000 period simulation after dropping 500 periods. Parameter values: $\overline{x} = 0.019$, $\sigma_\varepsilon = 0.030$, $\rho = -0.166$, $\alpha = 2$, and $\beta = 0.96$.

7 Concluding Remarks

Theories involving departures from fully-rational behavior have long played a role in efforts to account for the behavior of asset prices. Keynes (1936, p. 156) likened the stock market to a "beauty contest" where participants devoted their efforts not to judging the underlying concept of beauty, but instead to "anticipating what average opinion expects the average opinion to be."

There are many examples in history of asset prices exhibiting sustained run-ups that are difficult to justify on the basis of economic fundamentals. The typical transitory nature of these run-ups should perhaps be viewed as a long-run victory for fundamental asset pricing theory. Still, it remains a challenge for fundamental theory to explain the ever-present volatility of asset prices within a framework of efficient capital markets. Rational bubbles are an attractive modeling device because the framework allows asset prices to exceed fundamentals while imposing a no-arbitrage condition over short time horizons. In a rational bubble solution, an asset is valued not for its cash flows, but rather for its potential to deliver capital gain—a feature that seems to fit the prevailing psychology during historical bubble episodes.

This paper demonstrated the existence of a continuum of intrinsic rational bubble solutions that involve an equilibrium trade-off between the mean and volatility of the bubble drift rate.

[18] Boswijk, et al. (2007) interpret the end-of-sample run-up in the U.S. price-dividend ratio as an increase in the prevalence of trend-chasing agents versus fundamentalists. In their model, the shift in the composition of agents is itself a long swing that is driven by a series of random shocks.

When the mean drift rate of the bubble is zero by construction, the short-term prospects for capital gain derive solely from the high volatility of the bubble component. A driftless rational bubble exhibits irregularly-spaced expansions and collapses that are wholly endogenous.

Strictly speaking, rational bubbles are not fully rational because the transversality condition is not satisfied. In a world where agents' computational resources are limited, further movements away from full rationality would seem plausible. The near-rational asset pricing solution developed here is based on a parsimonious and versatile forecast rule: a geometric random walk without drift, where innovations to the random walk are linked to consumption/dividend growth. When the agent's forecast rule is parameterized to match the moments of observable data, the resulting forecast errors are close to white noise. The near-rational solution does a good job of matching many quantitative features of U.S. stock market data and allows the equity price to occasionally dip below the fundamental price.

A Appendix: Derivations and Proofs

A.1 Separating Consumption from Dividends

The Lucas (1978) model implies $c_t = d_t$ for all t. This section outlines a version of the model that allows $c_t \neq d_t$. The agent's first-order condition is

$$\frac{p_t}{d_t} = E_t\left\{\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\alpha}\left(\frac{d_{t+1}}{d_t}\right)\left(\frac{p_{t+1}}{d_{t+1}}+1\right)\right\}. \tag{A.1}$$

The separate growth rates of consumption and dividends are now given by

$$\log\left(c_t/c_{t-1}\right) \equiv x_t^c = \overline{x}^c + \rho\left(x_{t-1}^c - \overline{x}^c\right) + \varepsilon_t, \qquad \varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right), \quad |\rho| < 1, \tag{A.2}$$

$$\log\left(d_t/d_{t-1}\right) \equiv x_t^d = \overline{x}^d + \gamma\left(x_t^c - \overline{x}^c\right) + v_t \qquad v_t \sim N\left(0, \sigma_v^2\right), \tag{A.3}$$

where v_t is uncorrelated with ε_t. As before, consumption growth is described by a univariate AR(1) process. Deviations of dividend growth from consumption growth are governed by the parameters $\overline{x}^d$, γ, and σ_v^2. The original Lucas model with $c_t = d_t$ can be recovered by setting $\overline{x}^d = \overline{x}^c$, $\gamma = 1$, and $\sigma_v^2 = 0$. For the model with $c_t \neq d_t$, these parameters are calibrated to match three moments: (1) the unconditional mean of dividend growth $E\left[\log\left(d_t/d_{t-1}\right)\right]$, (2) the contemporaneous correlation between dividend growth and consumption growth $Corr\left(x_t^d,\, x_t^c\right)$, and (3) the unconditional variance of dividend growth $Var\left(x_t^d\right)$. The resulting calibration formulas are

$$\overline{x}^d = E\left[\log\left(d_t/d_{t-1}\right)\right], \tag{A.4}$$

$$\gamma = Corr\left(x_t^d,\, x_t^c\right)\left[Var\left(x_t^d\right)/Var\left(x_t^c\right)\right]^{1/2}, \tag{A.5}$$

$$\sigma_v^2 = Var\left(x_t^d\right) - \gamma^2 Var\left(x_t^c\right). \tag{A.6}$$

The agent's first-order condition can be written in terms of the price-dividend ratio y_t as follows:

$$\begin{aligned} y_t &= E_t\left\{\beta\exp\left[-\alpha x_{t+1}^c + \overline{x}^d + \gamma\left(x_{t+1}^c - \overline{x}^c\right) + v_{t+1}\right]\left(y_{t+1}+1\right)\right\}, \\ &= E_t\left\{\widetilde{\beta}\exp\left(\widetilde{\theta}\,\widetilde{x}_{t+1}\right)\left(y_{t+1}+1\right)\right\}, \end{aligned} \tag{A.7}$$

$$\text{where} \qquad \widetilde{\beta} \equiv \beta\exp\left(\overline{x}^d - \gamma\overline{x}^c\right), \qquad \widetilde{\theta} \equiv \gamma - \alpha, \qquad \widetilde{x}_t \equiv x_t^c + v_t/\widetilde{\theta}.$$

Making use of the above definitions, equations (A.2) and (A.3) yield the following transformed version of equation (3):

$$\widetilde{x}_t = \overline{x}^c + \rho\left(\widetilde{x}_{t-1} - \overline{x}^c\right) + \omega_t, \qquad \omega_t \equiv \varepsilon_t + \left(v_t - \rho v_{t-1}\right)/\widetilde{\theta}, \tag{A.8}$$

where $\omega_t \sim N\left(0, \sigma_\omega^2\right)$, and $\sigma_\omega^2 = \sigma_\varepsilon^2 + \left(1+\rho^2\right)\sigma_v^2/\widetilde{\theta}^2$.

Finally, we define $\widetilde{z}_t \equiv \widetilde{\beta} \exp\left(\widetilde{\theta}\, \widetilde{x}_t\right)(y_t + 1)$ to obtain the following transformed version of equation (8):

$$\widetilde{z}_t = \widetilde{\beta} \exp\left(\widetilde{\theta}\, \widetilde{x}_t\right)\left[E_t \widetilde{z}_{t+1} + 1\right]. \tag{A.9}$$

Thus, by an appropriate change of variables, equations (A.8) and (A.9) retain the same basic forms as equations (3) and (8).

A.2 Proof of Proposition 1: Approximate Fundamental Solution

Iterating ahead the conjectured law of motion for z_t^{f} and taking the conditional expectation yields

$$E_t z_{t+1}^{\mathrm{f}} = \exp\left[a_0 + \rho a_1 (x_t - \overline{x}) + \tfrac{1}{2}(a_1)^2 \sigma_\varepsilon^2\right]. \tag{A.10}$$

Substituting the above expression into the first order condition (8) and then taking logarithms yields

$$\begin{aligned}\log\left(z_t^{\mathrm{f}}\right) = F(x_t) &= \log(\beta) + \theta x_t \\ &\quad + \log\left\{\exp\left[a_0 + \rho a_1 (x_t - \overline{x}) + \tfrac{1}{2}(a_1)^2 \sigma_\varepsilon^2\right] + 1\right\}, \\ &\simeq a_0 + a_1 (x_t - \overline{x}),\end{aligned} \tag{A.11}$$

where a_0 and a_1 are the Taylor-series coefficients for an approximation of $F(x_t)$ around the non-stochastic steady state $\overline{x}$. The Taylor-series coefficients are given by

$$F(\overline{x}) = a_0 = \log(\beta) + \theta\overline{x} + \log\left\{\exp\left[a_0 + \tfrac{1}{2}(a_1)^2 \sigma_\varepsilon^2\right] + 1\right\} \tag{A.12}$$

$$F'(\overline{x}) = a_1 = \theta + \frac{\rho a_1 \exp\left[a_0 + \tfrac{1}{2}(a_1)^2 \sigma_\varepsilon^2\right]}{\exp\left[a_0 + \tfrac{1}{2}(a_1)^2 \sigma_\varepsilon^2\right] + 1}. \tag{A.13}$$

Solving equation (A.12) for a_0 yields

$$a_0 = \log\left\{\frac{\beta \exp(\theta\overline{x})}{1 - \beta \exp\left[\theta\overline{x} + \tfrac{1}{2}(a_1)^2 \sigma_\varepsilon^2\right]}\right\}, \tag{A.14}$$

which can be substituted into equation (A.13) to yield the following nonlinear equation that determines a_1:

$$a_1 = \theta + \rho a_1 \beta \exp\left[\theta\overline{x} + \tfrac{1}{2}(a_1)^2 \sigma_\varepsilon^2\right]. \tag{A.15}$$

Solving equation (A.15) for a_1 yields the nonlinear equation shown in Proposition 1. There are two solutions, but only one solution satisfies the condition $\beta \exp\left[\theta\overline{x} + \tfrac{1}{2}(a_1)^2 \sigma_\varepsilon^2\right] < 1$ such that $\exp(a_0) = \exp\left[E \log\left(z_t^{\mathrm{f}}\right)\right] > 0$. ■

A.3 Asset Pricing Moments: Fundamental Solution

This section briefly outlines the derivation of equations (16) through (19).

Equation (16) follows directly from equation (15) by taking the unconditional expectation of $\log\left(y_t^{\mathrm{f}}\right)$. We have

$$\log\left(y_t^{\mathrm{f}}\right)-E\left[\log\left(y_t^{\mathrm{f}}\right)\right]=a_1\rho_1\left(x_t-\overline{x}\right), \tag{A.16}$$

which implies $Var\left[\log\left(y_t^{\mathrm{f}}\right)\right]=\left(a_1\right)^2\rho^2 Var\left(x_t\right)$, as given by equation (17).

The fundamental equity return can be written as

$$\begin{aligned} R_{t+1}^{\mathrm{f}} &= \left(\frac{y_{t+1}^{\mathrm{f}}+1}{y_t^{\mathrm{f}}}\right)\exp\left(x_{t+1}\right), \\ &= \left(\frac{z_{t+1}^{\mathrm{f}}}{\beta E_t z_{t+1}^{\mathrm{f}}}\right)\exp\left(\alpha\, x_{t+1}\right), \end{aligned} \tag{A.17}$$

where I have eliminated y_t^{f} using the equilibrium relationship $y_t^{\mathrm{f}}=E_t z_{t+1}^{\mathrm{f}}$ and eliminated y_{t+1}^{f} using the definitional relationship $y_{t+1}^{\mathrm{f}}+1=\beta^{-1}\exp\left(-\theta x_{t+1}\right)z_{t+1}^{\mathrm{f}}$. Substituting in $z_{t+1}^{\mathrm{f}}=\exp\left[a_0+a_1\left(x_t-\overline{x}\right)\right]$ from Proposition 1 and $E_t z_{t+1}^{\mathrm{f}}$ from equation (15) and then taking the unconditional expectation of $\log\left(R_{t+1}^{\mathrm{f}}\right)$ yields equation (18). We have

$$\log\left(R_{t+1}^{\mathrm{f}}\right)-E\left[\log\left(R_{t+1}^{\mathrm{f}}\right)\right]=\alpha\left(x_{t+1}-\overline{x}\right)+a_1\varepsilon_{t+1}, \tag{A.18}$$

which in turns implies

$$Var\left[\log\left(R_{t+1}^{\mathrm{f}}\right)\right]=\alpha^2 Var\left(x_t\right)+\left(a_1\right)^2\sigma_\varepsilon^2+2\alpha a_1 Cov\left(x_t,\varepsilon_t\right), \tag{A.19}$$

as given by equation (19).

A.4 Proof of Proposition 5: Continuum of Intrinsic Rational Bubbles

The proof of Proposition 5 covers Propositions 2, 3, and 4 since these are just special cases of Proposition 5.

First consider the case where the agent can make use of the contemporaneous realization z_t^{b} when forming the rational expectation $E_t z_{t+1}^{\mathrm{b}}$. Iterating ahead the conjectured law of motion for z_t^{b} by one period and then taking the conditional expectation yields

$$E_t z_{t+1}^{\mathrm{b}}=z_t^{\mathrm{b}}\exp\left[\lambda_0+\left(\rho\lambda_1+\lambda_2\right)\left(x_t-\overline{x}\right)+\tfrac{1}{2}\left(\lambda_1\right)^2\sigma_\varepsilon^2\right]. \tag{A.20}$$

Substituting the above expression into the no-arbitrage condition (21) and then taking logarithms yields

$$0=\log\left(\beta\right)+\theta x_t+\lambda_0+\left(\rho\lambda_1+\lambda_2\right)\left(x_t-\overline{x}\right)+\tfrac{1}{2}\left(\lambda_1\right)^2\sigma_\varepsilon^2, \tag{A.21}$$

where $\log\left(z_t^{\mathrm{b}}\right)$ has been cancelled from both sides. For equation (A.21) to hold, the constant terms and the coefficients on x_t must separately sum to zero. Equilibrium therefore requires

$$\tfrac{1}{2}\left(\lambda_1\right)^2\sigma_\varepsilon^2 \;-\; \underbrace{\left(\rho\lambda_1+\lambda_2\right)}_{-\theta}\overline{x} \;+\; \log\left(\beta\right)+\lambda_0 \;=\; 0, \tag{A.22}$$

$$\theta \;+\; \rho\lambda_1+\lambda_2 \;=\; 0, \tag{A.23}$$

which represent a system of two equations in three unknown constants λ_0, λ_1, and λ_2. The solutions to equations (A.22) and (A.23) define a continuum of intrinsic rational bubble equilibria. Now consider the case where the agent can only make use of the lagged realization z_{t-1}^{b} when forming $E_t z_{t+1}^{\mathrm{b}}$. Iterating ahead the conjectured law of motion for z_t^{b} by one period and then substituting out z_t^{b} using the same law of motion yields

$$z_{t+1}^{\mathrm{b}} = z_{t-1}^{\mathrm{b}} \exp\left\{2\lambda_0 + \left[\lambda_1 (1+\rho) + \lambda_2\right](x_t - \overline{x}) + \lambda_2 (x_{t-1} - \overline{x}) + \lambda_1 \varepsilon_{t+1}\right\}, \qquad \text{(A.24)}$$

where I have eliminated $(x_{t+1} - \overline{x})$ using the law of motion for dividend growth (35). Taking the conditional expectation of the above expression yields

$$E_t z_{t+1}^{\mathrm{b}} = z_{t-1}^{\mathrm{b}} \exp\left\{2\lambda_0 + \left[\lambda_1 (1+\rho) + \lambda_2\right](x_t - \overline{x}) + \lambda_2 (x_{t-1} - \overline{x}) + \tfrac{1}{2} (\lambda_1)^2 \sigma_\varepsilon^2\right\}. \qquad \text{(A.25)}$$

Substituting the above expression into the no-arbitrage condition (21) and then taking logarithms yields

$$\begin{aligned} \log\left(z_t^{\mathrm{b}}\right) \;=\; & \log\left(z_{t-1}^{\mathrm{b}}\right) + \log(\beta) + \theta x_t + 2\lambda_0 + \left[\lambda_1 (1+\rho) + \lambda_2\right](x_t - \overline{x}) \\ & + \lambda_2 (x_{t-1} - \overline{x}) + \tfrac{1}{2} (\lambda_1)^2 \sigma_\varepsilon^2, \end{aligned} \qquad \text{(A.26)}$$

which can be compared to the following expression for the logarithm of the conjectured law of motion:

$$\log\left(z_t^{\mathrm{b}}\right) = \log\left(z_{t-1}^{\mathrm{b}}\right) + \lambda_0 + \lambda_1 (x_t - \overline{x}) + \lambda_2 (x_{t-1} - \overline{x}). \qquad \text{(A.27)}$$

Equation (A.26) will coincide exactly with equation (A.27) when the following equilibriums conditions are satisfied

$$\log(\beta) + 2\lambda_0 - \left[\lambda_1 (1+\rho) + 2\lambda_2\right]\overline{x} + \tfrac{1}{2} (\lambda_1)^2 \sigma_\varepsilon^2 \;=\; \lambda_0 - (\lambda_1 + \lambda_2)\,\overline{x}, \qquad \text{(A.28)}$$

$$\theta + \lambda_1 (1+\rho) + \lambda_2 \;=\; \lambda_1, \qquad \text{(A.29)}$$

which are isomorphic to the equilibrium conditions (A.22) and (A.23). ■

A.5 Asset Pricing Moments: Near-Rational Solution

Starting from the approximate ALM (38), the law of motion of $\Delta \log(z_t)$ can be written as:

$$\Delta \widehat{z}_t = (k-1)\left[\widehat{z}_{t-1} - E(\widehat{z}_t)\right] + m (x_t - \overline{x}), \qquad \text{(A.30)}$$

where $\widehat{z}_t \equiv \log(z_t)$. The above equation implies:

$$Cov(\Delta \widehat{z}_t,\, x_t) = (k-1)\, Cov(\widehat{z}_{t-1},\, x_t) + m Var(x_t). \qquad \text{(A.31)}$$

From (38), we have $Cov(\widehat{z}_{t-1},\, x_t) = [\rho m/(1-\rho k)]\, Var(x_t)$, which can be substituted into (A.31) to yield equation (42) in the text.

The ALM for the price-dividend ratio, equation (35), can be rewritten as follows:

$$
\begin{aligned}
y_t &= (y_{t-1}+1)\,\beta \exp\left[b\,(1+\rho)\,(x_t-\overline{x})+\theta x_{t-1}+\tfrac{1}{2}b^2\sigma_\varepsilon^2\right], \\
&= (y_{t-1}+1)\,k \exp\left[\left(\frac{m-\theta}{k}\right)(x_t-\overline{x})+\theta\,(x_{t-1}-\overline{x})\right], \qquad \text{(A.32)}
\end{aligned}
$$

where I have eliminated b and b^2 using the expressions for the Taylor series coefficients k and m, as given by equations (39) and (40). Taking logarithms of the above expression yields

$$
\begin{aligned}
\widehat{y}_t &= \log\left[\exp\left(\widehat{y}_{t-1}\right)+1\right]+\log(k)+\left(\frac{m-\theta}{k}\right)(x_t-\overline{x})+\theta\,(x_{t-1}-\overline{x}), \\
&\simeq n_0+n_1\left[\widehat{y}_{t-1}-E\left(\widehat{y}_t\right)\right]+\left(\frac{m-\theta}{k}\right)(x_t-\overline{x})+\theta\,(x_{t-1}-\overline{x}), \qquad \text{(A.33)}
\end{aligned}
$$

where $\widehat{y}_t \equiv \log(y_t)$, and n_0 and n_1 are Taylor series coefficients. Straightforward computations yield $n_0 = \log\left[k/(1-k)\right]$ and $n_1 = k$. The unconditional expectation of the above expression yields $E(\widehat{y}_t) = n_0$, as given by equation (43).

Using equation (A.33), the unconditional variance can be computed as follows:

$$
\begin{aligned}
Var\left(\widehat{y}_t\right) &= E\left\{\left[\widehat{y}_t-E\left(\widehat{y}_t\right)\right]^2\right\}, \\
&= \left(\frac{1}{1-k^2}\right)\left[\left(\frac{m-\theta}{k}\right)^2+\theta^2+2\left(\frac{m-\theta}{k}\right)\theta\rho\right] Var\left(x_t\right), \\
&\quad + \left[\frac{2\,(m-\theta)\,\rho+2\theta k}{1-k^2}\right] Cov\left(\widehat{y}_t, x_t\right), \qquad \text{(A.34)}
\end{aligned}
$$

where $Cov(\widehat{y}_t, x_t)$ can also be computed from equation (A.33).

The equity return is given by

$$
\begin{aligned}
R_{t+1} &= \left(\frac{z_{t+1}}{\beta \widehat{E}_t z_{t+1}}\right)\exp\left(\alpha\, x_{t+1}\right) \\
&= \frac{z_t^k\, \overline{z}^{\,1-k}\, \exp\left[m\ (x_{t+1}-\overline{x})+\alpha\, x_{t+1}\right]}{\beta z_{t-1}\exp\left[b\,(1+\rho)\,(x_t-\overline{x})+\frac{1}{2}b^2\sigma_\varepsilon^2\right]}, \qquad \text{(A.35)}
\end{aligned}
$$

where I have substituted in the approximate ALM (38) and the subjective expectation (32). Taking the unconditional expectation of $\widehat{R}_{t+1} \equiv \log(R_{t+1})$ yields equation (44). From (A.35), we have

$$
\begin{aligned}
\widehat{R}_{t+1}-E\left(\widehat{R}_{t+1}\right) &= \underbrace{k\left[\widehat{z}_t-E\left(\widehat{z}_t\right)\right]}_{k^2\left[\widehat{z}_{t-1}-E\left(\widehat{z}_t\right)\right]+km(x_t-\overline{x})} - \left[\widehat{z}_{t-1}-E\left(\widehat{z}_t\right)\right] \\
&\quad + (m+\alpha)\ (x_{t+1}-\overline{x}) - b\,(1+\rho)\,(x_t-\overline{x}), \qquad \text{(A.36)}
\end{aligned}
$$

which can be used to compute an analytical expression for $Var\left(\widehat{R}_{t+1}\right)$.

From equation (45), the law of motion for the percentage forecast error is given by

$$\begin{aligned} err_{t+1} - E\left(err_{t+1}\right) &= -\left(1-k^2\right)\left[\widehat{z}_{t-1} - E\left(\widehat{z}_t\right)\right] \\ &\quad + \left[km + m\rho - b\left(1+\rho\right)\right]\left(x_t - \overline{x}\right) + m\varepsilon_{t+1}, \end{aligned} \tag{A.37}$$

where I have eliminated $\left[\widehat{z}_t - E\left(\widehat{z}_t\right)\right]$ using the approximate ALM (38). Equation (A.37) is used to compute $Corr\left(err_{t+1}, err_t\right) = Cov\left(err_{t+1}, err_t\right)/Var\left(err_{t+1}\right)$, as plotted in Figure 5.

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